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# GROUND STATE ENERGY OF MANY-BODY FERMI SYSTEM BY USING FEENBERG PERTURBATION THEORY

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In this paper we derive an expression for the ground state energy of $N$ identical interacting fermions by using Brillouin-Wigner-Feenberg perturbation theory. It represents a generalization of the Gell-Mann-Brueckner formula which was obtained for dense electron gas.

## 1. Introduction

In order to understand completely the interacting fermion fluid as a model for liquid $^3$He, electron gas, nuclear and neutron matter, different approaches have been developed. Different approaches to the problem have provided different insights.

New experiments concerning high temperature superconductivity, surface and interface phenomena rekindled interest in these systems. In this paper we focus upon the ground state of the homogeneous interacting fermion system. We apply a powerful many-body perturbation method which was first developed by Feenberg (F)[1,2,3] in order to improve the convergence of Brillouin-Wigner (BW) perturbation theory. It was then shown by Ljolje and coworkers[4] that the F perturbation theory had an additional advantage, i.e. it has proper behaviour on the number of particles $N$ through second order. They further rearranged the Feenberg formula for energy and found a new expression which has proper behaviour through the fifth order. They also applied it to the derivation of the ground and single particle excited state energies (Bogoliubov spectrum) of a weakly interacting boson fluid. This approach has been recently used for the calculation of the interaction energy between a pair of quasiparticles in a bose fluid[5].

Hitherto Brillouin-Wigner-Feenberg perturbation method was applied only to boson systems. In the present work we apply this general method to fermion systems. In this way we expect to get new insights in these systems.

In Section 2 we present BWF perturbation energy formula and related topics. Section 3 is devoted to the derivation of an energy expression for fermions which is a generalization of Gell-Mann-Brueckner formula[6,7]. In Section 4 the relation of our result with Rayleigh-Schrödinger (RS) perturbation theory and some evaluations are presented: it was done for electron gas and for a system with short range repulsive and longer range attractive potential. The matrix elements are calculated in Appendix A. An important part of the proof of the energy expression is carried out in Appendix B.

## 2. *Survey of Brilloun-Wigner-Feenberg perturbation theory*

Let us assume that the Hamiltonian of the system being studied can be put in the form

$$H = H_0 + V \, , \tag{1}$$

and that the eigenstates and eigenvalues of the unperturbed Hamiltonian $H_0$ are known,

$$H_0 |\varphi_n\rangle = \varepsilon_n |\varphi_n\rangle \, . \tag{2}$$

The Brillouin-Wigner solution of the equation

$$H|\varphi\rangle = E|\varphi\rangle \tag{3}$$

for energies is

$$E = \varepsilon_l + V_{ll} + \sum_{n\,(\neq l)} \frac{|V_{ln}|^2}{E-\varepsilon_n} + \sum_{nn'\,(\neq l)} \frac{V_{ln}V_{nn'}V_{n'l}}{(E-\varepsilon_n)(E-\varepsilon_{n'})} + \sum_{nn'n''\,(\neq l)} \frac{V_{ln}V_{nn'}V_{n'n''}V_{n''l}}{(E-\varepsilon_n)(E-\varepsilon_{n'})(E-\varepsilon_{n''})} + \ldots \, , \tag{4}$$

where

$$V_{ij} + \langle \varphi_i | V | \varphi_j \rangle \, .$$

Eq. (4) is solved by successive approximations.

The perturbation theory was Feenberg's favoured object of study. Feenberg considered the convergence of the BW perturbation theory and noticed that in Eq. (4) some matrix elements may occur more than once in a given order. For instance $V_{nn'}$ in the fourth order term for $n'' = n$ appears as $V_{nn'}^2$. Feenberg found a way to avoid such repetition of matrix elements[1]. The equation for energy which he obtained reads

$$E = \varepsilon_l + V_{ll} + \sum_{n}^{\odot} \frac{V_{ln}V_{nl}}{E - E_n^F} + \sum_{nn'}^{\odot} \frac{V_{ln}V_{nn'}V_{n'l}}{(E - E_n^F)(E - E_{nn'}^F)}$$

$$+ \sum_{nn'n''}^{\odot} \frac{V_{ln}V_{nn'}V_{n'n''}V_{n''l}}{(E - E_n^F)(E - E_{nn'}^F)(E - E_{nn'n''}^F)} + \ldots, \qquad (5)$$

where Feenberg's energies are

$$E_n^F = \varepsilon_n + V_{nn} + \sum_{n'\,(\neq ln)} \frac{V_{nn'}V_{n'n}}{E - E_{nn'}^F} + \sum_{\substack{n' \neq ln \\ n'' \neq lnn'}} \frac{V_{nn'}V_{n'n''}V_{n''n}}{(E - E_{nn'}^F)(E - E_{nn'n''}^F)} + \ldots$$

$$E_{nn'}^F = \varepsilon_{n'} + V_{n'n'} + \sum_{n''\,(\neq lnn')} \frac{V_{n'n''}V_{n''n'}}{E - E_{nn'n''}^F} + \ldots \qquad (6)$$

The mark $\odot$ means that all indices are different. Equation (5) with a corresponding equation for the wave function means what has been called Feenberg perturbation[3].

Studying the applicability of BW perturbation method in the many-body theory we came to the conclusion that its terms could be rearranged to improve the dependence on the number of particles. The result of such rearrangement was a new BW formula which was later recognized as Feenberg formula[4]. Because of this the relation (5) was called BWF perturbation energy. Feenberg's formula (5) was applied to the theory of many-bosons in the Bogoliubov approximation[4]. It was shown that only the second order term of the F formula was sufficient to reproduce the ground state and elementary excitation spectrum of the Bogoliubov boson system. A relation with Jastrow wave function was established as well[8]. Regarding some new experiments concerning the interaction between two rotons[9], the interaction between two quasiparticles in the Bogoliubov boson system was studied and it was found that it changed sign, becoming repulsive when density increased[5,10].

## *3. An application of BWF perturbation theory to a fermion system*

We consider $N$ identical formions in the volume $\Omega$. Let the Hamiltonian (1) in the second quantization form reads

$$H_0 = \sum_{ks} e_{ks} a_{ks}^{+} a_{ks} \,, \qquad (7)$$

$$V = \frac{1}{2\Omega} \sum_{\substack{k_1 k_2 q \\ s_1 s_2}} V_q a^+_{k_1+qs_1} a_{k_2-qs_2} a_{k_2 s_2} a_{k_1 s_1} \,, \tag{8}$$

where

$$e_k = \frac{\hbar^2 k^2}{2m} \,.$$

The operators $a^+_{ks}$ and $a_{ks}$ satisfy the anticommutation rules

$$\left[a_{ks}, a^+_{k's'}\right]_+ = \delta_{kk'}\delta_{ss'} \ , \quad \left[a_{ks}, a_{k's'}\right]_+ = \left[a^+_{ks}, a^+_{k's'}\right]_+ = 0 \,. \tag{9}$$

Because of the relations (9) it becomes essential to keep track of signs. The effect of destruction and creation operators on a state is given with relations

$$a_{ks}|\ldots\ n_{ks}\ \ldots\rangle = (-1)^{S_{ks}} n_{ks}|\ldots\ n_{ks}-1\ \ldots\rangle \,,$$

$$a^+_{ks}|\ldots\ n_{ks}\ \ldots\rangle = (-1)^{S_{ks}} (1-n_{ks})|\ldots\ n_{ks}+1\ \ldots\rangle \,, \tag{10}$$

where $n_{ks} = 0$, 1 and

$$S_{ks} = n_1 + n_2 + \ldots + n_{(ks)-1} \tag{11}$$

is the number of occupied states preceding the state $n_{(ks)}$; it is supposed that the single particle quantum numbers of the occupied states are assumed to be ordered.

We will evaluate the ground state energy of the perturbed system (3). In order to get the ground state energy one has to take $l = 0$ in the Feenberg equation (5). The ground state vector of the unperturbed free fermion system is

$$|0\rangle = |\ldots\ 1_{k_1} \ldots\ 1_{k_2} \ldots\ k_f; 0\rangle \,,$$

where

$$k_f = (6\pi^2\rho/\nu)^{1/3}, \qquad \rho = N/\Omega \,. \tag{12}$$

is the radius of the Fermi sphere ($\nu$ denotes the number of spin orientations represented in $k$).

From now on only the holes in the Fermi sea and the particles out of the Fermi sea will be denoted explicitely in all states. In a vector ket the particles above the Fermi sea are placed on the right side of the semicolon sign.

