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Analytical bench tests for numerical methods for bending of beams and frames undergoing large displacements

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SUMMARY

The research of numerical methods for bending of beams and frames undergoing large displacement requires reliable reference analytical solutions. Although these solutions can be found spread in the literature, we still need to have them solved in a systematic way up to known accuracy. This paper is an attempt to fit this need.

The bench test examples include cantilever beam with concentrated force at the free end and diamond shaped beam frame with diagonal forces. Numerical solutions of bench test examples are compared to the analytical solutions. The analyzed examples show that the numerical solutions obtained by these methods converge monotonically towards an exact analytical solution. For all numerical methods good agreement is indicated.

The numerical methods are compared with each other, using criteria of accuracy, reliability, and numerical efficiency, in order to find out which methods are more suitable for engineering application.

Key words: *beam bending, large displacements, nonlinear numerical analysis, line systems, analytical solutions, accuracy, reliability, numerical efficiency.*

1. INTRODUCTION

Since the first applications of computers to nonlinear numerical analysis of structures, various nonlinear beam elements have been presented by Argyris and Dune [1], Bazant and Nimeiri [2], Oran and Kassimaili [3], Reissner [4] and Crisfield [5, 6]. A matrix displacement approach is developed for the numerical analysis of elastic problems of beams and frames by Yung [7].

The total Lagrangian formulation based on the Reisner kinematic relations is developed by Haefner and Willam [8].

An updated Lagrangian and a total Lagrangian formulation of a three-dimensional beam element are developed by Bathe and Bolourcki [9, 10].

The analysis of an engineering system requires the idealization of the system into a form that can be solved, the formulation of the mathematical model, the solution of this, and the interpretation of the results. The most effective mathematical model for the analysis is surely that one which yields the required response to a sufficient accuracy and, at least, cost.

The chosen mathematical model is reliable if the required response is known to be predicted within a selected level of accuracy measured on the response of the very-comprehensive mathematical model. The methods for numerical analysis of nonlinear line structural problems are presented in chronological order.

We start the analysis with method that uses a classical stiffness matrix for small displacements. The other methods for solution of geometrically nonlinear structural line problems comprise development of respective finite element matrices based on continuum mechanics. A virtual work approach and total and updated Lagrangian approach for large displacement of line structures are presented and compared with other approaches from the literature. A comparative analysis of methods for numerical solution of nonlinear line system is performed in order to find out what methods are more suitable for engineering application in terms of accuracy, numerical efficiency, and robustness.

Analysis is performed on standard bench test examples using software package Matlab and finite element program, which is developed in Ref. [11].

The results obtained for all numerical methods, are compared to the analytical solutions. Studies of large defections, which require nonlinear analytical solutions, have been concerned mainly with single members. Large defections in cantilever beams subjected to concentrated loads were studied by Bisshopp and Drucher [12]. The extensions of the above development to the analytical solutions of frame problems are, however, limited to a few specially idealized cases. Jenkins, Seitz and Przemieniecki [13] analyzed a diamond – shaped frame loaded diagonally at two corners.

2. ANALYTICAL SOLUTIONS OF BEAMS AND FRAMES UNDERGOING LARGE DISPLACEMENTS

2.1 Large deflection of cantilever beams [3]

The derivation is based on the fundamental Bernoulli-Euler theorem. Considering a long, thin cantilever leaf spring, let L to denote the length of beam, u the horizontal component of the displacement of loaded end of the beam, v the corresponding vertical displacement, P the concentrated vertical load at the free end, EI the flexural rigidity as shown in Figure 1. The exact expression for the curvature of the elastic line may be stated conveniently in terms of arc length and slope angle denoted by s and β, respectively, so that x is the horizontal coordinate measured from the fixed end of the beam.

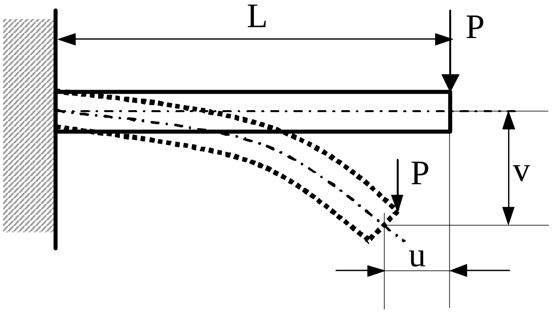

Fig. 1 A cantilever beam with concentrated vertical load

The product of EI and the curvature of the beam equals the bending moment M:

$$EI\frac{d\beta}{ds} = P\left(L - x - u\right) = M \Big/ \frac{d}{ds} \qquad (1)$$

or:

$$\frac{d^2\beta}{ds^2} = -\frac{P}{EI}\frac{dx}{ds} = -\frac{P}{EI}\cos\beta \qquad (2)$$

Integrating Eq. (2):

$$\frac{1}{2}\left(\frac{d\beta}{ds}\right)^2 = -\frac{P}{EI}\sin\beta + C,\ P=0 \Rightarrow C=\sin\beta_0 \qquad (3)$$

and:

$$\frac{d\beta}{ds} = \sqrt{\frac{2P}{EI}}\left(\sin\beta_o - \sin\beta\right)^{\frac{1}{2}} \qquad (4)$$

$$\sqrt{\frac{2P}{EI}}\int_0^L ds = \int_0^{\beta_0}\left(\sin\beta_o - \sin\beta\right)^{-\frac{1}{2}} = \sqrt{2}\left(\frac{PL^2}{EI}\right)^{\frac{1}{2}} \qquad (5)$$

