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The accuracy analysis of fully nonlinear Boussinesq equations

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SUMMARY

The accuracy analysis of fully nonlinear Boussinesq equations is made in this study. We first introduce the modified Boussinesq equations expressed in a recursion form. It is helpful for engineers to derive the higher order of equations in the future if needed. Almost all linear and nonlinear wave properties are derived subsequently. By comparing the error magnitude of each wave property, one can determine the optimal model and find that for wave propagation in shallow water, the higher order of Boussinesq equations provides a better prediction. But, for wave properties in deep water, the $O\left(\mu^4\right)$ *model behaves better in nonlinear properties and worse in some particle characteristics.*

Key words: *Boussinesq equations, accuracy, nonlinear property, linear property.*

1. INTRODUCTION

When the classical Boussinesq equations first appeared in 19th century [1], it had profoundly influenced the analysis of long waves propagating in coastal zones for almost one century. In late 20th century, many researchers were interested in rederiving Boussinesq-type equations again because there were still many restrictions worth being overcome. For example, the classical Boussinesq equations cannot be applied very well in deeper water or extremely shallow water.

In recent years, many studies and efforts have apparently enhanced the Boussinesq equations and extended its applicable region from shallow water to deep water. Peregrine [2] improved the classical Boussinesq equations for wave propagation over an uneven seabed. This is a key opening for new researches on old equations. For most researchers, numbers of efforts they made are for the sake of extending the applicable depth range of Boussinesq equations wider. Consequently, various kinds of the optimal wave equations were evaluated in the following two decades. Witting [3] applied conservative equations to analyze wave mechanics in a constant-depth channel. The wave equations were expressed in terms of the depth-averaged velocity and the mean free surface velocity respectively. He expanded the dispersion relation into a Taylor series form. Then the corresponding coefficients were determined to yield a Padé approximation to the Taylor expansion of the dispersion relation given by the linear Stokes wave theory. By applying the (2,2) Padé approximation, Witting obtained good results for both deep and shallow water waves.

In 1991, Madsen et al. [4] formulated the conventional Boussinesq equations for the flat bottom in terms of volume flux components instead of the depth-averaged velocity. They introduced some higher-order terms in the momentum equations, which were conventionally neglected in the process of deriving the Boussinesq equations. An additional third-order term was also added to momentum equations to improve the linear dispersion properties of Boussinesq-type equations. Madsen and Sorensen [5] further extended this set of Boussinesq-type equations for a slowing varying topography and introduced the linear shoaling gradient as another quantity to evaluate the improvement of wave equations. Nwogu [6] derived a new set of modified Boussinesq equations in terms of a horizontal velocity at an arbitrary elevation. His study is a crucial beginning for other consequent studies. The linear dispersion property in his study can approach very close to that of the first-order Stokes wave if a velocity near the middle depth is selected as the velocity variable. This makes the new set of equations applicable to wave propagation from relatively deep water to shallow water. Chen and Liu [7] rederived the $O\left(\mu^2\right)$ Boussinesq-type equations in terms of an arbitrary-depth velocity potential instead of the horizontal velocity adopted by Nwogu. The optimal elevation where the velocity potential should be evaluated is determined by comparing the phase velocity, group velocity and the shoaling gradient with those given by the linear Stokes theory. Their investigation provides an optimal wave model at a specific elevation near the middle depth that is lightly different from Nwogu's result.

Gobbi et al. [8] derived the $O\left(\mu^4\right)$ Boussinesq equations by introducing a new variable defined as a weighted average of the velocity potential at two distinct water depths. They determined the values of two parameters in the weighted velocity potential and so obtained the best Boussinesq-type model by comparing the coefficients of the Taylor-like expansion of dispersion relation with those of Padé (4,4) approximation of the exact linear solution. Some linear kinematic properties were also introduced to evaluate the wave equations. Madsen et al. [9] introduce finite series (Boussinesq-type) approximations involving up to fifth-derivative operators. By using the Padé approximants, they make some wave characteristics very accurate within the range of μ from *0* to *40*.

As mentioned in previous retrospection, almost each model was derived by satisfying the dispersion relation to the Padé approximation. These expansions lead to almost the best accuracy of the wave celerity not only in shallow water but in deeper water. However, these models based on the satisfaction of the Padé approximation are still weak in describing the velocity profile of water particles. In the other way, the models which can predict the velocity profile well often can not make the celerity close to the exact solution. To overcome such a natural weakness of these Padé-based models, we derive a set of the sixth-order Boussinesq-type equations in terms of a velocity potential at an arbitrary water depth for wave propagation over an uneven bottom. In our derivation, almost each variable and coefficient of the higher-order equations can be developed in the recursion form. These expressions greatly simplify the procedures of deriving the higher-order Boussinesq equations.

