UDC 519.8:339.1:336.645:519.61
Original scientific paper
Received: 02.09.2004.

Optimal order policy for EOQ model under shortage level constraint

Singa Wang Chiu(1), **Chia-Kuan Ting**(2) and **Yuan-Shyi Peter Chiu**(2)
(1)Department of Business Administration, Chaoyang University of Technology, 168 Jifong East Road, Wufong, Taichung 413, TAIWAN, e-mail: swang@mail.cyut.edu.tw
(2)Department of Industrial Engineering and Management, Chaoyang University of Technology, 168 Jifong East Road, Wufong, Taichung 413, TAIWAN, e-mail: djk@cyut.edu.tw, ypchiu@mail.cyut.edu.tw

SUMMARY

This paper studies the optimal order policy for the economic order quantity (EOQ) inventory model under maximal shortage level constraint. We first prove that total inventory cost per unit time for the classic EOQ model with backlogging permitted is less than or equal to that of the EOQ model with shortage not allowed. Secondly, the relationship between the "imputed backorder cost" and the maximal proportion of shortage permitted time per cycle is derived for decision-making on whether the service level can be accomplished or not. Finally, an equation for calculating the intangible backorder cost is proposed for the situation when the required service level is not achievable. By incorporating this intangible backorder cost, the newly derived optimal order policy is able to achieve the necessary service level as well as to minimize the overall inventory costs. Numerical example is provided to demonstrate its practical usage.

***Key words**: inventory, allowable shortage time, service level constraint, EOQ model, intangible backorder cost.*

1. INTRODUCTION

The aim of this paper is to investigate the effect of maximal allowable shortage level constraint on the economic order quantity (EOQ) model. The EOQ model was first introduced several decades ago to assist corporations in minimizing overall inventory costs; it employs a mathematical modelling to balance the inventory holding and setup costs, and derives an optimal order quantity that minimizes total inventory costs. Regardless of its simplicity, the economic order quantity model is still applied industry-wide today [1]. In real-life inventory control and management, due to certain internal orders of parts/materials and various operating considerations, the planned backlogging is the strategy to effectively minimize overall inventory costs. While allowing backlogging, abusive shortage in an inventory model, however, may cause an unacceptable service level and turn into possible loss of future sales (because of the loss of customer goodwill). Therefore, the maximal allowable shortage level per cycle is always set as an operating constraint of the business in order to attain the minimal service level while deriving the optimal order policy for inventory model.

A considerable amount of research has been carried out to address the service level constraint issue. Examples of them are surveyed below.

Schneider [2] examined a (Q,s) model, he determined the optimal value of the order quantity Q and the reorder point s in which the average annual costs of inventory and orders are minimal under the condition that a certain service level is reached. De Kok [3] considered a lost-sales production/inventory control model with two adjustable production rates to meet demand. He obtained the practical approximations for optimal switch-over levels to such a model under the service level constraints. Kelle [4] derived the optimal service levels in multi-item inventory systems. He formulated exact mathematical models and developed an inventory control program package with many modules to deal with the multi-item inventory systems with aggregate service levels. Boylan and Johnston [5] studied the relationships between different service level measures for inventory systems. They analyzed six of the most frequently used service measures, their relationships, and possible conversion to a common measure. Hopp et al. [6] found an effective stocking policy for a part distribution center supporting field maintenance of mail processing equipment. The objectives of their policies are to minimize overall inventory investment at the distribution center subject to constraints on customer service and order frequency. Ouyang and Wu [7] developed an algorithm to find the optimal order quantity and optimal lead time for an inventory model with a service level constraint and when the probability distribution of the lead time demand is normal. Bashyam and Fu [8] studied the optimization of (s,S) inventory systems with random lead times and a service level constraint. Their work was based on simulation and presented computational results for a large number of test cases, and they concluded that the vast majority of cases come within *5%* of estimated optimality. Metters and Vargas [9] considered a single product, single level, stochastic master production scheduling (MPS) policies on costs, service level, and schedule changes. The data envelopment analysis (DEA) methodology is extended to aid the evaluation of the simulation results and they concluded that the dual-buffer control systems outperform the existing policies. Chen and Krass [10] investigated inventory models with minimal service constraints. They showed that the minimal service level constraint (SLC) model was qualitatively different from their shortage cost counterparts and the transformation from SLC model to a shortage cost model may not be always possible.