In order to evaluate this elliptic integral, denote $\frac{PL^2}{EI}$ by α^2 and let:

$$1 + \sin\beta = 2k^2\sin^2\theta = \left(1 + \sin\beta_0\right)\sin^2\theta \qquad (6)$$

Then:

$$\alpha = \int_{\theta_1}^{\pi/2}\left(1 - k^2\sin^2\theta\right)^{-\frac{1}{2}}d\theta,\quad \sin\theta_1 = \frac{\sqrt{2}}{2k} \qquad (7)$$

The next step is to represent the deflection v in terms of α and elliptic integral.

Since:

$$\frac{dy}{d\beta}\cdot\frac{d\beta}{ds} = \frac{dy}{ds} = \sin\beta \qquad (8)$$

and since we have $\frac{d\beta}{ds}$ from Eq. (4):

$$\frac{dy}{d\beta}\sqrt{\frac{2P}{EI}}\left(\sin\beta_o - \sin\beta\right)^{\frac{1}{2}} = \sin\beta \qquad (9)$$

Thus:

$$v = \int_0^y dy = \sqrt{\frac{EI}{2P}}\int_0^{\beta}\frac{\sin\beta d\beta}{\left(\sin\beta_o - \sin\beta\right)^{\frac{1}{2}}} \qquad (10)$$

With the aid of Eq. (6) we obtain:

$$\frac{v}{L} = \frac{\sqrt{2}}{2\alpha}\int_0^{\beta_0}\frac{\sin\beta d\beta}{\left(\sin\beta_o - \sin\beta\right)^{\frac{1}{2}}} = \frac{1}{\alpha}\int_{\beta_1}^{\pi/2}\frac{\left(2k^2\sin^2\theta - 1\right)d\theta}{\left(1 - k^2\sin^2\theta\right)^{\frac{1}{2}}} \qquad (11)$$

This equation can be split up into complete and incomplete elliptic integrals of the first and second kinds. In the notation of Jahnke and Emde, the vertical displacement is:

$$\frac{v}{L}=\frac{1}{\alpha}\left[F\left(k\right)-F\left(k,\theta_1\right)-2E\left(k\right)+2E\left(k,\theta_1\right)\right] \quad (12)$$

$$\alpha=F\left(k\right)-F\left(k,\theta_1\right) \quad (13)$$

So that:

$$\frac{v}{L}=1-\frac{2}{\alpha}\left[E\left(k\right)-E\left(k,\theta_1\right)\right] \quad (14)$$

The horizontal displacement of the loaded end is calculated from Eqs. (1) and (4) with $x=0$ when $\beta=0$. Thus:

$$P\left(L-u\right)=EI\left(\frac{d\beta}{ds}\right)_{\beta=0}=EI\sqrt{\frac{2P}{EI}}\left(\sin\beta_0\right)^{\frac{1}{2}} \quad (15)$$

or:

$$u=1-\frac{\sqrt{2}}{\alpha}\sqrt{\sin\beta_0} \quad (16)$$

From Eq. (6) we have:

$$\sin\beta_0=2k^2-1 \quad (17)$$

The solution for vertical displacement by linear analysis is:

$$v_{lin}=\frac{PL^3}{3EI} \quad (18)$$

The exact analytical solution of the large displacements of a cantilever beam are shown in Tables 1 and 2 (see next pages) and Figure 2.

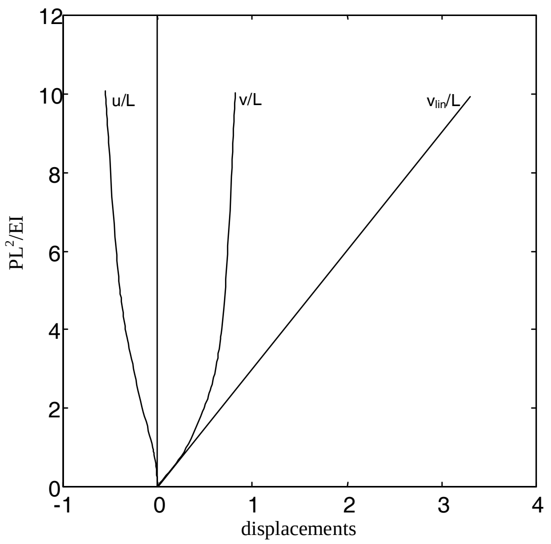

Fig. 2 Analytical solutions of the large displacements of a cantilever beam with a concentrated force at the free end

2.2 Large deflections of diamond-shaped frames [4]

The nonlinear solutions for large deflections of diamond – shaped frames are derived. The exact solutions are based on the assumption that the material is perfectly elastic and that the shear deformations are negligible.