Let us analyse Eq. (5) for the fermions with respect to grouping of an infinite number of terms which describe a special type of physical processes. The zero order term is proportional to $N$,

$$\varepsilon_0 = \sum_{k_s < k_f} e_{k_s} = \sum_{k_s < k_f} \frac{\hbar^2 k^2}{2m} = N \frac{3}{5} e_{k_f} \,. \tag{13}$$

The first order term $V_{00}$ (and all required matrix elements) is calculated in Appendix A and it is also proportional to $N$,

$$V_{00} = \frac{1}{2\Omega} \sum_{t_1 t_2 < k_f} \sum_{s_1 s_2} (V_0 - V_{t_1 - t_2} \delta_{s_1 s_2}) \,. \tag{14}$$

Let us suppose presently that the order of denominators in the energy relation (5) is one.

The second order term contains summing over all possibilities of the state

$$n = |k_1 \sigma_1 \ k_2 \sigma_2; k_1 + q \sigma_1' \ k_2 - q \sigma_2' \rangle \,, \tag{15}$$

and it reads

$$E_2 = \frac{1}{\Omega^2} \sum_{\substack{k_1 k_2 q \\ k_1 k_2 < k_f \\ |\vec{k}_1 + \vec{q}|, |\vec{k}_2 - \vec{q}| > k_f}} \sum_{\substack{\sigma_1 \sigma_2 \\ \sigma_1' \sigma_2'}} \frac{(V_q \delta_{\sigma_1 \sigma_1'} \delta_{\sigma_2 \sigma_2'} - V_{k_1 - k_2 + q} \delta_{\sigma_2 \sigma_1'} \delta_{\sigma_1 \sigma_2'})^2}{E - E_n^F} \,. \tag{16}$$

We see that this term contains a factor $V_q^2$ which is a not mixed product of the matrix elements like $V_q V_{k_1 - k_2 + q}$ or a product of the matrix elements with mixed arguments like $V_{k_1 - k_2 + q}^2$.

Our experience with boson systems show that no mixed terms in the Feenberg formula lead to the Bogoliubov theory of weakly interacting bosons. Let us do the same here and find all terms in the Feenberg perturbation formula that in the numerators contain only $V_q^n$ ($n$ = 2, 3, ...) matrix elements. For this purpose we will analyse in detail the Feenberg formula including the fifth order term.

In the third order term we find two states

$$n = |k_1 \sigma_1 \ k_2 \sigma_2; k_1 + q \sigma_1' \ k_2 - q \sigma_2' \rangle \,,$$

$$n' = |k_1 \sigma_1 \ k_3 \sigma_3; k_1 + q \sigma_1' \ k_3 - q \sigma_3' \rangle \,. \tag{17}$$

Using the matrix elements from Appendix A, one finds

$$E_3 = \frac{1}{\Omega^3} \sum_{\substack{k_1 k_2 k_3 q \\ k_i < k_f \\ |\vec{k}_1 - \vec{q}| > k_f \\ |\vec{k}_2 - \vec{q}|, |\vec{k}_3 - \vec{q}| > k_f}} \sum_{\substack{\sigma_1 \sigma_2 \sigma_3 \\ \sigma_1' \sigma_2' \sigma_3'}} \frac{V_q^3 \delta_{\sigma_1 \sigma_1'} \delta_{\sigma_2 \sigma_2'} \delta_{\sigma_3 \sigma_3'}}{(E - E_n^F)(E - E_{nn'}^F)} \,. \tag{18}$$

In the fourth order term we have the folowing states:

$$n = |k_1 \sigma_1 \ k_2 \sigma_2; \ k_1 + q \sigma_1' \ k_2 - q \sigma_2' \rangle \,,$$

$$m = |k_1\sigma_1\ k_3\sigma_3;\ k_1 + q\sigma_1'\ k_3 - q\sigma_3'\rangle\,,$$

$$m_1 = |k_2\sigma_2\ k_3\sigma_3;\ k_2 - q\sigma_2'\ k_3 + q\sigma_3'\rangle\,,$$

$$m_2 = |k_1\sigma_1\ k_2\sigma_2\ k_3\sigma_3\ k_4\sigma_4;\ k_1 + q\sigma_1'\ \ k_2 - q\sigma_2'\ k_3 + q\sigma_3'\ \ k_4 - q\sigma_4'\rangle\,,$$

$$r = |k_1\sigma_1\ k_4\sigma_4;\ k_1 + q\sigma_1'\ k_4 - q\sigma_4'\rangle\,,$$

$$r_1 = |k_3\sigma_3\ k_4\sigma_4;\ k_3 - q\sigma_3'\ k_4 + q\sigma_4'\rangle\,,$$

$$r_2 = |k_2\sigma_2\ k_4\sigma_4;\ k_2 - q\sigma_2'\ k_4 + q\sigma_4'\rangle\,,$$

$$r_3 = |k_3\sigma_3\ k_4\sigma_4;\ k_3 + q\sigma_3'\ k_4 - q\sigma_4'\rangle\,,$$

$$r_4 = |k_3\sigma_3\ k_4\sigma_4;\ k_3 + q'\sigma_3'\ k_4 - q'\sigma_4'\rangle\,.$$

There are seven combinations of the triplets which can give the factor $V_q^4$:

$$\begin{array}{ll} 1)\ nmr, & 5)\ nm_2r_4, \\ 2)\ nmr_1, & 6)\ nm_2r, \\ 3)\ nm_1r_2, & 7)\ nm_2m_1, \\ 4)\ nm_1r_3. & \end{array} \tag{19}$$

The triplet 5) leads to the term of the order $N^2$ and it should be cancelled by a corresponding part of the "second" order term. The triplets 1) and 3) are indeed the same; it can be seen after writing the whole expression and changing $k_2 \leftrightarrow k_1$ in 3). This is the case with 2) and 4), then with 6) and 7). In this way we are led to the conclusion that in the fourth order term only the triplets 1), 2) and 6) are in the game. Keeping only unmixed products in the matrix elements of the numerator, we find

$$E_4 = \frac{1}{\Omega^4} \sum_{\substack{k_1k_2k_3k_4q \\ k_i < k_f}} \sum_{\substack{\sigma_1\sigma_2\sigma_3\sigma_4 \\ \sigma_1'\sigma_2'\sigma_3'\sigma_4'}} V_q^4 \left\{ \frac{1}{(E-E_n^F)(E-E_m^F)(E-E_r)} \right.$$

$$\left. + \frac{1}{(E-E_n^F)(E-E_m^F)(E-E_{r_1})} + \frac{1}{(E-E_n^F)(E-E_{m_2}^F)(E-E_r)} \right\}$$

$$\times \delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}\delta_{\sigma_3\sigma_3'}\delta_{\sigma_4\sigma_4'}\,. \tag{20}$$

In the fifth order term we find the following states:

$$n = |k_1\sigma_1\ k_2\sigma_2;\ k_1 + q\sigma_1'\ k_2 - q\sigma_2'\rangle\,,$$

$$m = |k_1\sigma_1\ k_3\sigma_3;\ k_1 + q\sigma_1'\ k_3 - q\sigma_3'\rangle\,,$$

$$m_1 = |k_1\sigma_1\ k_2\sigma_2\ k_3\sigma_3\ k_4\sigma_4;\ k_1 + q\sigma_1'\ \ k_2 - q\sigma_2'\ k_3 + q\sigma_3'\ \ k_4 - q\sigma_4'\rangle\,,$$

$$m_2 = |k_1\sigma_1\ k_2\sigma_2\ k_3\sigma_3\ k_4\sigma_4;\ k_1 + q\sigma_1'\ \ k_2 - q\sigma_2'\ k_3 + q'\sigma_3'\ \ k_4 - q'\sigma_4'\rangle\,,$$

$$r = |k_1\sigma_1\ k_4\sigma_4;\ k_1 + q\sigma_1'\ k_4 - q\sigma_4'\rangle\,,$$

$$r_1 = |k_3\sigma_3\ k_4\sigma_4;\ k_3 - q\sigma_3'\ k_4 + q\sigma_4'\rangle\,,$$

$$r_2 = |k_2\sigma_2\ k_3\sigma_3;\ k_2 - q\sigma_2'\ k_3 + q\sigma_3'\rangle\,,$$

$$r_3 = |k_1\sigma_1\ k_3\sigma_3\ k_4\sigma_4\ k_5\sigma_5;\ k_1 + q\sigma_1'\ \ k_3 - q\sigma_3'\ k_4 + q\sigma_4'\ \ k_5 - q\sigma_5'\rangle\,,$$