This paper is inspired by the work of Chen and Krass [10] and it investigates the optimal order policy for the economic order quantity model with maximal permitted shortage level. The relationship between the imputed backorder cost and the maximal shortage level is derived for judging whether or not the service level is achievable. In the case that the required service level is not attainable, the intangible backorder cost is introduced to enable the newly derived optimal order policy to accomplish the necessary service level as well as to minimize overall inventory costs.

2. THE BASIC MODELS AND MATHEMATICAL ANALYSIS

The economic order quantity model for single commodity is the simplest and most fundamental of all inventory models. As mentioned earlier, regardless of its simplicity, the EOQ model is still applied industry-wide today [1]. This paper studies the effects of maximal allowable shortage level constraint on EOQ model. The following notations are used in our analysis:

λ = demand rate, in units per unit time,
Q = order quantity per cycle in the EOQ model with shortage not permitted,
Q_b = order quantity per cycle in the EOQ model with shortage allowed and backordered,
B = allowable backorder level in the EOQ model with backlogging permitted,
K = fixed ordering cost per order,
C = purchasing cost per item ($/item),
H = holding cost per item per unit time ($/item/unit time),
B = backordering cost per item per unit time,
$TCU(Q)$ = total inventory costs per unit time in the EOQ model,
$TCU(Q_b$,B) = total inventory costs per unit time in the EOQ model with backlogging permitted.

2.1 Formulation of the classic EOQ model with backlogging permitted

The EOQ model assumes that all items purchased are received instantaneously. Figure 1 depicts its on-hand inventory level and allowable backorder level.

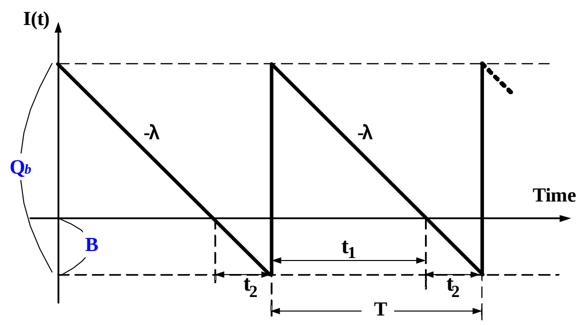

Fig. 1 On-hand inventory of the EOQ model with shortage allowed and backordered

The total inventory cost per unit time, $TCU(Q_b,B)$ is presented by equation:

$$TCU\left(Q_b\,,\,B\right)=\lambda\cdot c+\frac{K\lambda}{Q_b}+\frac{hQ_b}{2}-h\cdot B+\frac{B^2}{2Q_b}\left(b+h\right) \tag{1}$$

and by minimizing the cost function $TCU(Q_b,B)$, one obtains the optimal order policy, Q_b* and B* [11, 12, 13] as shown in equations:

$$Q_b^* = \sqrt{\frac{2K\lambda}{h}} \cdot \sqrt{\frac{b+h}{b}} \tag{2}$$

$$B^* = \frac{h}{b+h} \cdot Q_b^* = \sqrt{\frac{2K\lambda}{b}} \cdot \sqrt{\frac{h}{b+h}} \tag{3}$$

2.2 Formulation of the classic EOQ model with shortage not permitted

For the EOQ model with shortage not permitted, the cycle length simply is $T=t_1$ (for t_1 please refer to Figure 1). The total inventory cost per unit time, $TCU(Q)$, is presented by equation:

$$TCU(Q) = \lambda \cdot c + \frac{K\lambda}{Q} + \frac{h \cdot Q}{2} \tag{4}$$

By minimizing the cost function $TCU(Q)$, one obtains the optimal order quantity Q^* [11, 12, 13] as shown in the next equation:

$$Q^* = \sqrt{\frac{2K\lambda}{h}} \tag{5}$$

2.3 The effects of the backlogging and the service level constraint

Property 1. The total inventory cost per unit time for the EOQ model with shortage not permitted is always greater than or equal to that of the EOQ model with shortage allowed and backordered. That is $TCU(Q) \geqq TCU(Q_b,B)$, for any given $Q=Q_b$.