Large deflection analysis: Pinned-fixed frame

Because of symmetry of the frame it is sufficient to analyze only the frame member. The deformed configuration is shown in Figure 3 so that the Euler – Bernoulli equation for bending due to tensile loading must be expressed as:

$$\frac{d\beta}{ds}=\frac{M}{EI}=\frac{P}{EI}\left(L\cos\beta_0-u-x\right) \quad (19)$$

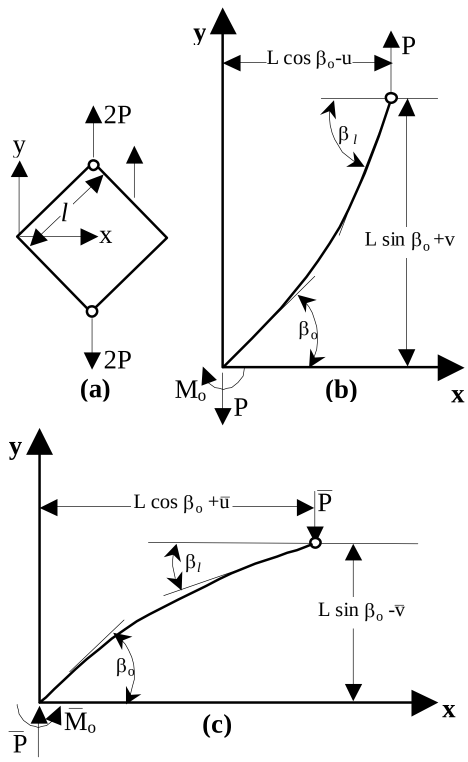

Fig. 3 Pinned – fixed frame: a) undeformed frame; b) large deflections in tension; c) large deflections in compression

Differentiating Eq. (19) with respect to s and introducing a nondimensional load parameter $\eta^2=\frac{PL^2}{EI}$ we obtain:

$$\frac{d^2\beta}{ds^2}=-\frac{P}{EI}\cdot\frac{dx}{ds}=-\frac{\eta^2}{L^2}\cos\beta \quad (20)$$

Table 1 Elliptic integrals of the first and second kinds

α	**k**	**sin β_0**	**b β_1**	**F (k)**	**F (k, β_1)**	**E (k)**	**E (k, β_1)**	α_1
45	0.707107	7.17856×10^{-9}	89.9992	1.85407	1.85405	1.35064	1.35063	0.0000202
46	0.71934	0.0348995	79.4183	1.86915	1.60487	1.34181	1.21274	0.264281
47	0.731354	0.0697565	75.2052	1.88481	1.51081	1.33287	1.15457	0.374
48	0.743145	0.104528	72.0835	1.90108	1.44252	1.32384	1.11055	0.458568
49	0.75471	0.139173	69.5415	1.918	1.38766	1.31475	1.0742	0.530341
50	0.766044	0.173648	67.378	1.93558	1.34144	1.30554	1.04294	0.594141
51	0.777146	0.207912	65.4883	1.95386	1.3014	1.29628	1.01542	0.652466
52	0.788011	0.241922	63.8095	1.97288	1.26606	1.28695	0.990802	0.706823
53	0.798636	0.275637	62.3	1.99267	1.23446	1.27757	0.96855	0.758212
54	0.809017	0.309017	60.9306	2.01327	1.20593	1.26815	0.94827	0.807341
55	0.819152	0.34202	59.6798	2.03472	1.17997	1.25868	0.929672	0.854747
56	0.822925	0.374607	58.5312	2.05706	1.15622	1.24918	0.912534	0.900845
57	0.838671	0.406737	57.4719	2.08036	1.13438	1.23966	0.896681	0.945977
58	0.848048	0.438371	56.4916	2.10466	1.11423	1.23013	0.88197	0.99043
59	0.857167	0.469472	55.5818	2.13002	1.09557	1.22059	0.868285	1.03445
60	0.866025	0.50000	54.7356	2.15652	1.07826	1.21106	0.855528	1.07826
61	0.87462	0.529919	53.947	2.18421	1.06216	1.20154	0.843617	1.12206
62	0.882948	0.559193	53.2112	2.21319	1.04716	1.19205	0.832483	1.16604
63	0.891007	0.587785	52.5236	2.24355	1.03317	1.18259	0.822064	1.21038
64	0.898794	0.615661	51.8808	2.27538	1.02011	1.17318	0.812308	1.25527
65	0.906308	0.642788	51.2794	2.30879	1.0079	1.16383	0.803169	1.30088
66	0.913545	0.669131	50.7167	2.3439	0.9965	1.15455	0.794607	1.34741
67	0.920505	0.694658	50.1901	2.38087	0.985841	1.14535	0.786586	1.39503
68	0.927184	0.71934	49.6974	2.41984	0.97879	1.13624	0.779074	1.44396
69	0.93358	0.743145	49.2367	2.461	0.966573	1.12725	0.772043	1.49443
70	0.939693	0.766044	48.8063	2.50455	0.957886	1.11838	0.765469	1.54666
71	0.945519	0.788011	48.4045	2.55073	0.949784	1.10964	0.75327	1.60095
72	0.951057	0.809017	48.0301	2.59982	0.942239	1.10106	0.753599	1.65758
73	0.956305	0.829038	47.6817	2.65214	0.935223	1.09265	0.748266	1.71692
74	0.961262	0.848048	47.3582	2.70807	0.928713	1.08443	0.743311	1.77935
75	0.965926	0.866025	47.0586	2.76806	0.922688	1.07641	0.73872	1.84538
76	0.970296	0.882948	46.782	2.83267	0.917128	1.06861	0.734479	1.91554
77	0.97437	0.898794	46.5277	2.90256	0.912018	1.06106	0.730577	1.99055
78	0.978148	0.913545	46.2948	2.97857	0.907341	1.05378	0.727003	2.07123
79	0.981627	0.927184	46.0827	3.06173	0.903083	1.04679	0.723747	2.15865
80	0.984808	0.939693	45.8908	3.15339	0.899233	1.04011	0.7208	2.25415
81	0.987688	0.951057	45.7187	3.2553	0.895781	1.03379	0.718156	2.35952
82	0.990268	0.961262	45.5659	3.36987	0.892715	1.02784	0.715808	2.47715
83	0.992546	0.970296	45.4319	3.50042	0.890029	1.02231	0.713749	2.61039
84	0.994522	0.978148	45.3165	3.65186	0.887715	1.01724	0.711974	2.76414
85	0.996195	0.984808	45.2193	3.83174	0.885767	1.01266	0.710479	2.94597
86	0.997564	0.990268	45.1401	4.05276	0.88418	1.00865	0.709261	3.16858
87	0.99863	0.994522	45.0787	4.33865	0.88295	1.00526	0.708317	3.4557
88	0.999391	0.997564	45.0349	4.74272	0.882073	1.00258	0.707644	3.86064
89	0.999848	0.999391	45.0087	5.43491	0.881548	1.00075	0.707241	4.55336
90	1.000000	1.000000	45.0000	∞	0.881374	1.00000	0.707107	∞