$$r_4 = |k_1\sigma_1\ k_2\sigma_2\ k_3\sigma_3\ k_5\sigma_5;\ k_1 + q\sigma_1'\ \ k_2 - q\sigma_2'\ k_3 + q\sigma_3'\ \ k_5 - q\sigma_5'\rangle\,,$$

$$r_5 = |k_1\sigma_1\ k_3\sigma_3\ k_4\sigma_4\ k_5\sigma_5;\ k_1 + q\sigma_1'\ \ k_3 + q\sigma_3'\ k_4 - q\sigma_4'\ \ k_5 - q\sigma_5'\rangle\,,$$

$$r_6 = |k_1\sigma_1\ k_2\sigma_2\ k_3\sigma_3\ k_5\sigma_5;\ k_1 + q\sigma_1'\ \ k_2 - q\sigma_2'\ k_3 + q'\sigma_3'\ \ k_5 - q'\sigma_5'\rangle\,,$$

$$r_7 = |k_1\sigma_1\ k_3\sigma_3\ k_4\sigma_4\ k_5\sigma_5;\ k_1 + q\sigma_1'\ \ k_3 + q\sigma_3'\ k_4 - q'\sigma_4'\ \ k_5 - q'\sigma_5'\rangle\,,$$

$$s = |k_1\sigma_1\ k_5\sigma_5;\ k_1 + q\sigma_1'\ k_5 - q\sigma_5'\rangle\,,$$

$$s_1 = |k_4\sigma_4\ k_5\sigma_5;\ k_4 - q\sigma_4'\ k_5 + q\sigma_5'\rangle\,,$$

$$s_2 = |k_4\sigma_4\ k_5\sigma_5;\ k_4 + q\sigma_4'\ k_5 - q\sigma_5'\rangle\,,$$

$$s_3 = |k_2\sigma_2\ k_5\sigma_5;\ k_2 - q\sigma_2'\ k_5 + q\sigma_5'\rangle\,,$$

$$s_4 = |k_3\sigma_3\ k_5\sigma_5;\ k_3 + q'\sigma_3'\ k_5 - q'\sigma_5'\rangle\,,$$

$$s_5 = |k_3\sigma_3\ k_4\sigma_4;\ k_3 + q'\sigma_3'\ k_4 - q'\sigma_4'\rangle\,,$$

$$s_6 = |k_3\sigma_3\ k_5\sigma_5;\ k_3 + q'\sigma_3'\ k_5 - q'\sigma_5'\rangle\,.$$

There are sixteen different combinations of the states for the numerator:

$$
\begin{array}{ll}
1)\ nmrs, & 9)\ nmr_3r_1, \\
2)\ nmrs_1, & 10)\ nmr_3s, \\
3)\ nm_1r_1s_2, & 11)\ nmr_4r_2, \\
4)\ nm_1rs, & 12)\ nm_1r_5r, \\
5)\ nm_1rs_1, & 13)\ nm_1r_5s_4, \\
6)\ nm_1r_2s_3, & 14)\ nm_2r_6s_6, \\
7)\ nm_1r_2s_4, & 15)\ nm_2r_7s, \\
8)\ nmr_3s, & 16)\ nm_2r_5s_5.
\end{array}
\tag{21}
$$

The combinations 14), 15) and 16) give the terms which are proportional to $N^2$ and therefore they should be cancelled by some terms which appear in the second and third order. Keeping again only unmixed terms, one finds for the fifth order term

$$
E_5 = \frac{1}{\Omega^5} \sum_{\substack{k_1k_2k_3 \\ k_4k_5q \\ k_i < k_f}} \sum_{\substack{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5 \\ \sigma_1'\sigma_2'\sigma_3'\sigma_4'\sigma_5'}} V_q^5 \left\{ \sum_{i=1}^{13} \frac{1}{\mu_i} \right\} \delta_{\sigma_1\sigma_1'} \delta_{\sigma_2\sigma_2'} \delta_{\sigma_3\sigma_3'} \delta_{\sigma_4\sigma_4'} \delta_{\sigma_5\sigma_5'} \,, \tag{22}
$$

where

$$
\mu_i = (E - E_j^F)(E - E_k^F)(E - E_l^F)(E - E_p^F) \,,
$$

and $j$, $k$, $l$ and $p$ represent four states denoting one combination from the set (21).

For higher order terms of the Feenberg perturbation formula one can, in principle, do the same.

Let us now consider the denominators, where the differences of the energies appear. In the zero order term the differences $E^0 - E^{F0}$ are just the same as in the Rayleigh-Schrödinger (RS) perturbation theory. For example if $n = |k_1\sigma_1, k_2\sigma_2; k_1 + q\sigma_1' k_2 - q\sigma_2'\rangle$,

$$
E^0 - E^{F0} = \sum_{k\sigma<k_f} \frac{\hbar^2 k^2}{2m} - \left\{ \sum_{k\sigma<k_f} \frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 k_1^2}{2m} - \frac{\hbar^2 k_2^2}{2m} + \frac{\hbar^2 (k_1^2 + q^2)}{2m} \right.
$$

$$
\left. + \frac{\hbar^2 (\vec{k}_2 - \vec{q})^2}{2m} \right\} = -\frac{\hbar^2}{m} \left[ \left( \frac{q^2}{2} + \vec{k}_1 \vec{q} \right) + \left( \frac{q^2}{2} - \vec{k}_2 \vec{q} \right) \right] .
$$

Changing $\vec{k}_2 \to -\vec{k}_2$ (of course this should be done everywhere in the sum over $k_2$), one finds a symmetrical form

$$
E^0 - E^{F0} = -\frac{\hbar^2}{m} \left[ \left( \frac{q^2}{2} + \vec{k}_1 \vec{q} \right) + \left( \frac{q^2}{2} + \vec{k}_2 \vec{q} \right) \right] = -\frac{\hbar^2}{m} \big( D_1^{(0)} + D_2^{(0)} \big) \,,
$$

where

$$D_i^{(0)} = \frac{q^2}{2} + \vec{k}_i\vec{q}\,.$$

In the first order, using the matrix elements from Appendix A and neglecting the terms of the order $1/N$, one finds

$$E^1 - E_n^{F1} = E^0 - E_n^{F0} + \frac{1}{2\Omega}\sum_{t_1t_2}\sum_{s_1s_2}(V_0 - V_{t_2-t_1}\delta_{s_2s_1}$$

$$-\frac{1}{2\Omega}\sum_{t_1t_2}\sum_{s_1s_2}(V_0 - V_{t_2-t_1}\delta_{s_1s_2} - \frac{1}{\Omega}\sum_{ts_1s_2}\Big[V_{t-k_1}\delta_{s_1s_2}\delta_{s_1\sigma_1'}$$

$$+V_{t-k_2}\delta_{s_1s_2}\delta_{s_1\sigma_2'} - V_{t-k_1-q}\delta_{s_1s_2}\delta_{s_1\sigma_1'}\delta_{s_2\sigma_1'} - V_{t-k_2+q}\delta_{s_1s_2}\delta_{s_1\sigma_2'}\delta_{s_2\sigma_2'}\Big]\,,$$

and again in the symmetrical form

$$E^1 - E_n^{F1} = -\frac{\hbar^2}{m}\big(D_1^{(1)} + D_2^{(1)}\big)\,,$$

where it is introduced

$$D_1^{(1)} = \frac{q^2}{2} + \vec{k}_i\vec{q} + \frac{m}{\hbar^2}\frac{1}{\Omega}\sum_{ts<k_f}\big(V_{t+k_i} - V_{t+k_i+q}\delta_{s\sigma_i'}\big)\,. \tag{23}$$

If we consider this difference for the state

$$n = |k_1\sigma_1\ k_2\sigma_2k_3\sigma_3\ k_4\sigma_4;\ k_1+q\sigma_1'\ k_2-q\sigma_2'k_3+q'\sigma_3'\ k_4-q'\sigma_4'\rangle$$

in a similar procedure, one finds

$$E^1 - E_n^{F1} = D_1^{(1)} + D_2^{(1)} + D_3^{(1)} + D_4^{(1)}\,.$$

Thus we showed that any energy difference (including first order approximation) can be writen as a sum of terms $D_i$. Let us suppose that it is generally correct.