Proof. Assume that $Q=Q_b$, employing Eqs. (1) and (4), one obtains:

$$TCU(Q) - TCU(Q,B) =$$
$$= hB - \frac{B^2}{2Q}(b+h) = B\left[h - \frac{B}{2Q}(b+h)\right];$$

Then from Eq. (3), because:

$$B = \frac{h}{(b+h)}Q \ ,$$

therefore, one obtains:

$$TCU(Q) - TCU(Q,B) = \frac{h^2}{2(b+h)}Q \geq 0.$$

Property 1 confirms that it is better off to permit shortage. While allowing backlogging, abusive shortage in an inventory model, however, may cause an unacceptable service level and turn into possible loss of future sales. Hence, the maximal allowable shortage level per cycle is always set as an operating constraint of the business in order to attain the minimal service level. Suppose that we set α to be the maximum proportion of shortage permitted time per cycle (that is the service level = $(1-\alpha)$ %), then:

$$\alpha = \frac{t_2}{t_1 + t_2} \tag{6}$$

$$\frac{\alpha}{1-\alpha} = \frac{t_2}{t_1} \tag{7}$$

From Figure 1, we obtain $t_1 = \frac{Q_b - B}{\lambda}$ and $t_2 = \frac{B}{\lambda}$. Hence, Eq. (7) becomes:

$$\frac{\alpha}{1-\alpha} = \frac{B}{Q_b - B} \tag{8}$$

$$B = \frac{h}{b+h} \cdot Q_b$$

By substituting B in Eq. (8), one obtains the following:

$$\frac{\alpha}{1-\alpha} = \frac{h}{b} \tag{9}$$

or:
$$b = h\left(\frac{1}{\alpha} - 1\right) \tag{10}$$

Assume that:

$$f(\alpha) = h\left(\frac{1}{\alpha} - 1\right). \tag{11}$$

Equation (10) represents the relationship between the imputed backorder cost and the maximum proportion of shortage permitted time per cycle. In other words, when the service level $(1-\alpha)\%$ of the EOQ model is set, the corresponding backorder cost can be obtained. Hence one can utilize this information to determine whether or not the service level is achievable.

Let b_t be the tangible backorder cost per item, if $b_t > f(\alpha)$ then the service level $(1-\alpha)\%$ is achievable; otherwise, we need to increase the tangible backorder cost to the "imputed backorder cost" and then use it to derive the new optimal operating policy (in terms of Q_b^* and B^*), so that the overall inventory costs can be minimized and the service level constraint will be attained. Let b_i be the adjustable intangible backorder cost (per item per unit time), then b_i should satisfy the following condition in order to attain the $(1-\alpha)\%$ service level:

$$b_i \geq \left[f(\alpha) - b_t\right] \tag{12}$$

Therefore, by using $b=f(\alpha)$ one can derive the new optimal order quantity Q_b^* and the optimal backorder level B^*. Figure 2 depicts the relationship between $f(\alpha)$ and α, and pinpoints the value of $f(\alpha=0.3)$ in accordance with service level constraint set at minimum of *70%*. Here, *70%* of service level is selected arbitrarily for demonstration purpose. Practitioners should choose a correct service level set by his/her firm.

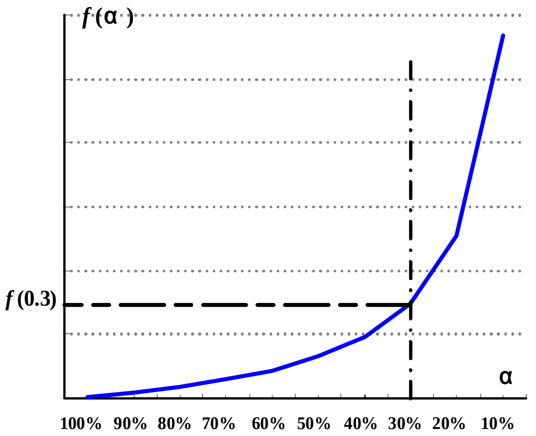

Fig. 2 Variation of permitted shortage rate effects on backorder cost

3. NUMERICAL EXAMPLES AND DISCUSSION

A firm purchases an item from its supplier. It wants to determine the relationship between the adjustable intangible shortage cost and the maximum allowable proportion shortage time per cycle. This item has experienced a relatively flat demand of *4,000* units per year. The accounting department has estimated that it costs *$90* to initiate a purchase order and each unit costs the company *$2.4*. The service level of this item, according to the company's policy, is set at *70%* or above (i.e. the maximum proportion of shortage permitted time per cycle α is *0.3*). The allowable shortage items are backordered at a cost of *$0.2* per item per year and the cost of holding is *$0.6* per item per year. Thus, we have the following:

- λ = *4,000* units per year,
- K = *$90* per order purchased,
- C = *$2.4*; unit purchase price,
- H = *$0.6* per item per unit time,
- Bt = *$0.2* per item backordered per unit time (the tangible backorder cost),
- α = *0.3*; the maximum proportion of shortage permitted time per cycle.

First let $b=b_t$, from Eqs. (1) through (3), one obtains the overall costs $TCU(Qb^*,B^*) = \$9{,}929$, the optimal order quantity $Q_b^*=2{,}191$, and the optimal backorder level $B^*=1{,}643$. Convexity of the total cost function for this example is displayed in Figure 3.

Also if shortages are not allowed, from Eqs. (4) and (5) we obtain the total cost $TCU(Q^*)=\$10{,}257$ and the optimal order quantity $Q^*=1{,}095$. One notices that the EOQ model with backlogging permitted has a lower overall cost than that of the EOQ model with no shortages allowed, as proved by the Property 1 (see Figure 4).

In this example, suppose we ignore the *70%* service level constraint for now, then from Eq. (6) the proportion of shortage time per cycle is $\alpha=0.75$. This represents a *25%* service level only. In order to achieve *70%* service level, one can use the proposed Eqs. (11) and (12) and find that $b_i=f(\alpha)-b_t=1.4-0.2=1.2$.

Then, by using $b=(b_t+b_i)=1.4$ and Eqs. (1) through (3), we can re-compute the optimal order quantity $Qb^*=1{,}309$, the optimal backorder level $B^*=393$, and the optimal overall costs $TCU(Qb^*,B^*)=\$10{,}150$.

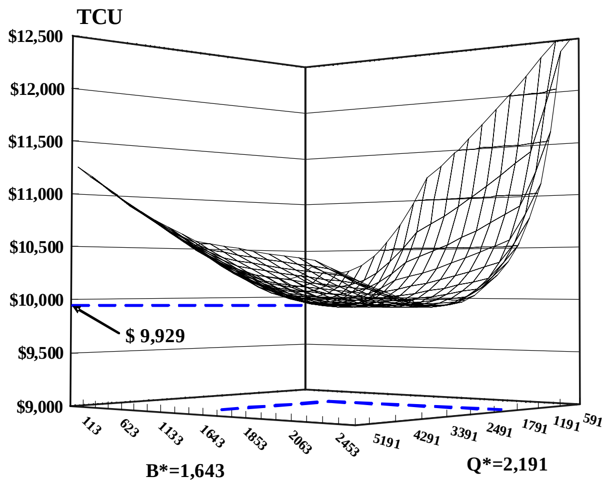

Fig. 3 Convexity of the total cost function for the EOQ model with backlogging permitted

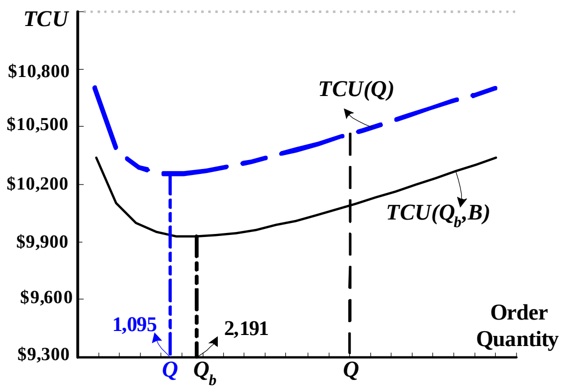

Fig. 4 A comparison between cost functions TCU(Q) and $TCU(Q_b,B)$

Table 1 shows variation of α effects on the optimal operating policy, $TCU(Q_b^*,B^*)$, the $TCU(Q_b^*,B^*)$ excluding an intangible backorder cost, and the price for raising the service level from *25%*.