Based on the above reasons, the $O\left(\mu^2\right)$, $O\left(\mu^4\right)$ and $O\left(\mu^6\right)$ Boussinesq equations, which are described by the velocity potential at an arbitrary *z*-level, are introduced in Section 2. In order to derive the higher-order equations more conveniently in the coming future, we express some formulations and variables with the iteration forms. Some linear and nonlinear properties of waves are consequently derived. The Stokes solutions are also provided for comparison. In Section 3, the accuracy analysis is made by investigating all wave properties. Conclusions are made in Section 4.

2. MATHEMATICAL FORMULATION

First, we introduce the velocity potential Φ_m that the subscript *m* denotes the expressed by the velocity potential at an arbitrary *z*-level as:

$$
\begin{aligned}
\Phi = \Phi_m &+ \mu^2\left[\left(z-z_m\right)\Psi_1 - \frac{\left(z^2-z_m^2\right)}{2!}\nabla^2\Phi_{00}\right] + \\
&+\mu^4\left[\left(z-z_m\right)\Psi_2 - \frac{\left(z^2-z_m^2\right)}{2!}\nabla^2\Phi_{10} - \frac{\left(z^3-z_m^3\right)}{3!}\nabla^2\Psi_1 + \frac{\left(z^4-z_m^4\right)}{4!}\nabla^2\nabla^2\Phi_{00}\right] + \\
&+\mu^6\Bigg[\left(z-z_m\right)\Psi_3 - \frac{\left(z^2-z_m^2\right)}{2!}\nabla^2\Phi_{20} - \frac{\left(z^3-z_m^3\right)}{3!}\nabla^2\Psi_2 + \frac{\left(z^4-z_m^4\right)}{4!}\nabla^2\nabla^2\Phi_{10} + \\
&+\frac{\left(z^5-z_m^5\right)}{5!}\nabla^2\nabla^2\Psi_1 - \frac{\left(z^6-z_m^6\right)}{6!}\nabla^2\nabla^2\nabla^2\Phi_{00}\Bigg] + O\left(\mu^8\right)
\end{aligned}
\tag{1}
$$

Inserting Eq. (1) into the following equations:

$$\frac{\partial \Phi}{\partial t}+\eta+\frac{\varepsilon}{2}\left[\left(\nabla \Phi\right)^2+\frac{1}{\mu^2}\left(\frac{\partial \Phi}{\partial z}\right)^2\right]=0 \ \ at \quad z=\varepsilon\eta \tag{2}$$

$$\nabla\cdot\int_{-h}^{\varepsilon\eta}\nabla\Phi dz+\frac{\partial \eta}{\partial t}=0 \tag{3}$$

leads to the fully nonlinear Boussinesq equations of the accuracy of the sixth order. The linear forms of Eqs. (2) and (3) are as follows:

$$\begin{aligned}&\frac{\partial \eta}{\partial t}+G_1 h\nabla^2\Phi_m+\mu^2\left(G_2 h^3\nabla^2\nabla^2\Phi_m\right)+\mu^4\left(G_3 h^5\nabla^2\nabla^2\nabla^2\Phi_m\right)+\\&+\mu^6\left(G_4 h^7\nabla^2\nabla^2\nabla^2\nabla^2\Phi_m\right)+O\left(\mu^8\right)=0\end{aligned} \tag{4}$$

$$\begin{aligned}&\eta+\frac{\partial \Phi_m}{\partial t}+\mu^2\left(H_1 h^2\nabla^2\frac{\partial \Phi_m}{\partial t}\right)+\mu^4\left(H_2 h^4\nabla^2\nabla^2\frac{\partial \Phi_m}{\partial t}\right)+\\&+\mu^6\left(H_3 h^6\nabla^2\nabla^2\nabla^2\frac{\partial \Phi_m}{\partial t}\right)+O\left(\mu^8\right)=0\end{aligned} \tag{5}$$

where:

$$\begin{cases}H_1=m+\dfrac{m^2}{2}\\ H_2=\dfrac{m}{3}+m^2+\dfrac{5m^3}{6}+\dfrac{5m^4}{24}\\ H_3=\dfrac{2m}{15}+\dfrac{2m^2}{3}+\dfrac{23m^3}{18}+\dfrac{7m^4}{6}+\dfrac{61m^5}{120}+\dfrac{61m^6}{720}\end{cases} \tag{6}$$

and:

$$\begin{cases}G_1=1\\ G_2=\dfrac{1}{3}+m+\dfrac{m^2}{2}\\ G_3=\dfrac{2}{15}+\dfrac{2m}{3}+\dfrac{7m^2}{6}+\dfrac{5m^3}{6}+\dfrac{5m^4}{24}\\ G_4=\dfrac{17}{315}+\dfrac{17m}{45}+\dfrac{16m^2}{15}+\dfrac{14m^3}{9}+\dfrac{89m^4}{72}+\dfrac{61m^5}{120}+\dfrac{61m}{720}\end{cases} \tag{7}$$

The above linear equations are of the sixth order. By assigning the different value of *m*, the different type of Boussinesq equations are obtained. To study the optimal Boussinesq equations, five linear properties are introduced as:

$$C=\left[\frac{1-G_2\mu^2+G_3\mu^4-G_4\mu^6}{1-H_1\mu^2+H_2\mu^4-H_3\mu^6}\right]^{0.5} \tag{8}$$

$$C_g=\frac{\partial}{\partial\mu}\left\{\mu\left[\frac{1-G_2\mu^2+G_3\mu^4-G_4\mu^6}{1-H_1\mu^2+H_2\mu^4-H_3\mu^6}\right]^{0.5}\right\} \tag{9}$$

$$\begin{aligned}F_V=\frac{w(z,m)}{w(o,m)}=&\left\{\left(-G_1-z\right)\mu^2-\left(-G_2-zH_1+\frac{z^2}{2!}G_1+\frac{z^3}{3!}\right)\mu^4+\right.\\&\left.+\left(-G_3-zH_2+\frac{z^2}{2!}G_2+\frac{z^3}{3!}H_1-\frac{z^4}{4!}G_1-\frac{z^5}{5!}\right)\mu^6\right\}/\left[\left(-G_1\right)\mu^2-\left(-G_2\right)\mu^4+\left(-G_3\right)\mu^6\right]\end{aligned} \tag{10}$$

$$F_H = \frac{u(z,m)}{u(0,m)} = \left\{ 1 - \left[(m-z)G_1 - \frac{z^2-m^2}{2!} \right]\mu^2 + \left[(m-z)G_2 - \frac{(z^2-m^2)}{2!}H_1 + \frac{(z^3-m^3)}{3!}G_1 + \frac{(z^4-m^4)}{4!} \right]\mu^4 - \right.$$
$$\left. - \left[(m-z)G_3 - \frac{(z^2-m^2)H_2}{2!} + \frac{(z^3-m^3)G_2}{3!} + \frac{(z^4-m^4)H_1}{4!} - \frac{(z^5-m^5)G_1}{5!} - \frac{(z^6-m^6)}{6!} \right]\mu^6 \right\} /$$
$$/ \left\{ 1 - \left[mG_1 + \frac{m^2}{2!} \right]\mu^2 + \left[mG_2 + \frac{m^2}{2!}H_1 - \frac{m^3}{3!}G_1 - \frac{m^4}{4!} \right]\mu^4 - \left[mG_3 + \frac{m^2}{2!}H_2 - \frac{m^3}{3!}G_2 - \frac{m^4}{4!}H_1 + \frac{m^5}{5!}G_1 + \frac{m^6}{6!} \right]\mu^6 \right\}$$
(11)

$$F_T = \frac{w(0,m)}{u(0,m)} \cdot \frac{1}{\mu} = \left[(-G_1)\mu - (-G_2)\mu^3 + (-G_3)\mu^5 \right] /$$
$$/ \left\{ 1 - \left(mG_1 + \frac{m^2}{2!} \right)\mu^2 + \left(mG_2 + \frac{m^2}{2!}H_1 - \frac{m^3}{3!}G_1 - \frac{m^4}{4!} \right)\mu^4 - \right. \qquad (12)$$
$$\left. - \left(mG_3 + \frac{m^2}{2!}H_2 - \frac{m^3}{3!}G_2 - \frac{m^4}{4!}H_1 + \frac{m^5}{5!}G_1 + \frac{m^6}{6!} \right)\mu^6 \right\}$$