The expression $D_i$ depends upon the wave vector $\vec{k}_i$, transferred wave vector $\vec{q}$ and the potential of the interaction. After this the third, the fourth and the fifth order terms become

$$E_3 = \frac{1}{\Omega^3}\sum_{\substack{k_1k_2k_3q\\ k_i\,<\,k_f}}\ \sum_{\substack{\sigma_1\sigma_2\sigma_3\\ \sigma_1'\sigma_2'\sigma_3'}}\left(-\frac{m}{\hbar^2}\right)V_q^3\frac{1}{(12)(13)}\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}\delta_{\sigma_3\sigma_3'}\,, \tag{24}$$

$$E_4 = \frac{1}{\Omega^4} \sum_{\substack{k_1 k_2 k_3 \\ k_4 q \\ k_i < k_f}} \sum_{\substack{\sigma_1\sigma_2\sigma_3\sigma_4 \\ \sigma_1'\sigma_2'\sigma_3'\sigma_4'}} \left(-\frac{m}{\hbar^2}\right) V_q^4 \left\{ \frac{1}{(12)(13)(14)} \right.$$

$$\left. + \frac{1}{(12)(13)(34)} + + \frac{1}{(12)(1234)(14)} \right\} \prod_{i=1}^{4} \delta_{\sigma_i \sigma_i'} \,, \tag{25}$$

$$E_5 = \frac{1}{\Omega^5} \sum_{\substack{k_1 k_2 k_3 \\ k_4 k_5 q \\ k_i < k_f}} \sum_{\substack{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5 \\ \sigma_1'\sigma_2'\sigma_3'\sigma_4'\sigma_5'}} \left(-\frac{m}{\hbar^2}\right) V_q^5 \left\{ \frac{1}{(12)(13)(14)(15)} \right.$$

$$+ \frac{1}{(12)(13)(14)(45)} + \frac{1}{(12)(13)(34)(45)} + \frac{1}{(12)(1234)(14)(15)}$$

$$+ \frac{1}{(12)(1234)(14)(45)} + \frac{1}{(12)(1234)(23)(25)} + \frac{1}{(12)(1234)(23)(35)}$$

$$+ \frac{1}{(12)(13)(1345)(15)} + \frac{1}{(12)(13)(1345)(34)} + \frac{1}{(12)(1234)(1235)(15)}$$

$$+ \frac{1}{(12)(1234)(1235)(23)} + \frac{1}{(12)(1234)(1345)(14)}$$

$$\left. + \frac{1}{(12)(1234)(1345)(35)} \right\} \prod_{i=1}^{5} \delta_{\sigma_i \sigma_i'} \,, \tag{26}$$

where

$$(ij) = D_i + D_j \,,$$

$$(ijkl) = D_i + D_j + D_k + D_l \,.$$

The factors including four sums of $D_i$ in denominators can be transformed into the form where each factor in any denominator has two sumands only. For instance, by the use of the symmetry in the sums, we find

$$\frac{1}{(12)(1234)(14)} = \frac{1}{2} \left\{ \frac{1}{(12)(1234)(14)} + \frac{1}{(12)(1234)(23)} \right\} = \frac{1}{2} \frac{1}{(12)(14)(23)} \,.$$

Let us consider the interactions which are spin independent. The relations (24), (25) and (26) after some algebra, performed in Appendix B, get the form

$$E_n = \frac{S^n}{2n\Omega^n}\left(-\frac{m}{\hbar^2}\right)^{n-1}\left(\frac{\Omega}{8\pi^3}\right)^{n+1}\int \mathrm{d}\vec{q}\, V_q^n \int \mathrm{d}\vec{k}_1 \ldots \int \mathrm{d}\vec{k}_n I_n \,, \tag{27}$$

where $n = 3$, 4, 5 and

$$I_n = \int\limits_{-\infty}^{\infty} \mathrm{d}t_1 \ldots \int\limits_{-\infty}^{\infty} \mathrm{d}t_n \; \mathrm{e}^{-D_1|t_1|-\ldots-D_n|t_n|}\delta(t_1+\ldots+t_n)\,. \tag{28}$$

The factor $S^n$ arises from the spin sums.

The expressions (27) and (28) are formally the same as those in the RS perturbation theory of the electron system when the ring approximatinn is treated[6,7]. The ring approximation of RS perturbation theory in the relation to our "diagrams of the states" is discussed in Section 4.

Having in mind our result (27), then the structure of the complete Feenberg perturbation energy formula and the formal similarity with RS perturbation theory, we can conclude that the expression (27) is valid for any $n \geq 2$.

Let us find a closed form for the energy, summing up the terms of all orders in the relation (27). If we define

$$B_n(q) = \frac{1}{n}\int \mathrm{d}\vec{k}_1 \ldots \int \mathrm{d}\vec{k}_n \int\limits_{-\infty}^{\infty} \mathrm{d}t_1 \ldots \int\limits_{-\infty}^{\infty} \mathrm{d}t_n \; \mathrm{e}^{-D_1|t_1|-\ldots-D_n|t_n|}\delta(t_1+\ldots+t_n)\,, \tag{29}$$

the relation (27) becomes

$$E_n = \frac{S^n}{2n\Omega^n}\left(-\frac{m}{\hbar^2}\right)^{n-1}\left(\frac{\Omega}{8\pi^3}\right)^{n+1}\int \mathrm{d}\vec{q}\, V_q^n B_n(q)\,. \tag{30}$$

Since

$$\delta(t_1+\ldots+t_n) = \frac{q}{2\pi}\int\limits_{-\infty}^{\infty} \mathrm{d}u \; \mathrm{e}^{\mathrm{i}q(t_1+\ldots+t_n)u}\,,$$

that is

$$B_n(q) = \frac{q}{2\pi n}\int\limits_{-\infty}^{\infty} \mathrm{d}u \left\{ \int \mathrm{d}\vec{k}_1 \int\limits_{-\infty}^{\infty} \mathrm{d}t_1 \; \mathrm{e}^{-D_1|t_1|+\mathrm{i}gut_1} \right.$$

$$\int \mathrm{d}\vec{k}_2 \int\limits_{-\infty}^{\infty} \mathrm{d}t_2 \; \mathrm{e}^{-D_2|t_2|+\mathrm{i}gut_2}$$

$$\vdots$$

$$\int \mathrm{d}\vec{k}_n \int_{-\infty}^{\infty} \mathrm{d}t_n \; \mathrm{e}^{-D_n|t_n|+\mathrm{i}gut_n} \Bigg\}; \qquad \begin{array}{c} k_i < k_f \\ |\vec{k}_i + \vec{q}| > k_f \end{array}$$

$$= \frac{q}{2\pi n} \int_{-\infty}^{\infty} \mathrm{d}u \; \big[Q_q(u)\big]^n \, , \tag{31}$$

where

$$Q_q(u) = \int \mathrm{d}\vec{k} \int_{-\infty}^{\infty} \mathrm{d}t \; \mathrm{e}^{-D|t|+\mathrm{i}gut}; \quad k_i < k_f, \quad |\vec{k}_i + \vec{q}| > k_f \, . \tag{32}$$

Summing up all terms $n \geq 2$ of the relation (30) gives

$$\sum_{n=2}^{\infty} E_n = \frac{\Omega}{16\pi^3} \int \mathrm{d}\vec{q} \frac{q}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}u \sum_{n=2}^{\infty} \frac{S^n}{n} \left(-\frac{m}{\hbar^2}\right)^{n-1} \left[\frac{V_q Q_q(u)}{8\pi^3}\right]^n$$

$$= \frac{\hbar^2 Q}{16\pi^3 m} \int \mathrm{d}\vec{q} \frac{q}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}u \left\{ \ln\left[1 + \frac{V_q Q_q(u) ms}{8\pi^3\hbar^2}\right] - \frac{V_q Q_q(u) ms}{8\pi^3\hbar^2} \right\} , \tag{33}$$