Table 2 Analytical solutions of the large displacements of a cantilever beam with a concentrated force at the free end

FORCE	Loading parameter	Displacements of a beam		
P	PL^2/EI	u	v_{lin}	v
7.17 x 10^{-9}	4.1 x 10^{-10}	-4.28 x 10^{-6}	1.367 x 10^{-10}	-9.388 x 10^{-6}
1.22228	0.06984	0.00032	0.02328	0.02327
2.44783.	0.13988	0.00129	0.04663	0.04652
3.67998	0.21028	0.00292	0.07009	0.06974
4.92208	0.28126	0.00519	0.09375	0.09292
6.17755	0.35300	0.00812	0.11767	0.11604
7.44997	0.42571	0.01168	0.14190	0.13907
8.74299	0.49959	0.01589	0.16653	0.16202
10.06050	0.57489	0.02075	0.19163	0.18486
11.40650	0.65180	0.02625	0.21727	0.20758
12.78540	0.73059	0.03238	0.24353	0.23016
14.20160	0.81152	0.03916	0.27051	0.25260
15.66030	0.89487	0.04657	0.29829	0.27487
17.16660	0.98095	0.05461	0.32698	0.29696
18.72650	1.07009	0.06328	0.35669	0.31886
20.34620	1.16264	0.07258	0.38755	0.34055
22.03270	1.25901	0.08250	0.41967	0.36203
23.79380	1.35965	0.09305	0.45322	0.38328
25.63800	1.46503	0.10422	0.48834	0.40428
27.57480	1.57571	0.11601	0.52524	0.42503
29.61520	1.69230	0.12841	0.56410	0.44552
31.77130	1.81550	0.14144	0.60517	0.46573
34.05690	1.94611	0.15508	0.64870	0.48566
36.48800	2.08503	0.16934	0.69510	0.50529
41.86300	2.39217	0.19971	0.79739	0.54365
44.85310	2.56303	0.21584	0.85434	0.56236
48.08260	2.74758	0.23261	0.91586	0.58076
51.58650	2.94780	0.25001	0.98259	0.59883
55.40680	3.16610	0.26808	1.05537	0.61659
59.59470	3.40541	0.28683	1.13514	0.63402
64.21290	3.66931	0.30627	1.22310	0.65114
69.33990	3.96228	0.32645	1.32076	0.66795
75.07480	4.28999	0.34739	1.43000	0.68446
81.54560	4.65975	0.36916	1.55325	0.70070
88.92100	5.08120	0.39183	1.69373	0.71669
97.42850	5.56735	0.41549	1.85578	0.73246
107.38500	6.13629	0.44026	2.04543	0.74807
119.24800	6.81415	0.46634	2.27138	0.76359
133.70800	7.64047	0.49399	2.54682	0.77913
151.87800	8.67877	0.52361	2.89292	0.79485
185.69800	10.03990	0.55585	3.34663	0.81103
208.98300	11.94190	0.59188	3.98063	0.82815
260.83000	14.90460	0.63413	4.96819	0.84721
362.73310	20.73310	0.68951	6.91103	0.87108
∞	∞	1.00000	∞	1.00000

Integrating Eq. (20) we obtain:

$$\left(\frac{d\beta}{ds}\right)^2 = -\frac{2\eta^2}{L^2}\sin\beta + C \qquad (21)$$

where the constant *C* may be determined from the boundary condition:

$$\frac{d\beta}{ds} = 0 \quad at \quad \beta = \beta_l \qquad (22)$$

Hence:

$$\pm\frac{d\beta}{ds} = \frac{\eta}{L}\sqrt{2\left(\sin\beta_l - \sin\beta\right)} \qquad (23)$$

The positive sign must be taken in the ambiguity on the left of Eq. (23):

$$\frac{d\beta}{ds} = \frac{d\beta}{dy}\cdot\frac{dy}{ds} = \frac{d\beta}{dy}\sin\beta \qquad (24)$$