where C, C_g, F_H, F_V and F_T indicate the phase velocity, the group velocity, the horizontal velocity of particle, the vertical velocity of particle and the particle trajectory, respectively. The corresponding Stokes solutions are expressed as:

$$C_S = \left(\frac{tanh\,\mu}{\mu} \right)^{0.5}$$

$$C_{gS} = \frac{1}{2}\left(\frac{tanh\,\mu}{\mu} \right) \cdot \left(1 + \frac{2\mu}{sinh\,2\mu} \right)$$

$$F_{HS} = \frac{cosh\left[\mu(1+z) \right]}{cosh\,\mu}$$

$$F_{VS} = \frac{sinh\left[\mu(1+z) \right]}{sinh\,\mu}$$

$$F_{TS} = tanh\,\mu$$

As for the study of nonlinear characteristics, consider the following perturbations:

$$\Phi_m = \tilde{\Phi}_0 + \varepsilon\tilde{\Phi}_1 + \varepsilon^2\tilde{\Phi}_2 + ... \qquad (13)$$

$$\eta = \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + ... \qquad (14)$$

Inserting Eqs. (13) and (14) into the fully nonlinear equations gives:

$$\frac{\partial \eta_n}{\partial t} + L_1\tilde{\Phi}_n = -\nabla \cdot F_n \qquad (15)$$

where:

$$L_1 = \left(G_1\nabla^2 + \mu^2 G_2\nabla^2\nabla^2 + \mu^4 G_3\nabla^2\nabla^2\nabla^2 + \mu^6 G_4\nabla^2\nabla^2\nabla^2\nabla^2 \right)$$

and:

$$F_1 = \left(\nabla + \mu^2 H_1\nabla\nabla^2 + \mu^4 H_2\nabla\nabla^2\nabla^2 + \mu^6 H_3\nabla\nabla^2\nabla^2\nabla^2 \right)\eta_0\tilde{\Phi}_0 \qquad (16)$$

$$
\begin{aligned}
F_2 = & \left(\eta_0 \nabla \tilde{\Phi}_1 + \eta_1 \nabla \tilde{\Phi}_0\right) + \\
& + \mu^2 \left[\left(mG_1 + \frac{m^2}{2!} \right) \left(\eta_1 \nabla \nabla^2 \tilde{\Phi}_0 + \eta_0 \nabla \nabla^2 \tilde{\Phi}_1 \right) - \frac{G_1}{2} \eta_0^2 \nabla \nabla^2 \tilde{\Phi}_0 \right] + \\
& + \mu^4 \left[\left(mG_2 + \frac{m^2}{2!} H_1 - \frac{m^3}{3!} G_1 - \frac{m^4}{4!} \right) \left(\eta_1 \nabla \nabla^2 \nabla^2 \tilde{\Phi}_0 + \eta_0 \nabla \nabla^2 \nabla^2 \tilde{\Phi}_1 \right) - \frac{G_2}{2} \eta_0^2 \nabla \nabla^2 \nabla^2 \tilde{\Phi}_0 \right] + \\
& + \mu^6 \left[\left(mG_3 + \frac{m^2}{2!} H_2 - \frac{m^3}{3!} G_2 - \frac{m^4}{4!} H_1 + \frac{m^5}{5!} G_1 + \frac{m^6}{6!} \right) \left(\eta_1 \nabla \nabla^2 \nabla^2 \nabla^2 \tilde{\Phi}_0 + \eta_0 \nabla \nabla^2 \nabla^2 \nabla^2 \tilde{\Phi}_1 \right) - \right. \\
& \left. - \frac{G_3}{2} \eta_0^2 \nabla \nabla^2 \nabla^2 \nabla^2 \tilde{\Phi}_0 \right] \}
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
F_3 = & \left(\eta_0 \nabla \tilde{\Phi}_2 + \eta_1 \nabla \tilde{\Phi}_1 + \eta_2 \nabla \tilde{\Phi}_0\right) + \\
& + \mu^2 \left[\left(mG_1 + \frac{m^2}{2!} \right) \left(\eta_0 \nabla \nabla^2 \tilde{\Phi}_2 + \eta_1 \nabla \nabla^2 \tilde{\Phi}_1 + \eta_2 \nabla \nabla^2 \tilde{\Phi}_0 \right) - \left(G_1 \eta_0 \eta_1 + \frac{1}{6} \eta_0^3 \right) \nabla \nabla^2 \tilde{\Phi}_0 - \frac{G_1}{2} \eta_0^2 \nabla \nabla^2 \tilde{\Phi}_1 \right] + \\
& + \mu^4 \left[\left(mG_2 + \frac{m^2}{2!} H_1 - \frac{m^3}{3!} G_1 - \frac{m^4}{4!} \right) \left(\eta_0 \nabla \nabla^2 \nabla^2 \tilde{\Phi}_2 + \eta_1 \nabla \nabla^2 \nabla^2 \tilde{\Phi}_1 + \eta_2 \nabla \nabla^2 \nabla^2 \tilde{\Phi}_0 \right) - \right. \\
& \left. - \left(G_2 \eta_0 \eta_1 + \frac{H_1}{6} \eta_0^3 \right) \nabla \nabla^2 \nabla^2 \tilde{\Phi}_0 - \frac{G_2}{2} \eta_0^2 \nabla \nabla^2 \nabla^2 \tilde{\Phi}_1 \right] + \\
& + \mu^6 \left[\left(mG_3 + \frac{m^2}{2!} H_2 - \frac{m^3}{3!} G_2 - \frac{m^4}{4!} H_1 + \frac{m^5}{5!} G_1 + \frac{m^6}{6!} \right) \left(\eta_0 \nabla \nabla^2 \nabla^2 \nabla^2 \tilde{\Phi}_2 + \eta_1 \nabla \nabla^2 \nabla^2 \nabla^2 \tilde{\Phi}_1 + \eta_2 \nabla \nabla^2 \nabla^2 \nabla^2 \tilde{\Phi}_0 \right) - \right. \\
& \left. - \left(G_3 \eta_0 \eta_1 + \frac{H_2}{6} \eta_0^3 \right) \nabla \nabla^2 \nabla^2 \nabla^2 \tilde{\Phi}_0 - \frac{G_3}{2} \eta_0^2 \nabla \nabla^2 \nabla^2 \nabla^2 \tilde{\Phi}_1 \right]
\end{aligned}
\tag{18}
$$