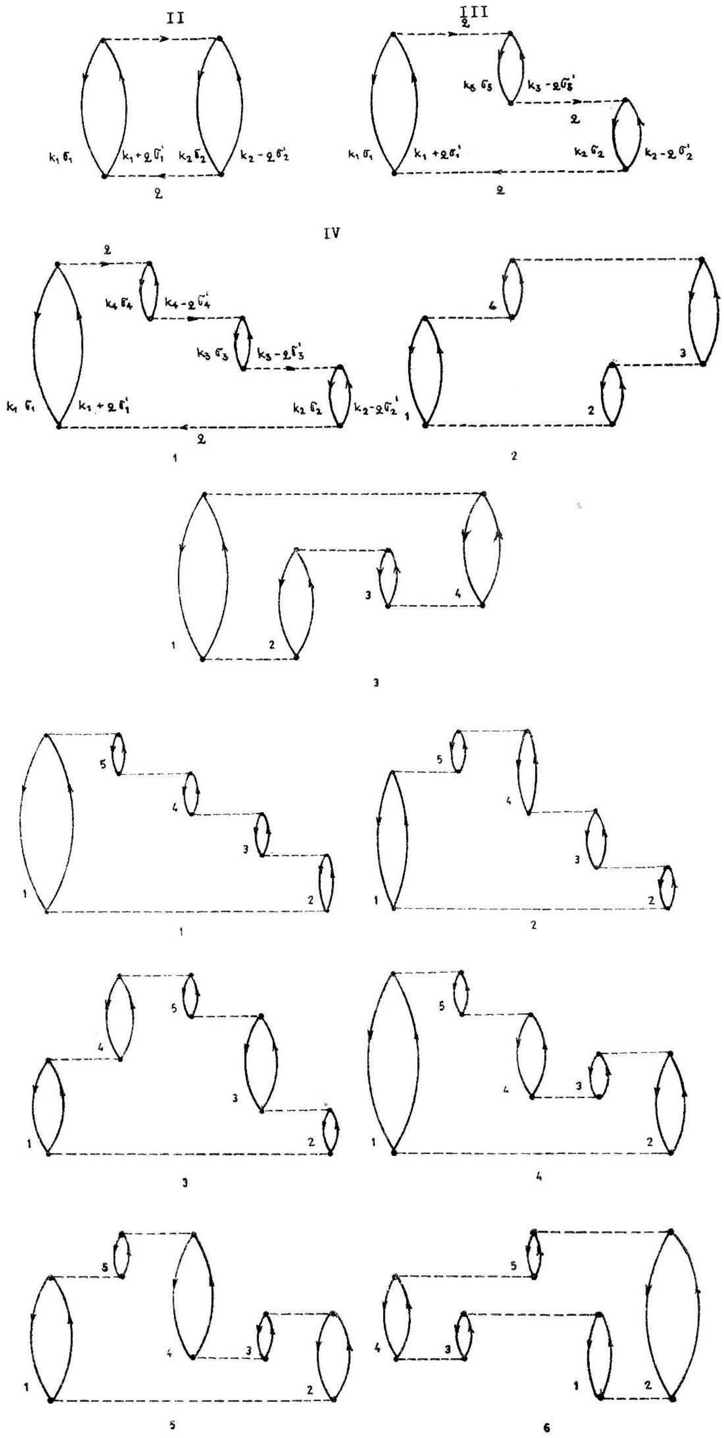
with the constraint

$$-1 < \frac{V_q Q_q(u) ms}{8\pi^3\hbar^2} \leq 1 \, . \tag{34}$$

The total energy in our approximation is then the sum of the relations (13), (14) and (33).

## 4. *The relation with RS perturbation theory and some evaluations*

The states which occur in the combination for the second, third, fourth and fifth order term can be related to the Goldstone diagrams. Figure 1 presents, respectively, the second, the third and the fourth order ring diagrams which belong to the states given by the relations (15), (17) and (19). In Fig. 2 we plotted the fifth order diagrams which correspond to the combinations of the states (21). Let us mention that the unconnected diagrams (the combinations 5 in (19) and 14, 15 and 16 in (21)) and the combinations 3, 4 and 7 in (19) which do not represent new possibilities with the respect to the combinations 1, 2 and 6 are not drawn.


II
III
IV

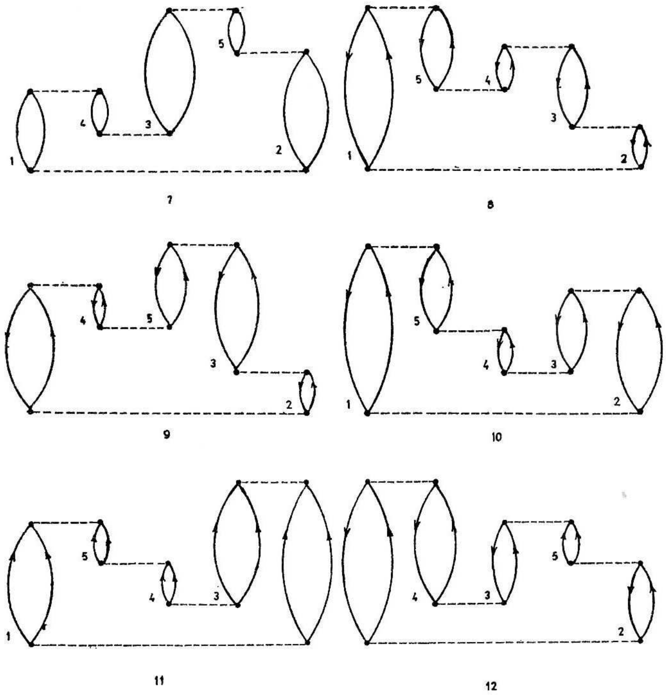



*Fig. 1. The ring diagrams of the second (II), third (III) and fourth (IV) order.*

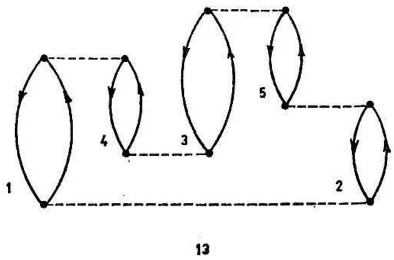



*Fig. 2. The ring diagrams of the fifth order.*

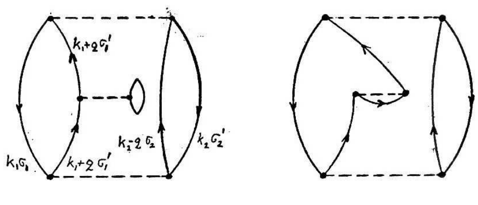


*Fig. 3. Third order Goldstone diagrasms included (among others) in the first order of $D^{(1)}$.*

The diagrams have different structure and regardless of denoting the lines and holes they must not be reduced to each other; this is the consequence of the role of ordering of the states (for instance the diagram 6 in Fig. 2 can be transformed into diagram 8 after the substitution $2 \to 1$, $1 \to 5$, $3 \to 4$, $4 \to 3$, $5 \to 2$; but they have different ordering and represent the different possibilities of summing in the fifth order term of BWF formula). In our approach they could be used to define the states which amount to the energy. Which states belong to a diagram? A state is defined by the particle-hole lines between two neighbouring interaction lines. For instance, in the fourth order diagram 1 the bottom state is $n = |k_1\sigma_1\ k_2\sigma_2;\ k_1{+}q\sigma_1'\ k_2{-}q\sigma_2'\rangle$. All states which correspond to a diagram are formed in the same way. For example the fifth order term 1 defines the combination

$$|k_1\sigma_1\ k_2\sigma_2;\ k_1+q\sigma_1'\ k_2-q\sigma_2'\rangle\,, \qquad |k_1\sigma_1\ k_4\sigma_4;\ k_1+q\sigma_1'\ k_4+q\sigma_4'\rangle\,,$$

$$|k_1\sigma_1\ k_3\sigma_3;\ k_1+q\sigma_1'\ k_3-q\sigma_3'\rangle\,, \qquad |k_1\sigma_1\ k_5\sigma_5;\ k_1+q\sigma_1'\ k_5-q\sigma_5'\rangle\,.$$

Because of the above approach, the diagrams are sometimes called the “diagrams of the states”.

Of course in the above mentioned procedure there is no the difference between direct and exchange terms.

$$D^{(1)} = \frac{q^2}{2} + \vec{k}\vec{q} + \frac{m}{8\pi^3\hbar^2}\sum_{s\sigma'}\delta_{\sigma\sigma'}\int\limits_{\substack{k_p\,<\,k_f\\ |\vec{k_i}+\vec{q}|\,>\,k_f}} \mathrm{d}\vec{p}(V_{p+k} - V_{p+k+q}). \tag{35}$$

In addition let us consider the application of our result to 1. electron gas and 2. an appropriate model potential.

1. It is well known that the RS perturbation theory in the ring approximation is suitable for the description of the high density electron gas. Our approximation in which we sum up unmixed terms, $V_q$, (in the numerators), contains RS perturbation theory in the ring approximation; they are identical if all $D_i$ in the denominators of BWF formula are taken in the zero order approximation with respect to the interaction.