The projection of the deformed member on the *y* axis may be calculated from:

$$L\sin\beta_0 + v = \int_0^{l\sin\beta_0 + v} dy \qquad (25)$$

Using Eqs. (23), (24) and (25) we have:

$$\eta\left(\sin\beta_0 + \frac{v}{L}\right) = \int_{\beta_0}^{\beta_l}\frac{\sin\beta\, d\beta}{\sqrt{2\left(\sin\beta_l - \sin\beta\right)}} \qquad (26)$$

We can now introduce a new variable θ such that:

$$\sin^2\theta = \frac{\left(1+\sin\beta\right)}{2k^2} \qquad (27)$$

where:

$$2k^2 = 1 + \sin\beta_l \qquad (28)$$

Equation (26) may be transformed into:

$$\eta\left(\sin\beta_0 + \frac{v}{L}\right) = \int_{\theta_1}^{\theta_2}\frac{\left(2k^2\sin^2\theta - 1\right)}{\sqrt{\left(1-k^2\sin^2\theta\right)}}\, d\theta \qquad (29)$$

where:

$$\sin^2\theta_1 = \left(1+\sin\beta_0\right)\left(1+\sin\beta_l\right) \qquad (30)$$

and:

$$\theta_2 = \frac{\pi}{2} \qquad (31)$$

The integral on the right side of the Eq. (29) is expressible in terms of elliptic integrals and it can be shown that:

$$\eta\left(\sin\beta_0 + \frac{v}{L}\right) = K\left(k\right) - F\left(\theta_1,k\right) - 2E\left(k\right) + 2E\left(\theta_1,k\right) \qquad (32)$$

Then, the vertical displacement is:

$$\frac{v}{L} = \frac{K\left(k\right) - F\left(\theta_1,k\right) - 2E\left(k\right) + 2E\left(\theta_1,k\right)}{\eta} - \sin\beta_0$$

Similarly we can develop expression:

$$\eta\left(\cos\beta_0 - \frac{u}{L}\right) = \int_{\theta_1}^{\theta_2} 2k\sin\beta\, d\beta = 2k\cos\theta_1 \qquad (33)$$

from where the horizontal displacement is:

$$\frac{u}{L} = \frac{\cos\beta_0 - 2k\cos\theta_1}{\eta}$$

and:

$$\eta = \int_{\theta_1}^{\theta_2}\frac{d\theta}{\sqrt{\left(1-k^2\sin^2\theta\right)}} = K\left(k\right) - F\left(\theta_1,k\right) \qquad (34)$$

For small deflection analysis we can develop expression:

$$\frac{u}{L} = \frac{v}{L} = \frac{1}{2^{\frac{1}{2}}}\left[1 - \frac{2^{\frac{1}{4}}}{\eta}\tanh\left(\frac{\eta}{2^{\frac{1}{4}}}\right)\right] \qquad (35)$$

For infinitesimal deflections (linear theory) the right-sides of the Eq. (35) reduce to $PL^2/6EI$:

$$\frac{u_{lin}}{L} = \frac{v_{lin}}{L} = \frac{PL^2}{6\,EI} \qquad (36)$$

Exact analytical solutions for vertical and horizontal deflection of pinned-fixed frame are shown in Table 3 and Figure 4.

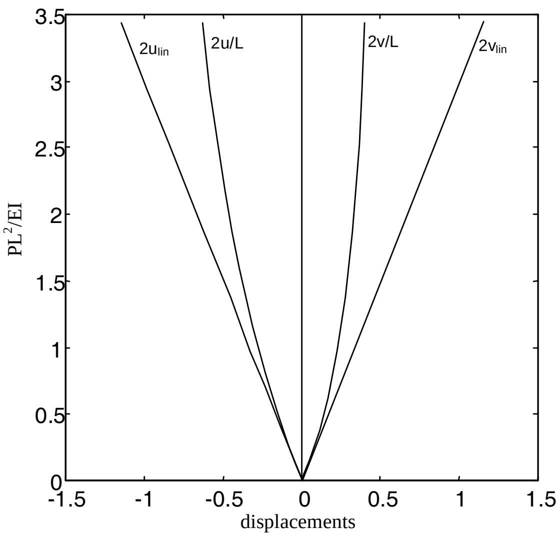

Fig. 4 Vertical and horizontal deflections of pinned-fixed frame under tensile loading

Table 3 Exact analytical solution for vertical and horizontal deflections of pinned-fixed frame under tensile loading