Similarly:

$$
\eta_n + L_2 \frac{\partial \tilde{\Phi}_n}{\partial t} = -E_n \tag{19}
$$

where:

$$
L_2 = 1 + \mu^2 H_1 \nabla^2 + \mu^4 H_2 \nabla^2 \nabla^2 + \mu^6 H_3 \nabla^2 \nabla^2 \nabla^2
$$

and:

$$
\begin{aligned}
E_1 = & -L_{2a} \frac{\partial \tilde{\Phi}_0}{\partial t} + \frac{1}{2} \left(\nabla L_2 \tilde{\Phi}_0 \right)^2 + \mu^2 \left[\frac{G_1^2}{2} \left(\nabla^2 \tilde{\Phi}_0 \right)^2 \right] + \mu^4 \left[G_1 G_2 \nabla^2 \tilde{\Phi}_0 \nabla^2 \nabla^2 \tilde{\Phi}_0 \right] + \\
& + \mu^6 \left[\frac{G_2^2}{2} \left(\nabla^2 \nabla^2 \tilde{\Phi}_0 \right)^2 + G_1 G_3 \nabla^2 \tilde{\Phi}_0 \nabla^2 \nabla^2 \nabla^2 \tilde{\Phi}_0 \right]
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
E_2 = & -L_{2a} \frac{\partial \tilde{\Phi}_1}{\partial t} - L_{2b} \frac{\partial \tilde{\Phi}_0}{\partial t} + \nabla L_2 \tilde{\Phi}_0 \left(\nabla L_2 \tilde{\Phi}_1 - \nabla L_{2a} \tilde{\Phi}_0 \right) + \mu^2 \left[G_1 \eta_0 \left(\nabla^2 \tilde{\Phi}_0 \right)^2 + G_1^2 \nabla^2 \tilde{\Phi}_0 \nabla^2 \tilde{\Phi}_1 \right] + \\
& + \mu^4 \left[G_1 G_2 \left(\nabla^2 \tilde{\Phi}_0 \nabla^2 \nabla^2 \tilde{\Phi}_1 + \nabla^2 \tilde{\Phi}_1 \nabla^2 \nabla^2 \tilde{\Phi}_0 \right) + \left(G_1 H_1 + G_2 \right) \eta_0 \nabla^2 \tilde{\Phi}_0 \nabla^2 \nabla^2 \tilde{\Phi}_0 \right] + \mu^6 \left[G_2 H_1 \eta_0 \left(\nabla^2 \nabla^2 \tilde{\Phi}_0 \right)^2 + \right. \\
& \left. + G_2^2 \nabla^2 \nabla^2 \tilde{\Phi}_0 \nabla^2 \nabla^2 \tilde{\Phi}_1 + G_1 G_3 \left(\nabla^2 \tilde{\Phi}_0 \nabla^2 \nabla^2 \nabla^2 \tilde{\Phi}_1 + \nabla^2 \tilde{\Phi}_1 \nabla^2 \nabla^2 \nabla^2 \tilde{\Phi}_0 \right) + \left(G_1 H_2 + G_3 \right) \eta_0 \nabla^2 \tilde{\Phi}_0 \nabla^2 \nabla^2 \nabla^2 \tilde{\Phi}_0 \right]
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