It is also possible to obtain an analytic form of $D^{(1)}$, which is a step beyond the ring approximation. Solving the integrals in the relation, we find

$$D^{(1)} = \frac{q^2}{2} + \vec{k}\vec{q} + \frac{e^2 m}{8\pi^3\varepsilon_0 q\hbar^2}\left\{\frac{1}{k}(k_f^2 - k^2)\ln\frac{k_f - k}{k_f + k}\right.$$

$$\left. -\frac{1}{|\vec{k}+\vec{q}|}(k_f^2 - |\vec{k}+\vec{q}|^2)\ln\frac{k_f + |\vec{k}+\vec{q}|}{k_f - |\vec{k}+\vec{q}|}\right\}\,. \tag{36}$$

It is a pity that the substitution of this equation into the relation for $Q_q(u)$ does not make it easily solvable in an analytic form.

We emphasize that our approach in the first order of $D^{(1)}$ includes "direct and exchange scattering with unexcited particles" which were not contained in the Gell-Alann-Brueckner theory, (Fig. 3), i.e.

$$1 + 2 \to 1' + 2' ,$$

$$1' + 3 \to 1' + 3 ,$$

$$1' + 2' \to 1 + 2 .$$

Numerical calculation including above processes in electron gas at high density will be presented in our following report.

2. A relatively simple analytic solution of the quantity $Q_q(u)$ is possible for a model potential

$$V_q = \begin{cases} V(-q^2 + g^2), & q \le g, \quad V > 0 \\ 0, & q \ge g. \end{cases} \tag{37}$$

$V$ and $g$ are the parameters of the potential. In $r$-space this potential reads

$$V(r) = \frac{Vg^3}{\pi^2 r^2} j_2(gr) , \tag{38}$$

where $j_2(gr)$ is the spherical Bessel function. The potential (38) has short range repulsion, longer range attraction and further decreasing oscillatory shape. To derive the solution, let us perform the integration in Eq. (35) under the constraints

$$\delta \gg k_f , \qquad q < \delta, \qquad G \ge 2k_f + \delta .$$

In this case the domains of integration for both integrals become the same and we find

$$\int\limits_{\substack{kp < k_f \\ |\vec{k}_i + \vec{q}| > k_f}} \mathrm{d}\vec{p}(V_{p+k} - V_{p+k+q}) = \frac{4\pi k_f^3}{3} V(q^2 + 2\vec{k}\vec{q}) ,$$

that is

$$D^{(1)} = K(q^2 + 2\vec{k}\vec{q}) , \tag{39}$$

where

$$K = 1 + \frac{mVk_f^3}{3\pi^2\hbar^2} . \tag{40}$$

The second term is the correction to the ring approximation. Substituting (39) into (32) and integrating, one finds

$$Q_q(u) = \frac{2k}{K} \left\{ k_f + \frac{1}{2q} \left( \frac{u^2}{K^2} + k_f^2 - \frac{q^2}{4} \right) \ln \frac{u^2/K^2 + \left(k_f + q/2\right)^2}{u^2/K^2 - \left(k_f + q/2\right)^2} \right.$$

$$-\frac{u}{K}\left[\arctan\left(K\frac{k_f+q/2}{u}\right)+\arctan\left(K\frac{k_f-q/2}{u}\right)\right]\Bigg\}\,. \tag{41}$$

The limit, when $q \to 0$, is

$$Q_0(u)=\lim_{q\to 0}Q_q(u)=\frac{4\pi k_f}{K}\;\frac{x-\arctan x}{x}\,, \tag{42}$$

where $x = k_f K/u$ and it satisfies the inequality

$$0 \le Q_0(u) \le \frac{4\pi k_f}{K}\,. \tag{43}$$

The relation (43) can be considered as a necessary condition. In order to estimate a correction to the ring approximation, let us find the constant $K$ for liquid $^3$He. We derive $V$ and $g$ from the fitting of Bruch-McGee potential[11]: $V = 2.87\ 10^{-71}$ Jm$^5$, $g = 2.399\ 10^{10}$ m$^{-1}$. The density of liquid $^3$He is 1.638 $10^{28}$ m$^{-3}$ at $T = 0.6$ K and the mass of an atom 5.008 $10^{-27}$ kg. So we find $K = 1+mV/(\hbar^2\rho) = 1+0.2117$. We see that the correction of $K$ is about 20%. Furthermore, for the above conditions we find $k_f = (3\pi^2\rho)^{1/3} = 7.856\ 10^9$ m$^{-1}$ and we have indeed $g > 2k_f$. (We took helium 3 parameters although we knew that the ring approximation was not enough to describe this system.)

We see that Feenberg perturbation theory offers a good basis for the investigation of different physical systems. Furthermore, the analysis in this paper has been performed using the free particle basis. This method in this form is not applicable to liquid helium because of the strong short range repulsive potential between atoms. But it is expected that it can be redeveloped within correlated basis[12–16] where it should be applicable to real systems.

*Acknowledgements*

It is a pleasure to thank Professors C. E. Campbell and K. Ljolje for suggestions and discussions.

*Appendix A*

Here we calculate the matrix elements for the states which are included in our calculation.

*Diagonal matrix elements*

1. In the state $|0\rangle$ we find

$$V_{00}=\frac{1}{2\Omega}\sum_{\substack{p_1p_2Q\\ s_1s_2}}\langle 0|a^+_{p_1+Qs_1}a^+_{p_2-Qs_2}a_{p_2s_2}a_{p_1s_1}|0\rangle\,,$$

and there are two possibilities of pairing off:

$$\text{a)}\ p_1 + Qs_1 = p_1 s_1\,, \qquad \text{b)}\ p_1 + Qs_1 = p_2 s_2\,,$$
$$p_2 - Qs_1 = p_2 s_2\,, \qquad p_2 + Qs_2 = p_1 s_1\,.$$

The first pairing gives $Q = 0$ and second $Q = p_2 - p_1$, $s_1 - s_2$. The matrix element becomes

$$V_{00} = \frac{1}{2\Omega} \sum_{p_1 p_2 < k_f} \sum_{s_1 s_2} (V_0 - V_{p_2 - p_1} \delta_{s_1 s_2})\,. \qquad (A.1)$$

2. The state

$$n = |k_1\sigma_1\ k_2\sigma_2; k_1 + q\sigma_1'\ k_2 - q\sigma_2'\rangle \qquad (A.2)$$

has two particles above the Fermi sea and two missing in the sea. Using a similar procedure, one finds

$$V_{nn} = \frac{1}{2\Omega} \sum_{\substack{s_1 s_2 \\ \neq k_1 k_2 \\ < k_f}} \sum_{s_1 s_2} (V_0 - V_{t_2 - t_1} \delta_{s_1 s_2}) + \frac{1}{\Omega} \sum_{\substack{s_1 \\ \neq k_1 k_2 \\ < k_f}} \sum_{s_1 s_2} \Big[ V_0 (\delta_{s_1 \sigma_1'} + \delta_{s_1 \sigma_2'})$$
$$- V_{t_1 - k_1 - q} \delta_{s_1 s_2} \delta_{s_1 \sigma_1'} \delta_{s_2 \sigma_1'} - V_{t_1 - k_2 - q} \delta_{s_1 s_2} \delta_{s_1 \sigma_2'} \delta_{s_2 \sigma_2'} \Big] + O(\frac{1}{N})\,. \qquad (A.3)$$

3. In the state

$$n' = |k_1\sigma_1\ k_2\sigma_2\ k_3\sigma_3\ k_4\sigma_4;\ k_1 + q\sigma_1'\ \ k_2 - q\sigma_2'\ k_3 + q\sigma_3'\ \ k_4 - q\sigma_4'\rangle$$

one finds (up to the order $1/N$)

$$V_{n'n'} = \frac{1}{2\Omega} \sum_{\substack{t_1 t_2 \\ \neq k_1 \dots k_4 \\ < k_f}} \sum_{s_1 s_2} (V_0 - V_{t_2 - t_1} \delta_{s_1 s_2}) + \frac{1}{\Omega} \sum_{\substack{t \\ \neq k_1 \dots k_4 \\ < k_f}} \sum_{s_1 s_2} \Big[ V_0 \big( \delta_{s_1 \sigma_1'}$$
$$+ \delta_{s_1 \sigma_2'} + \delta_{s_1 \sigma_3'} + \delta_{s_1 \sigma_4'} \big) - \big( V_{t - k_1 - q} \delta_{s_1 \sigma_1'}$$
$$+ V_{t - k_2 - q} \delta_{s_1 \sigma_2'} + V_{t - k_3 - q} \delta_{s_1 \sigma_3'} + V_{t - k_4 - q} \delta_{s_1 \sigma_4'} \big) \delta_{s_1 s_2} \Big] + O(\frac{1}{N})$$
$$= \frac{1}{2\Omega} \sum_{\substack{t_1 t_2 \\ < k_f}} \sum_{s_1 s_2} (V_0 - V_{t_2 - t_1} \delta_{s_1 s_2}) + \frac{1}{\Omega} \sum_{\substack{t \\ < k_f}} \sum_{s_1 s_2} \Big[ (V_{t - k_1} - V_{t - k_1 - q}) \delta_{s_1 \sigma_1'}$$
$$+ (V_{t - k_2} - V_{t - k_2 - q}) \delta_{s_1 \sigma_2'} + (V_{t - k_3} - V_{t - k_3 - q}) \delta_{s_1 \sigma_3'}$$

$$+(V_{t-k_4} - V_{t-k_4-q})\delta_{s_1\sigma'_4}\Big]\delta_{s_1s_2} + O(\frac{1}{N})\,. \tag{A.4}$$