From Table 1, one notices that though the $TCU(Q_b^*,B^*)$ for *70%* service level is *$10,150*, if we exclude the intangible backorder cost b_i (which merely is helping us to achieve the *70%* service level) from the computation of Eq. (1), we will obtain the actual cost *$10,079*. Comparing to *$9,929*, there is an increase of *$150* in cost. In other words, *$150* is the price that we pay for raising the service level from *25%* to *70%* (see both Table 1 and Figure 5 for details). One also notices that as the service level $(1-\alpha)\%$ increases, the cost function $TCU(Q_b^*,B^*)$ and the price for raising the service level increase too.

Table 1 Variation of α effects on optimal operating policy, $TCU(Q_b^*,B^*)$ and $TCU(Q_b^*,B^*)$ excluding the intangible backorder cost and price for raising the service level

Service level (1-α)%	α	f(α)	b_i	Q_b^*	B*	$TCU(Q_b^*,B^*)$	$TCU(Q_b^*,B^*)$ excluding intangible backorder cost	Price for raising service level
100%	0	∞	∞	1,095	-	$10,257	$10,257	$328
95%	0.05	11.40	11.20	1,124	56	$10,241	$10,225	$296
90%	0.10	5.40	5.20	1,155	115	$10,224	$10,194	$265
85%	0.15	3.40	3.20	1,188	178	$10,206	$10,163	$234
80%	0.20	2.40	2.20	1,225	245	$10,188	$10,134	$205
75%	0.25	1.80	1.60	1,265	316	$10,169	$10,106	$177
70%	0.30	1.40	1.20	1,309	393	$10,150	$10,079	$150
65%	0.35	1.11	0.91	1,359	476	$10,130	$10,054	$125
60%	0.40	0.90	0.70	1,414	566	$10,109	$10,030	$101
55%	0.45	0.73	0.53	1,477	665	$10,087	$10,008	$79
50%	0.50	0.60	0.40	1,549	775	$10,065	$9,987	$58
45%	0.55	0.49	0.29	1,633	898	$10,041	$9,969	$40
40%	0.60	0.40	0.20	1,732	1,039	$10,016	$9,953	$24
35%	0.65	0.32	0.12	1,852	1,204	$9,989	$9,941	$12
30%	0.70	0.26	0.06	2,000	1,400	$9,960	$9,932	$3
25%	0.75	0.20	0	2,191	1,643	$9,929	$9,929	$0

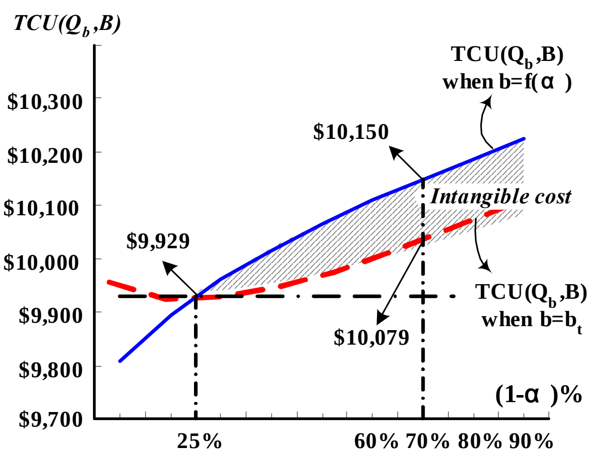

Fig. 5 Variation of α effects on the optimal costs $TCU(Q_b,B)$ and the $TCU(Q_b,B)$ excluding the intangible backorder cost

4. CONCLUDING REMARKS

In real-life inventory control and management, due to the existence of internal orders and other operating considerations, the planned backlogging is the strategy to effectively minimize overall inventory costs. While allowing backordering, abusive shortage in an inventory model, however, may cause an unacceptable service level and turn into possible loss of future sales.

This paper studies the optimal order policy for the EOQ inventory model with maximal shortage level constraint. As the result, we derive the relationship between the imputed backorder cost and the maximal proportion of shortage permitted time per cycle. According to this relationship, the practitioners can determine on whether or not the service level is achievable. Another equation is also presented in this study for calculating intangible backorder cost for the situation when the required service level is not accomplishable. By utilizing this intangible backorder cost, the newly derived optimal order policy is able to achieve the necessary service level as well as to minimize the overall inventory costs.