E_3 = & -L_{2a}\frac{\partial\tilde{\Phi}_2}{\partial t} - L_{2b}\frac{\partial\tilde{\Phi}_1}{\partial t} - L_{2c}\frac{\partial\tilde{\Phi}_0}{\partial t} + \nabla L_2\,\tilde{\Phi}_0\left(\nabla L_2\,\tilde{\Phi}_2 - \nabla L_{2a}\,\tilde{\Phi}_1 - \nabla L_{2b}\,\tilde{\Phi}_0\right) + \frac{1}{2}\left(\nabla L_2\,\tilde{\Phi}_1 - \nabla L_{2a}\,\tilde{\Phi}_0\right)^2 + \\
& + \mu^2\left[\left(G_1\eta_1 + \frac{\eta_0^2}{2}\right)\left(\nabla^2\tilde{\Phi}_0\right)^2 + 2G_1\eta_0\nabla^2\tilde{\Phi}_0\nabla^2\tilde{\Phi}_1 + \frac{G_1^2}{2}\left(\nabla^2\tilde{\Phi}_2\right)^2\right] + \mu^4\left[\left(\left(G_2 + G_1H_1\right)\eta_1 + \left(\frac{G_1^2}{2} + H_1\right)\eta_0^2\right)\nabla^2\tilde{\Phi}_0\nabla^2\nabla^2\tilde{\Phi}_0 + \right. \\
& \left. + \left(G_1H_1 + G_2\right)\eta_0\left(\nabla^2\tilde{\Phi}_0\nabla^2\nabla^2\tilde{\Phi}_1 + \nabla^2\tilde{\Phi}_1\nabla^2\nabla^2\tilde{\Phi}_0\right) + G_1G_2\left(\nabla^2\tilde{\Phi}_0\nabla^2\nabla^2\tilde{\Phi}_2 + \nabla^2\tilde{\Phi}_2\nabla^2\nabla^2\tilde{\Phi}_0\right)\right] + \\
& + \mu^6\left[\left(G_2H_1\eta_1 + \frac{H_1^2}{2}\eta_0^2\right)\left(\nabla^2\nabla^2\tilde{\Phi}_0\right)^2 + 2G_2H_1\eta_0\nabla^2\nabla^2\tilde{\Phi}_0\nabla^2\nabla^2\tilde{\Phi}_1 + \frac{G_2^2}{2}\left(\nabla^2\nabla^2\tilde{\Phi}_1\right)^2 + \right. \\
& + \left(\left(G_3 + G_1H_2\right)\eta_1 + \left(H_2 + \frac{G_1G_2}{2}\right)\eta_0^2\right)\left(\nabla^2\tilde{\Phi}_0\nabla^2\nabla^2\nabla^2\tilde{\Phi}_0\right) + \left(G_3 + G_1H_2\right)\eta_0\left(\nabla^2\tilde{\Phi}_0\nabla^2\nabla^2\nabla^2\tilde{\Phi}_1 + \nabla^2\tilde{\Phi}_1\nabla^2\nabla^2\nabla^2\tilde{\Phi}_0\right) + \\
& \left. + G_1G_3\left(\nabla^2\tilde{\Phi}_0\nabla^2\nabla^2\nabla^2\tilde{\Phi}_2 + \nabla^2\tilde{\Phi}_2\nabla^2\nabla^2\nabla^2\tilde{\Phi}_0\right)\right]
\end{aligned}
\tag{22}
$$

Now the possible obtained nonlinear and linear wave properties are shown above. In next section, the accuracy analysis will be made.

