ISSN 1330–0008
CODEN FIZAE4

# MECHANICAL RESPONSE OF PIEZOQUARTZ BAR WITH HEAT INFLUX AT ONE END

TAPAS K. MUNSHI

*Department of Physics, Kharagpur College, P.O. Kharagpur, Dist. Midnapore, West Bengal, India*

KARTIK K. KUNDU

*Department of Physics, City College, Calcutta 700009, India*

and

RANGIT K. MAHALANABIS

*Department of Mathematics, Jadavpur University, Calcuta 700032, India*

Received 11 May 1993

UDC 573.226.32

Original scientific paper

An attempt has been made to investigate analitically the mechanical behaviour of an open-circuited piezoquartz bar, one end of which is subjected to some prescribed electrical and thermal excitations while the other end is kept fixed. The method of operational calculus has been utilised and the numerical results are illustrated graphically. For time-scale ranging from 0 to 1 s variations of the displacement were found to be of the order of $10^{-6}$ to $10^{-8}$ m. Significantly, some disturbances persist at $t = 0$ for linear and exponential input signal while the displacement ceases at $t = 0$ in case of periodic input signal. The nature of the graph is found to be parabolic in case of exponential input whereas a linear relationship is obtained in case of periodic and linear input signals.

## 1. Introduction

The studies of piezoelectric transducers from the standpoint of mechanics of continuous media have been initiated by Mason [1], Haskins and Walsh [2], Cady [3], Redwood [4], etc. The relevant problems are of much importance in view of their applications in the field of ultrasonics and acoustic engineering. Earlier workers like Paria [5], Redwood [4], Windlin [6], etc. discussed the situations where the two fields, viz., mechanical and electrical interact with each other. Certainly the studies become more interesting if the above interaction is coupled with a thermal field. In fact, problems in thermo-piezoelectricity have not been considered extensively. The present study is an attempt to this end and is a follow-up of the papers by Paria [5], Kundu [7], Samoilov and Shchedrina [8] and Paul and Raman [9].

The present paper deals with a problem of mechanical behaviour of an open-circuited piezoquartz bar when one end is kept fixed and the other end is subjected to some electrical and thermal excitations, viz., with the input signals

i) varying linearly with time and acting for a finite interval,

ii) varying exponentially with time and acting for a finite interval, and

iii) varying periodically with time and acting for a finite interval.

## 2. Formulation of the problem, fundamental equations and boundary conditions

We consider here an open-circuited bar of piezoelectric material at one end of which is applied an electrical voltage as well as a time dependent flux of heat. Our object is to obtain the mechanical response exhibited by the bar. We shall consider here three different input signals varying with time and acting for a finite interval, viz., linear, exponential and periodic.

Since the problem involves the interaction of three fields, viz., mechanical, electrical and thermal, we must consider an equation involving all of them. To derive such an equation, we take the relevant piezoelectric equations, as in Mindlin [6]

$$T = c\frac{\partial \Psi}{\partial x} - hD - \lambda\theta \tag{1}$$

$$E = -h\frac{\partial \Psi}{\partial x} + \beta D - \gamma\theta \tag{2}$$

where $T$, $\Psi$ and $\theta$ are stress, displacement and temperature in the $x$-direction, $c$ is the elastic stress coefficient, $h$ the piezoelectric stress constant, $\beta$ the electromechanical coupling factor, $\lambda$ the thermoelastic compliance, $\gamma$ the thermo-piezoelectric modulus, $D$ and $E$ are the electric displacement and electric field, respectively.

The equation of motion in the $x$-direction is given by

$$\varrho \frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial T}{\partial x}\,, \qquad \varrho = \text{density of piezoquartz}\,. \tag{3}$$

The electric displacement $D$ satisfies the equation

$$\mathrm{div} D = 0\,. \tag{4}$$

To obtain the equation for the displacement $\Psi$, we must make some simplifying assumptions, viz.

i) The $X$-$Z$ faces of the bar are covered with conducting electrodes so that $\partial E/\partial x = 0$.

ii) The dimension $X$ of the bar is several times larger than that of $Y$ or $Z$.

iii) The $X$-$Z$ faces of the bar are thermally insulated so that $\partial \theta/\partial x = 0$.

iv) The end at $x = X$ is rigidly fixed.

From Eq. (3), with the aid of Eq. (1) andEq. (2) and the assumptions $\partial E/\partial x = 0$ and $\partial \theta/\partial x = 0$, we obtain the wave equation

$$\frac{\partial^2 \Psi}{\partial t^2} = \alpha^2 \frac{\partial^2 \Psi}{\partial x^2} \tag{5}$$

where

$$\alpha = \frac{\beta c - h^2}{\beta \varrho}\,.$$

Now, the fundamental Eqs. (1), (2) and (5) are to be solved subject to the following boundary conditions at $x = 0$ and $x = X$:

i) the displacement is continuous, i.e. $(\overline{\Psi})_0 = (\overline{\Psi}_1)_0$

ii) the force is continuous, i.e. $(\overline{F})_0 = (\overline{F}_1)_0$ and at $x = X$, the displacement is zero, i.e.

iii) $(\overline{\Psi})_x = 0$

together with

$$\begin{array}{llll} \text{(a)} & \Phi & = \Phi_0(1 - t/\tau), & 0 \le t \le \tau \\ & & = 0 & t > \tau \\ \text{(b)} & V & = V_0(1 - t/\tau), & 0 \le t \le \tau \\ & & = 0 & t > \tau \end{array} \tag{7}$$

for the case (i), and

$$\begin{aligned} &\text{(a)} \quad \Phi = \Phi_0\, e^{\omega t}\{H(t) - H(t-\tau)\}, \qquad \tau > 0 \\ &\text{(b)} \quad V = V_0\, e^{\omega t}\{H(t) - H(t-\tau)\}, \qquad \tau > 0 \end{aligned} \tag{8}$$

for the case (ii), and

$$\begin{aligned} &\text{(a)} \quad \Phi = \Phi_0 \sin\omega t, \qquad 0 \le t \le \tau \\ &\qquad\quad\ \ = 0 \qquad\qquad\quad\ t > \tau \\ &\text{(b)} \quad V = V_0 \sin\omega t, \qquad 0 \le t \le \tau \\ &\qquad\quad\ \ = 0 \qquad\qquad\quad\ t > \tau \end{aligned} \tag{9}$$

for the case (iii).