*Nondiagonal matrix elements*

4. In the matrix element between the ground state and the state $n$ (A.2)

$$V_{0n} = \frac{1}{2\Omega}\sum_{\substack{p_1p_2Q\\ s_1s_2}} V_q\langle 0|a^+_{p_1+Qs_1}a^+_{p_2-Qs_2}a_{p_2s_2}a_{p_1s_1}|n\rangle$$

there are four "pairings" different from zero, which give

$$V_{0n} = \frac{1}{2\Omega}\sum_{s_1s_2}\big(V_q\delta_{s_1\sigma_1}\delta_{s_2\sigma_2}\delta_{s_1\sigma'_1}\delta_{s_2\sigma'_2} + V_q\delta_{s_1\sigma_2}\delta_{s_2\sigma_1}\delta_{s_1\sigma'_2}\delta_{s_2\sigma'_1}$$

$$-V_{k_2-k_1-q}\delta_{s_1\sigma'_1}\delta_{s_2\sigma_1}\delta_{s_1\sigma_2}\delta_{s_2\sigma'_2} - V_{k_1-k_2-q}\delta_{s_1\sigma'_2}\delta_{s_2\sigma'_1}\delta_{s_1\sigma_1}\delta_{s_2\sigma_2}\,.$$

If the interaction is spin independent, then

$$V_{0n} = \frac{1}{\Omega}\big(V_q\delta_{\sigma_1\sigma'_1}\delta_{\sigma_2\sigma'_2} - V_{k_1-k_2+q}\delta_{\sigma_2\sigma'_1}\delta_{\sigma_1\sigma'_2}\,. \tag{A.5}$$

5. Similarly one finds

$$V_{nn'} = \frac{1}{2\Omega}\sum_{s_1s_2}\big(V_q\delta_{s_1\sigma'_3}\delta_{s_2\sigma'_4}\delta_{s_1\sigma_3}\delta_{s_2\sigma_4} + V_q\delta_{s_1\sigma'_4}\delta_{s_2\sigma'_3}\delta_{s_1\sigma_4}\delta_{s_2\sigma_3}$$

$$-V_{k_4-k_3-q}\delta_{s_1\sigma'_3}\delta_{s_1\sigma_4}\delta_{s_2\sigma'_4}\delta_{s_2\sigma_3} - V_{k_3-k_4-q}\delta_{s_1\sigma'_4}\delta_{s_1\sigma_3}\delta_{s_2\sigma'_3}\delta_{s_3\sigma_4}\,,$$

or for spin independent interaction

$$V_{nn'} = \frac{1}{\Omega}\big(V_q\delta_{\sigma_3\sigma'_3}\delta_{\sigma_4\sigma'_4} - V_{k_4-k_3-q}\delta_{\sigma_4\sigma'_3}\delta_{\sigma_3\sigma'_4}\,. \tag{A.6}$$

6. The matrix element between the states $n$ and

$$n'' = |k_1\sigma_1\ k_3\sigma_3; k_1+q\sigma'_1\ k_3-q\sigma'_3\rangle\,,$$

is

$$V_{nn''} = \frac{1}{2\Omega}\sum_{s_1s_2}\big(V_q\delta_{s_1\sigma'_3}\delta_{s_2\sigma_2}\delta_{s_1\sigma_3}\delta_{s_2\sigma'_2} + V_q\delta_{s_1\sigma_2}\delta_{s_2\sigma'_2}\delta_{s_2\sigma_3}\delta_{s_1\sigma'_2}$$

$$-V_{k_4-k_3}\delta_{s_1\sigma'_3}\delta_{s_2\sigma_2}\delta_{s_1\sigma'_2}\delta_{s_2\sigma_3} - V_{k_3-k_2}\delta_{s_1\sigma_2}\delta_{s_2\sigma'_3}\delta_{s_1\sigma_3}\delta_{s_2\sigma'_2}\,,$$

and for spin independent potential

$$V_{nn''} = \frac{1}{\Omega}\big(V_q\delta_{\sigma_3\sigma'_3}\delta_{\sigma_2\sigma'_2} - V_{k_3-k_2}\delta_{\sigma'_2\sigma'_3}\delta_{\sigma_2\sigma_3}\big)\,. \tag{A.7}$$

*Appendix B*

The expression (24) for the third order energy in the integral form reads

$$E_3 = \frac{S^3}{\Omega^3}\left(-\frac{m}{\hbar^2}\right)^2\left(\frac{\Omega}{8\pi^3}\right)^4\int \mathrm{d}\vec{q}\, V_q^3 J_3\,, \tag{B.1}$$

where

$$J_3 = \int \mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{k}_3\frac{1}{(D_1+D_2)(D_1+D_3)}\,, \tag{B.2}$$

and summation over spin is performed. Because of the integral symmetry it becomes

$$J_3 = \frac{1}{3}\int \mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{k}_3\left[\frac{1}{(D_1+D_2)(D_1+D_3)}+\frac{1}{(D_1+D_2)(D_2+D_3)}\right.$$

$$\left.+\frac{1}{(D_1+D_3)(D_2+D_3)}\right].$$

On the other hand the integral

$$I_3 = \int_{\infty}^{\infty}\mathrm{d}t_1\int_{\infty}^{\infty}\mathrm{d}t_2\int_{\infty}^{\infty}\mathrm{d}t_3\ \mathrm{e}^{-D_1|t_1|-D_2|t_2|-D_3|t_3|}\delta(t_1+t_2+t_3)$$

$$= 2\left[\frac{1}{(D_1+D_2)(D_1+D_3)}+\frac{1}{(D_1+D_2)(D_2+D_3)}\right.$$

$$\left.+\frac{1}{(D_1+D_3)(D_2+D_3)}\right]. \tag{B.3}$$

The relation (B.2) was proved earlier[6,7]. So we find

$$J_3 = \frac{1}{2\cdot 3}\int \mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{k}_3 I_3$$

$$= \frac{1}{2\cdot 3}\int \mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{k}_3\int_{\infty}^{\infty}\mathrm{d}t_1\int_{\infty}^{\infty}\mathrm{d}t_2$$

$$\int_{\infty}^{\infty}\mathrm{d}t_3\ \mathrm{e}^{-D_1|t_1|-D_2|t_2|-D_3|t_3|}\delta(t_1+t_2+t_3)\,. \tag{B.4}$$

This proves the relation (27) for $n = 3$.