3. THE ACCURACY ANALYSIS OF BOUSSINESQ EQUATIONS WITH DIFFERENT ORDERS

Due to the nature property of the convergence of Boussinesq equations in deep water, a further study for determining the accuracy is needed. Figure 1 shows the error magnitude of the phase velocity. The maximum error is up to 30%. It indicates that the sixth order equations have the wider valid range. The valid ranges of all models will decrease with the increase of μ. Figure 2 displays the error magnitude of the horizontal velocity of water particles. The results are the same as that of the phase velocity. The sixth order model has a better behavior than other models. Since there are many similar figures that describe different properties of wave motion, the results are very alike that the $O\left(\mu^6\right)$ owns more excellent behaviors than other two models. All results are concentrated to express in Table 1. Table 1 shows that, for nonlinear properties, the $O\left(\mu^4\right)$ model behaves better than other two models. But for the particle characteristics, the $O\left(\mu^6\right)$ and the $O\left(\mu^2\right)$ models provide a more excellent prediction than the $O\left(\mu^4\right)$ model. The reason is that the expansion of Boussinesq equations has an alternative sign in the subsequent term. In conclusion, for wave propagation in shallow water, the higher order of Boussinesq equations provide a better prediction. But, for wave properties in deep water, the optimal wave equations are referred to Table 1.

Table 1. The valid range of μ for the maximum error up to 30%

Maximum μ	$O\left(\mu^6\right)$ **model**	$O\left(\mu^4\right)$ **model**	$O\left(\mu^2\right)$ **model**
Phase velocity	*Over 20*	*Over 20*	*Over 20*
Group velocity	*18.6*	*15.5*	*12.3*
Horizontal velocity	*15.2*	*4.2*	*10.7*
Vertical velocity	*14.0*	*7.2*	*13.5*
Particle trajectory	*13.2*	*7.6*	*11.3*
F_1	*7.6*	*8.5*	*7.8*
F_2	*5.5*	*6.3*	*5.7*
E_1	*7.3*	*8.1*	*7.7*
E_2	*5.4*	*6.2*	*5.5*
Optimal m	*-0.611*	*-0.581*	*-0.346*