α	P	η^2	2 v/l large	2 u/l large	2 v/l = 2 u/l small	$2\,u_{lin} = 2\,v_{lin}$
67,5	0	0	0	0	0	0
68	0.876748	0.0500999	0.0163494	0.0165739	0.164667	0.0167
69	2.71288	0.155021	0.0484065	0.0504795	0.0495043	0.0516738
70	4.67269	0.267011	0.079592	0.0853604	0.0827591	0.0890036
71	6.77488	0.387136	0.109888	0.121214	0.116324	0.129045
72	9.04157	0.516661	0.139278	0.158036	0.150295	0.17222
73	11.4992	0.579099	0.167746	0.195824	0.184767	0.219033
74	14.1798	0.810277	0.195279	0.234579	0.219843	0.270092
75	17.1225	0.978426	0.221866	0.274309	0.255626	0.326142
76	20.3754	1.16431	0.247496	0.315029	0.292231	0.388103
77	23.9993	1.37139	0.272163	0.356761	0.329779	0.45713
78	28.0715	1.60409	0.295864	0.399539	0.368407	0.534696
79	32.6924	1.86813	0.318598	0.443417	0.408269	0.622711
80	37.995	2.17114	0.340373	0.488466	0.449543	0.723714
81	44.1605	2.52346	0.3612	0.534792	0.492444	0.841153
82	51.443	2.9396	0.381104	0.582545	0.537235	0.979866
83	60.212	3.44068	0.400121	0.631944	0.584259	1.14689
84	71.0314	4.05894	0.418312	0.683317	0.633978	1.35298
85	84.8154	4.84659	0.435774	0.737172	0.687057	1.61553
86	103.175	5.89569	0.452672	0.794352	0.744529	1.96523
87	129.31	7.38915	0.469307	0.856383	0.808185	2.46305
88	170.943	9.76816	0.486319	0.926483	0.881693	3.25605
89	255.289	14.5879	0.505538	1.01385	0.975313	4.86264
90	655.927	37.4815	0.536047	1.16419	1.13953	12.4938

3. SOME NUMERICAL SOLUTIONS OF BENCH PROBLEMS AND COMPARISON WITH ANALYTICS

We start the analysis with method that uses a classical stiffness matrix for small displacements.

This method is developed by T.Y. Yung [7]. The present development is based on the assumption that the material is linearly elastic and the displacements are not small in comparison with the length of the beam. The solution procedure includes first formulating the stiffness equations for a beam element based on the small deflection theory but with the inclusion of effect of axial force, then applying a linearized midpoint tangent incremental approach and coordinate transformation at every step. If the displacements obtained at every step are small with reference to the local coordinates such that the squares of the slope-increment are negligible in comparison with unity, the small deflection theory should hold.

A simple beam element is developed for the solution of large deflection problems by Haefner and Willam [8]. The total Lagrangian formulation is based on the kinematic relations proposed by Reissner for finite rotations and stretching as well as shearing of plane beams.

Third method is developed by Bathe and Bolourchi [9, 10]. An updated Lagrangian formulation of a three-dimensional beam element is presented for large displacement and large rotation analysis. The formulations are derived from the continuum mechanics based Lagrangian incremental equilibrium equations. The beam elements are assumed to be straight, and the conventional beam displacement functions are employed to express the displacements of the elements in convected coordinates. The element has been implemented for use in elastic, elastic-plastic, static and dynamic analysis.

Analysis is performed on standard bench test examples. The numerical solutions obtained for all methods are compared to exact analytical solutions by Bisshopp [12], and Jenkins [13]. In addition, in all analysis shear deformations were neglected.

3.1 A cantilever beam with a concentrated load at the free end

A cantilever beam, subjected at free end to a concentrated force, was analyzed with the results of vertical and horizontal displacements at the free end shown in Table 4, Figures 5 and 6. Eight finite elements were used. The results from an exact analytical solution in terms of elliptic integrals [12] are also shown in Figure 6 for comparison. For all methods good agreement is indicated.

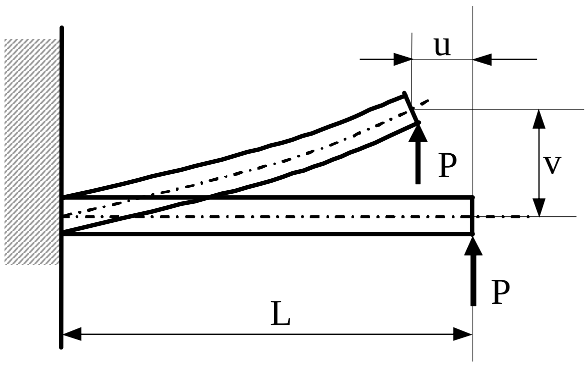

Fig. 5 A cantilever beam with a concentrated force at the free end

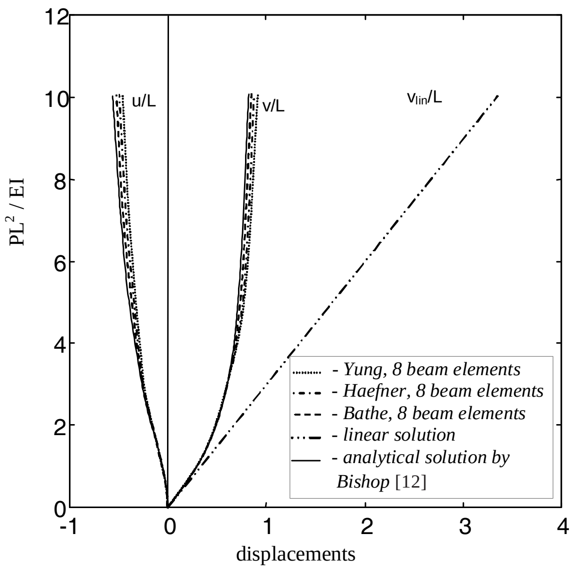

Fig. 6 Large displacements of a cantilever beam with a concentrated force at the free end

3.2 A diamond-shaped frame loaded diagonally at two joints

A diamond-shaped frame loaded by forces applied at a pair of diagonally opposite joints is shown in Figure 7. The two loaded joints are assumed to be hinged while the two joints are assumed to be rigid. It was also studied extensively by Jenkins, Seitz and Przemieniecki [13]. Jenkins et al. provided an analytical solution which was in a good agreement with their experimental results. Numerical solutions for the dimensionless horizontal elongation and vertical contraction of the diagonals for all methods are shown in Table 5 and Figure 8. The agreement is reasonable. Again, eight finite elements were used.