The proof of Eq. (27) for $n = 4$ is more complicated. Let us transform Eq. (25) into the integral form

$$E_4 = \frac{S^4}{\Omega^4}\left(-\frac{m}{\hbar^2}\right)^2\left(\frac{\Omega}{8\pi^3}\right)^5 \int \mathrm{d}\vec{q}\, V_q^4 J_4\,, \tag{B.5}$$

where the summation over spin gives the factor $S^4$ and

$$\begin{aligned} J_4 &= \int \mathrm{d}\vec{k}_1 \ldots \mathrm{d}\vec{k}_4 \left[\frac{1}{(12)(13)(14)} + \frac{1}{(16)(13)(34)} + \frac{1}{(12)(1234)(14)}\right] \\ &= \frac{1}{4}\int \mathrm{d}\vec{k}_1 \ldots \mathrm{d}\vec{k}_4 \left\{\frac{1}{(12)(13)(14)} + \frac{1}{(12)(23)(24)} + \frac{1}{(12)(23)(34)} + \right. \\ &+ \frac{1}{(14)(24)(34)} + \frac{1}{(12)(34)(13)} + \frac{1}{(12)(34)(23)} + \frac{1}{(12)(34)(14)} \\ &+ \frac{1}{(12)(34)(24)} + \frac{1}{2}\left[\frac{1}{(12)(14)(23)} + \frac{1}{(14)(23)(24)}\right. \\ &\left.\left. + \frac{1}{(14)(23)(13)} + \frac{1}{(14)(23)(34)}\right]\right\}. \end{aligned} \tag{B.6}$$

Defining again

$$I_4 = \int\limits_{\infty}^{\infty} \mathrm{d}t_1 \ldots \int\limits_{\infty}^{\infty} \mathrm{d}t_4\ \mathrm{e}^{-D_1|t_1|\ldots-D_4|t_4|}\delta(t_1 + \ldots t_4)\,, \tag{B.7}$$

it can be proved

$$I_4 = 2 \times \{\text{wiggle bracket in (B.6)}\}\,.$$

Hence we find

$$\begin{aligned} J_4 &= \frac{1}{2\cdot 4}\int \mathrm{d}\vec{k}_1 \ldots \mathrm{d}\vec{k}_4 I_4 \\ &= \frac{1}{2\cdot 4}\int \mathrm{d}\vec{k}_1 \ldots \mathrm{d}\vec{k}_4 \int\limits_{\infty}^{\infty} \mathrm{d}t_1 \ldots \\ &\times \int\limits_{\infty}^{\infty} \mathrm{d}t_4\ \mathrm{e}^{-D_1|t_1|\ldots-D_4|t_4|}\delta(t_1 + \ldots + t_4)\,, \end{aligned} \tag{B.8}$$

and with this we complete the proof of the relation (27) for $n = 4$.

Of course a similar procedure can be performed to prove Eq. (27) for $n = 5$. In the integral form Eq. (26) reads

$$E_5 = \frac{S^5}{\Omega^5}\left(-\frac{m}{\hbar^2}\right)^4\left(\frac{\Omega}{8\pi^3}\right)^6\int \mathrm{d}\vec{q}\, V_q^5 J_5\,, \tag{B.9}$$

where

$$J_5 = \int \mathrm{d}\vec{k}_1\ldots\mathrm{d}\vec{k}_4 \times \begin{Bmatrix}\text{the expression in the wiggle}\\ \text{bracket of the relation (26)}\end{Bmatrix}.$$

A more symmetric form of this relation is

$$J_5 = \frac{1}{5}\int \mathrm{d}\vec{k}_1\ldots\mathrm{d}\vec{k}_5\left\{\frac{1}{(12)(13)(14)(15)} + \frac{1}{(12)(23)(24)(25)}\right.$$

$$+\frac{1}{(13)(23)(34)(35)} + \frac{1}{(41)(42)(43)(45)} + \frac{1}{(51)(52)(53)(54)}$$

$$+\frac{1}{(12)(13)(14)(54)} + \frac{1}{(12)(13)(15)(45)} + \frac{1}{(12)(15)(14)(34)}$$

$$+\frac{1}{(12)(23)(25)(45)} + \frac{1}{(12)(23)(24)(45)} + \frac{1}{(52)(53)(51)(24)}$$

$$+\frac{1}{(52)(54)(51)(23)} + \frac{1}{(52)(54)(53)(12)} + \frac{1}{(41)(45)(43)(12)}$$

$$+\frac{1}{(51)(54)(53)(12)} + \frac{1}{(21)(24)(25)(35)} + \frac{1}{(23)(24)(25)(15)}$$

$$+\frac{1}{(13)(14)(15)(23)} + \frac{1}{(13)(34)(35)(21)} + \frac{1}{(42)(43)(45)(12)}$$

$$+\frac{1}{(32)(34)(35)(21)} + \frac{1}{(32)(35)(31)(45)} + \frac{1}{(42)(41)(45)(23)}$$

$$+\frac{1}{(32)(34)(31)(45)} + \frac{1}{(31)(32)(34)(42)} + \frac{1}{(12)(13)(34)(45)}$$

$$+\frac{1}{(12)(13)(35)(45)} + \frac{1}{(12)(23)(34)(45)} + \frac{1}{(12)(23)(35)(45)}$$

$$+\frac{1}{(12)(15)(34)(35)}+\frac{1}{(14)(15)(23)(24)}+\frac{1}{(31)(32)(14)(45)}$$

$$+\left.\frac{1}{(13)(23)(15)(45)}\right\}. \qquad (B.10)$$

Introducing the integral

$$I_5=\int\limits_{\infty}^{\infty}\mathrm{d}t_1\ldots\int\limits_{\infty}^{\infty}\mathrm{d}t_5\ \mathrm{e}^{-D_1|t_1|\ldots-D_5|t_5|}\delta(t_1+\ldots t_5)\,,$$

and solving it, we find

$$I_5=2\times\left\{\begin{matrix}\text{the expression in the wiggle}\\ \text{bracket of the relation (B.10)}\end{matrix}\right\}.$$

After this $J_5$ becomes

$$J_5=\frac{1}{2\cdot 5}\int\mathrm{d}\vec{k}_1\ldots\mathrm{d}\vec{k}_5 I_5$$

$$=\frac{1}{2\cdot 5}\int\mathrm{d}\vec{k}_1\ldots\mathrm{d}\vec{k}_5\int\limits_{\infty}^{\infty}\mathrm{d}t_1\ldots\int\limits_{\infty}^{\infty}\mathrm{d}t_5$$

$$\times\ \mathrm{e}^{-D_1|t_1|\ldots-D_5|t_5|}\delta(t_1+\ldots+t_5)\,.$$

In this way we complete our proof for $n=5$.

## References


1) E. Feenberg, Phys. Rev. **74** (1948) 206;
2) H. Feshbach, Phys. Rev. **74** (1948) 1548;
3) P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York (1953);
4) S. Kilić and K. Ljolje, Fizika **4** (1972) 195;
5) S. Kilić and C. E. Campbell, to be published;
6) M. Gell-Mann and K. A. Brueckner, Phys. Rev. **106** (1957) 364;
7) N. H. M. March, W. H. Young and S. Sampanthar, *The Many-Body Problem in Quantum Mechanics*, Cambridge (1967);
8) S. Kilić, RADOVI LII Akademije nauka i umjetnosti Bosne i Hercegovine, (1974) 71;

9) T. J. Greytak and J. Yan, Phys. Rev. Lett. **22** (1969) 987; T. J. Greytak, R. Woerner, J. Yan and R. Benjamin, Phys. Rev. Lett. **25** (1970) 1547; W. G. Stirling, J. Phys. C **6** (1978) 1341; K. Ohbayashi, Springer Series in Solid-State Sciences, Vol. 79, *Elementary Excitations in Quantum Fluids*, Editors: K. Ohbayashi and M. Watabe, Springer-Verlag Berlin, Heidelberg (1989);

10) S. Kilić and K. Rivier, Fizika **22** (1990) 399;

11) L. W. Bruch and I. J. McGee, J. Chem. Phys. **59** (1973) 409;

12) E. Feenberg, *Theory of Quantum Fluids*, Academic, New York (1969);

13) C. W. Woo, in *Physics of Liquid and Solid Helium*, edited by K. H. Bennemann and J. B. Ketterson, Wiley, New York (1976), Pt. 1, Chap. 5;

14) C. E. Campbell, *Progress in Liquid Physics*, edited by C. A. Croxton, Wiley, New York (1978), Chap. 6;

15) J. W. Clark in *Progress in Particle and Nuclear Physics*, edited by D. H. Wilkinson, Pergamon, Oxford (1979) Vol. II;

16) E. Krotscheck, Ann. Phys. **155** (1984) 1.

# ENERGIJA OSNOVNOG STANJA VIŠEČESTINOG FERMIONSKOG SISTEMA ODREĐENA POMOĆU FEENBERGOVE PERTURBACIONE TEORIJE

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U ovom radu određen je izraz za energiju osnovnog stanja sistema od $N$ identičnih fermiona koji međudjeluju dvočestičnim potencijalom. Izraz je dobiven upotrebom Brillouin-Wigner-Feenbergova računa smetnje. Primijenjen na elektronski plin, izraz predstavlja poopćenje Gell-Mann-Brucknerove formule za gust elektronski plin. Izvršena je i jedna procjena, korekcija u odnosu na Gell-Mann-Bruecknerov izraz, za tekući $^3$He. U našem saznanju ovo je prvi rad koji učinkovito opisuje fermionske sisteme koristeći se Feenbergovim računom smetnje.