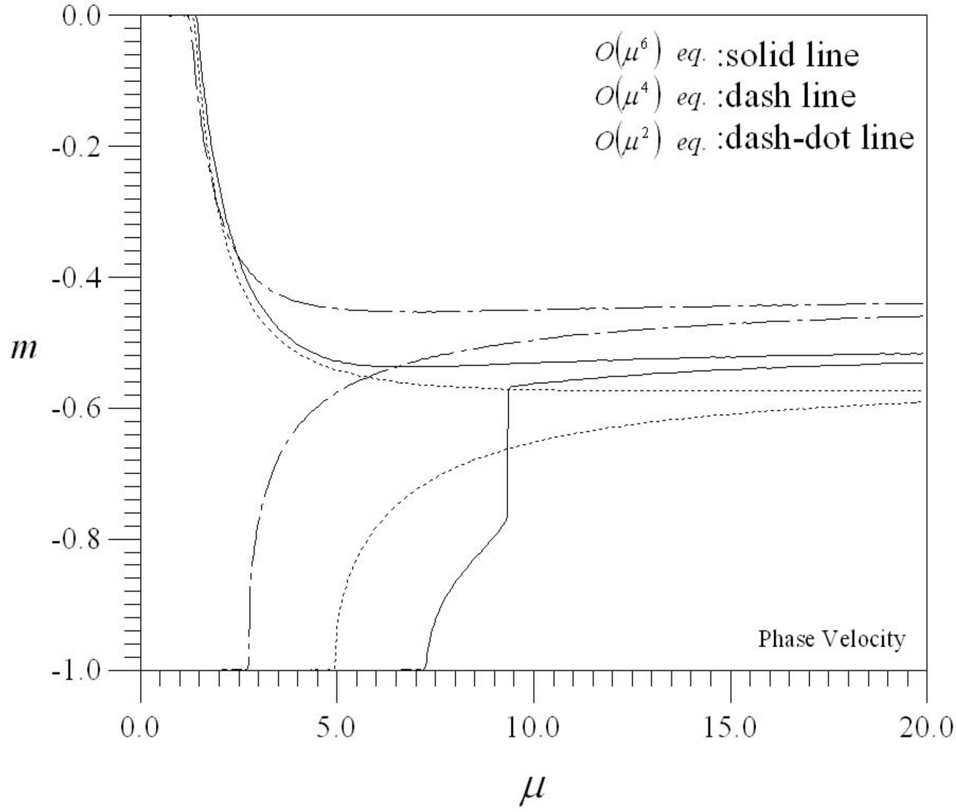

Fig. 1 The error magnitude of Boussinesq equations

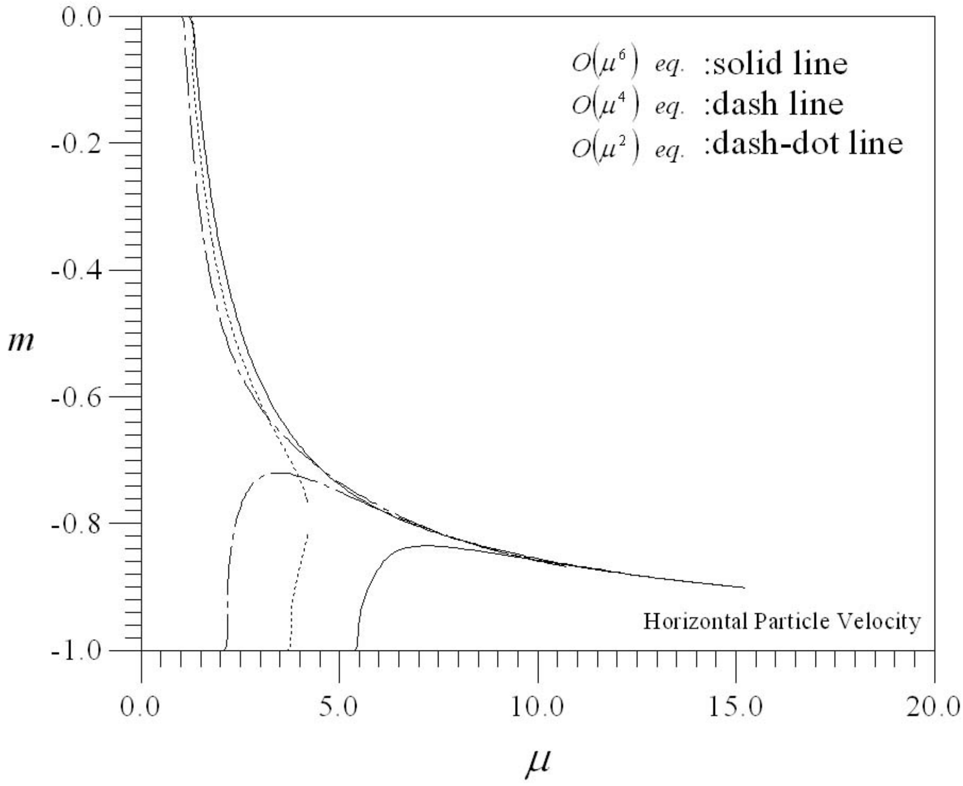

Fig. 2 The error magnitude of the horizontal velocity of water particles

4. CONCLUSION REMAKS

In our present study, we first introduce the modified Boussinesq equations that are accurate to $O\left(\mu^6\right)$. Almost all linear and nonlinear wave properties are derived subsequently. By comparing the optimal models of different order wave equations, we conclude that, for wave propagation in shallow water, the higher order of Boussinesq equations provide a better prediction. But, for wave properties in deep water, the optimal wave equations are referred to Table 1.

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ANALIZA TOČNOSTI POTPUNO NELINEARNIH BOUSSINESQ-OVIH JEDNADŽBI

SAŽETAK

U ovom radu izvršena je analiza točnosti potpuno nelinearnih Boussinesq-ovih jednadžbi. Najprije su uvedene modificirane Boussinesq-ove jednadžbe u rekurzivnom obliku. To pomaže inženjerima da u budućnosti izvedu jednadžbe višeg reda, ako bude potrebno. Gotovo sva linearna i nelinearna svojstva vala izvedena su kasnije. Uspoređujući veličinu greške svojstva svakog vala može se odrediti optimalni model i zaključiti da Boussinesq-ove jednadžbe omogućavaju da se bolje predvidi širenje valova u plićaku. Za svojstva vala u dubokoj vodi model $O\left(\mu^4\right)$ *se bolje ponaša s nelinearnim svojstvima, a lošije kod nekih karakteristika čestica.*

Ključne riječi: *Boussinesq-ove jednadžbe, točnost, nelinearno svojstvo, linearno svojstvo.*