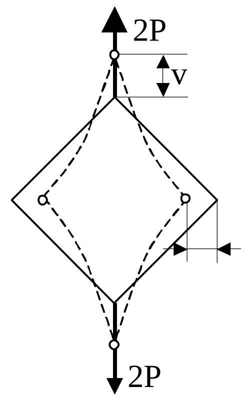

Fig. 7 A diamond-shaped frame loaded diagonally at two joints

Variation of the elongation of the frame diagonals with the applied loading is plotted in Figure 8.

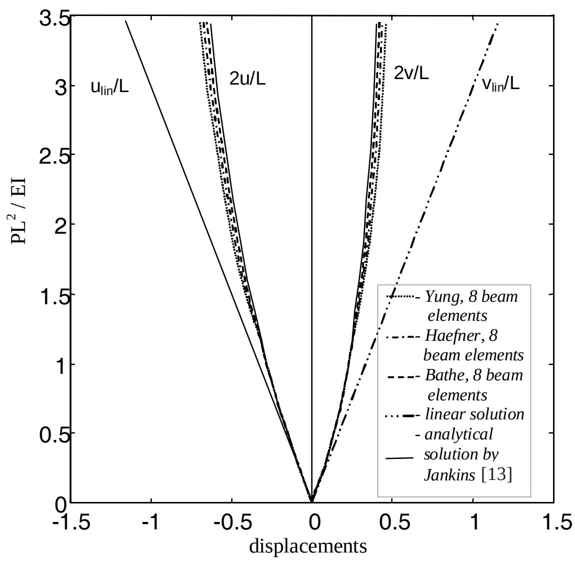

Fig. 8 Vertical and horizontal deflections of pinned-fixed frame under tensile loading

Typical deformed shapes for a square frame are shown in Figure 9.

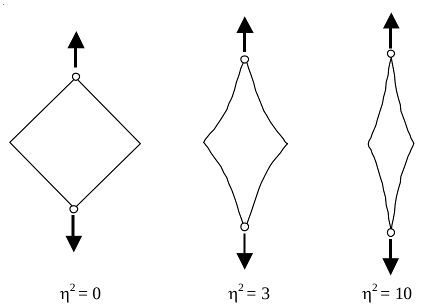

Fig. 9 Deformed shapes of a square pinned-fixed frame for different values of the loading parameter η^2

Table 4 Horizontal and vertical components of the free end displacements for a cantilever beam with a concentrated force

FORCE	LOAD PARAMETER	Cantilever beam with a concentrated force by Yung		Cantilever beam with a concentrated force by Haefner and Willam		Cantilever beam with a concentrated force by Bathe	
		Horizontal displacement	Vertical displacement	Horizontal displacement	Vertical displacement	Horizontal displacement	Vertical displacement
P	PL^2 / EI	u/L	v/L	u/L	v/L	u/L	v/L
0	0	0	0	0	0	0	0
10	0.57142	-0.02028	0.18226	-0.02072	0.18293	-0.02045	0.18386
20	1.14285	-0.06917	0.33186	-0.06992	0.33485	-0.07039	0.33618
30	1.71428	-0.127403	0.44261	-0.12977	0.44810	-0.13050	0.44962
40	2.28571	-0.18408	0.52292	-0.18820	0.53008	-0.189101	0.53170
50	2.85557	-0.23550	0.58209	-0.24093	0.59000	-0.24195	0.59168
60	3.42857	-0.29931	0.64781	-0.28642	0.63419	-0.28832	0.63658
70	4.00000	-0.33963	0.68205	-0.32754	0.66926	-0.32870	0.67108
80	4.57142	-0.27597	0.71301	-0.36271	0.69638	-0.36392	0.69826
90	5.14285	-0.40351	0.73332	-0.39353	0.718218	-0.38478	0.72018
100	5.71428	-0.43451	0.75005	-0.42071	0.73615	-0.42200	0.73820
110	6.28571	-0.45920	0.76650	-0.44484	0.75115	-0.44617	0.75328
120	6.85714	-0.48002	0.77929	-0.46640	0.76387	-0.46777	0.76609
130	7.42857	-0.50011	0.78203	-0.48578	0.77482	-0.48718	0.77713
140	8.00000	-0.51780	0.79980	-0.50331	0.78435	-0.50474	0.78675
150	8.57142	-0.53381	0.80830	-0.51924	0.79273	-0.52070	0.79522
160	9.14285	-0.54831	0.81581	-0.53379	0.80018	-0.53528	0.80274
170	9.71428	-0.55820	0. 82005	-0.54047	0.80351	-0.54512	0.80768
175	10.00000	-0.56806	0.82570	-0.54714	0.80684	-0.55495	0.81261

Table 5 Horizontal and vertical components of the displacements for a diamond shaped beam frame with a diagonal forces