For future research, one interesting direction among others will be to investigate the effect of service level constraint on an EOQ model with imperfect quality items produced.

5. REFERENCES

[1] S. Osteryoung, E. Nosari, D. McCarty, and W.J. Reinhart, Use of the EOQ model for inventory analysis, *Production and Inventory Management*, Vol. 27, pp. 39-45, 1986.

[2] H. Schneider, The service level in inventory control systems, *Engineering Costs and Production Economics*, Vol. 4, pp. 341-348, 1979.

[3] A.G. de Kok, Approximations for a lost-sales production/inventory control model with service level constraints, *Management Science*, Vol. 31, No. 6, pp. 729-737, 1985.

[4] P. Kelle, Optimal service levels in multi-item inventory systems, *Engineering Costs and Production Economics*, Vol. 15, pp. 375-379, 1989.

[5] J.E. Boylan and F.R. Johnston, Relationships between service level measures for inventory systems, *Journal of the Operational Research Society*, Vol. 45, No. 7, pp. 838-844, 1994.

[6] W.J. Hopp, M.L. Spearman and R.Q. Zhang, Easily implementable inventory control policies, *Operations Research*, Vol. 45, No. 3, pp. 327-340, 1997.

[7] L.-Y. Ouyang and K.-S. Wu, Mixture inventory model involving variable lead time with a service level constraint, *Computers & Operations Research*, Vol. 24, No. 9, pp. 875-882, 1997.

[8] S. Bashyam and M.C. Fu, Optimization of (*s*,*S*) inventory systems with random lead times and a service level constraint, *Management Science*, Vol. 44, No. 12, pp. 243-256, 1998.

[9] R. Metters and V. Vargas, A comparison of production scheduling policies on costs, service level and schedule changes, *Production and Operations Management*, Vol. 8, No. 1, pp. 76-91, 1999.

[10] F.Y. Chen and D. Krass, Inventory models with minimal service level constraints, *European Journal of Operational Research*, Vol. 134, No. 1, pp. 120-140, 2001.

[11] F.S. Hillier and G.J. Lieberman, *Introduction to Operations Research*, 7th Edition, McGraw Hill Co. Inc., pp. 941-958, 2001.

[12] S. Nahmias, *Production and Operations Analysis*, McGraw-Hill Book Company, pp. 203-214, 2001.

[13] E.A. Silver, D.F. Pyke and R. Peterson, *Inventory Management and Production Planning and Scheduling*, John Wiley & Sons, pp.151-172, 1998.

POLITIKA OPTIMALNOG NARUČIVANJA UTEMELJENA NA MODELU OPTIMALNE KOLIČINE NARUDŽBE UZ OGRANIČENJA MINIMALNE KOLIČINE NA SKLADIŠTU

SAŽETAK

Ovaj rad istražuje politiku optimalnog naručivanja utemeljenu na modelu optimalne količine narudžbe (EOQ - Economic Order Quantity) uz maksimalno ograničenje minimalne količine na skladištu. Prvo se dokazuje da je ukupan trošak zaliha po jedinici vremena u klasičnom EOQ modelu (uz dozvoljene ponovne narudžbe) manji ili jednak onom kod EOQ modela, pri čemu nije dozvoljeno spuštanje ispod minimalne razine zaliha. Drugo, odnosi između uračunatog troška ponovne narudžbe i maksimalnog udjela vremena u kojem se ne održava minimalna količina zaliha po proizvodnom ciklusu se izračunava za potrebe odlučivanja s obzirom na činjenicu može li se ostvariti tražena razina proizvodnje/pružanja usluga. Na kraju rada se predlaže jednadžba za izračun nevidljivog dijela troškova ponovnog naručivanja u slučaju kada nije moguće postići traženu razinu proizvodnje/pružanja usluga. Uključivanjem nevidljivog dijela ponovne narudžbe, novoformuliranom politikom optimalne narudžbe postiže se tražena razina proizvodnje/pružanja usluga, te minimiziraju ukupni troškovi zaliha. Numeričkim se primjerom pokazuje praktična uporaba predloženog modela.

Ključne riječi*: inventar, najkraći zakonski rok, razina usluge, EOQ model, nevidljivi troškovi.*