FORCE	LOAD PARAMETER	Diamond-shaped frame with diagonal forces by Yung		Diamond-shaped frame with diagonal forces by Haefner and Willam		Diamond-shaped frame with diagonal forces by Bathe	
		Horizontal displacement	Vertical displacement	Horizontal displacement	Vertical displacement	Horizontal displacement	Vertical displacement
P	PL^2 / EI	2u/L	2v/L	2u/L	2v/L	2u/L	2v/L
0	0	0	0	0	0	0	0
0.876	0.05009	0.01668	0.01651	0.01658	0.016403	0.01657	0.016387
2.713	0.15502	0.05084	0.04890	0.05053	0.04857	0.05051	0.04853
4.673	0.26701	0.08600	0.08042	0.08547	0.07989	0.08544	0.07982
6.775	0.38714	0.12215	0.11107	0.12141	0.11034	0.12137	0.11025
9.0416	0.51666	0.15931	0.14082	0.15833	0.13989	0.15829	0.13978
11.499	0.57909	0.19746	0.16965	0.19625	0.16853	0.19621	0.16841
14.180	0.81027	0.23661	0.19756	0.23515	0.19624	0.23512	0.19612
17.123	0.97843	0.27677	0.22452	0.27506	0.22302	0.27503	0.22289
20.375	1.16431	0.31795	0.25054	0.31598	0.24886	0.31596	0.24872
23.999	1.37139	0.36018	0.27559	0.35793	0.27374	0.35793	0.27360
28.0715	1.60409	0.40349	0.29968	0.40096	0.29766	0.40098	0.29752
32.692	1.86813	0.44793	0.32280	0.44512	0.32062	0.44517	0.32049
37.995	2.17114	0.49359	0.34496	0.49048	0.34263	0.49056	0.34251
44.161	2.52346	0.54056	0.36618	0.53715	0.36369	0.53726	0.36359
51.443	2.9396	0.58901	0.38647	0.58528	0.38384	0.58542	0.38375
60.212	3.44068	0.63914	0.40584	0.63509	0.40311	0.63528	0.40303
71.0314	4.05894	0.69131	0.42446	0.68691	0.42155	0.68715	0.42149

3.3 Analysis of results

The analysis of obtained results shown in Tables 4 and 5 is performed by the formula:

$$\text{deviation} = \left| \frac{v_{\text{method}}}{v_{\text{analitical}}} \cdot 100 - 100 \right| \quad [\%]$$

where v_{method} is a displacement toward any of methods, and $v_{analitical}$ is a displacement which is solved by analytical method. Deviation is absolutely a deviation displacement of beams and frames of analytical solutions expressed in percentage.

Analysis of the Bathe and Haefner methods for determining large displacements of a cantilever beam and diamond shaped beam frame reveals a variance of between *0%* and *1%* measured against the analytical solutions. Young's method results in a difference in the range of *0%* to *2%* compared with the analytical solutions.

4. CONCLUSION

Nonlinear numerical analysis was performed on a cantilever beam with a concentrated force at the free end and diamond shaped beam frame with diagonal forces.

The results obtained for all numerical methods are compared with the analytical solutions. The numerical methods are compared with each other, using criteria of accuracy, reliability, and numerical efficiency. The analyzed examples show that the numerical solutions obtained by these methods converge monotonically towards an exact analytical solution.

For discretisation with eight beam elements, in respect of the finite element mesh, the methods used by Bathe [9, 10] and Haefner [8], followed by Yung [7] correspond most closely to the analytical solution. The results for large displacements of a cantilever beam with a force at the end and diamond shaped beam frame with diagonal forces in terms of numerical iterations are most closely aligned to the analytical solution by method of Bathe, followed by the procedures of Yung, and then Haefner. Therefore, the Bathe method is more efficient than the methods used by Haefner and Yung.

The methods employed by Bathe and Haefner are based upon the full Newton-Raphson method for solving nonlinear problems. The Yung method is based upon incremental loads, and it has not removed control error.

To reduce the error it is necessary to repeat the process with other incremental loads. To summarize, the methods adopted by Bathe and by Haefner are extremely reliable, while the Yung method is less reliable. Taking into consideration the criteria of accuracy, reliability and numerical efficiency the Bathe method provides the best results.

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ANALITIČKI TESTOVI ZA NUMERIČKE METODE KOD SAVIJANJA GREDA I OKVIRA PRI VELIKIM POMACIMA

SAŽETAK

Istraživanje numeričkih metoda za rješavanje savijanja greda i okvira pri velikim pomacima zahtijevaju pouzdana referentna analitička rješenja. Mada se ova rješenja mogu naći u literaturi, potrebno je naći metode koje vode dovoljno točnom rješenju. Ovaj rad je pokušaj pronalaska tih metoda.

Prikazani test primjeri uključuju gredu s koncentriranom silom na slobodnom kraju i rombasti okvir s dijagonalnim opterećenjem. Numerička rješenja prikazanih test primjera su uspoređena s analitičkim rješenjima. Analizirani primjeri pokazuju da dobivena numerička rješenja monotono konvergiraju prema točnom analitičkom rješenju. Sve numeričke metode daju zadovoljavajuće rezultate.

Numeričke metode su uspoređene prema kriterijima točnosti, pouzdanosti i numeričkoj efikasnosti.

Ključne riječi: *savijanje greda, veliki pomaci, nelinearna numerička analiza, linijski sustavi, analitička rješenja, točnost, pouzdanost, numerička efikasnost.*