Here $V$ and $\Phi$ represent electrical voltage and heat influx, respectively, and $H(t)$ is the Heaviside unit function defined by

$$\begin{aligned} H(t) &= 1, \qquad t > 0 \\ &= 0, \qquad t < 0\,. \end{aligned}$$

## 3. *Method of solution*

Applying Laplace transform to Eqs. (5), (7), (8) and (9), we obtain

$$\frac{d^2\overline{\Psi}}{dx^2} = \frac{p^2}{\alpha^2}\overline{\Psi} \qquad (\mathrm{Re}\, p > 0) \tag{10}$$

where $p$ is the Laplace transform parameter

$$\begin{aligned} &\text{(a)} \quad \overline{\Phi} = \frac{\Phi_0}{p}\left\{1 - \frac{1}{\tau p} + \frac{1}{\tau p} e^{-p\tau}\right\}, \\ &\text{(b)} \quad \overline{V} = \frac{V_0}{p}\left\{1 - \frac{1}{\tau p} + \frac{1}{\tau p} e^{-p\tau}\right\}, \end{aligned} \tag{11}$$

$$\begin{aligned} &\text{(a)} \quad \overline{\Phi} = \Phi_0\left\{\frac{1}{p-\omega} - \frac{e^{-(p-\omega)\tau}}{p-\omega}\right\}, \\ &\text{(b)} \quad \overline{V} = V_0\left\{\frac{1}{p-\omega} - \frac{e^{-(p-\omega)\tau}}{p-\omega}\right\}, \end{aligned} \tag{12}$$

$$\begin{aligned} &\text{(a)} \quad \overline{\Phi} = \frac{\Phi_0}{p^2+\omega^2}\left\{\omega - p\, e^{-p\tau}\sin\omega\tau - \omega\, e^{-p\tau}\cos\omega\tau\right\}, \\ &\text{(b)} \quad \overline{V} = \frac{V_0}{p^2+\omega^2}\left\{\omega - p\, e^{-p\tau}\sin\omega\tau - \omega\, e^{-p\tau}\cos\omega\tau\right\}. \end{aligned} \tag{13}$$

The solution of Eq. (10) is given by

$$\overline{\Psi}(x,p) = A(p)\, e^{-px/\alpha} + B(p)\, e^{+px/\alpha} \tag{14}$$

where $A(p)$ and $B(p)$ are constants to be determined from the boundary conditions Eq. (6).

We may mention here that a wave equation of the form of Eq. (5) is satisfied even if the material is non-piezoelectric but with a different wave velocity. Therefore, we assume, after Redwood [4], that two mechanical systems labelled 1 and 2 are attached to the two extremities of the bar $x = 0$ and $x = X$. The displacements in those materials will be similar to Eq. (14) with different values of $A$ and $B$, say $A_1$, $B_1$ and $A_2$, $B_2$ in materials 1 and 2, respectively.

To develop next the relation between electrical, mechanical and thermal quantities, we put as in Paria [5]

$$F = TYZ\,, \qquad V = EY\,, \qquad \Phi = \frac{K\theta}{Y}$$

where $F$, $V$, $\Phi$ represent the mechanical force, electrical voltage and heat influx, respectively, and $K$ denotes the constant of diffusion. With these substitutions we obtain the required relation from Eqs. (1) and (2):

$$\overline{F} = p\alpha^2 YZ\varrho\left(-A\mathrm{e}^{-px/\alpha} + B\mathrm{e}^{+px/\alpha}\right) - \frac{hZ}{\beta}\,\overline{V} - \left(\lambda + \frac{h\nu}{\beta}\right)\frac{Y^2 Z}{K}\,\overline{\Phi}\,. \tag{15}$$

### *3.1. Displacement for the case (i) (the input signals varying linearly with time acting for a finite interval)*

From Eqs. (6), (11), (14) and (15) we obtain the following relations

$$B_1 = A + B \tag{16}$$

$$\begin{aligned} &p\alpha^2 YZ\varrho(-A+B) - \frac{hZ}{\beta}\frac{V_0}{p}\left\{1 - \frac{1}{\tau p} + \frac{1}{\tau p}\mathrm{e}^{-p\tau}\right\} - \\ &-\left(\lambda + \frac{h\nu}{\beta}\right)\frac{Y^2 Z}{K}\frac{\Phi_0}{p}\left\{1 - \frac{1}{\tau p} + \frac{1}{\tau p}\mathrm{e}^{-p\tau}\right\} = p\alpha_1^2 Y_1 Z_1 \varrho\, B_1 \end{aligned} \tag{17}$$

$$A\mathrm{e}^{-pX/\alpha} + B\mathrm{e}^{+pX/\alpha} = 0\,. \tag{18}$$

Solving Eqs. (16), (17) and (18) we get the values of $A$ and $B$ as

$$A = \mathrm{e}^{pX/\alpha}\mu\left\{\left(\frac{1}{p} - \frac{1}{\tau p^2} + \frac{1}{\tau p^2}\mathrm{e}^{-p\tau}\right)\frac{1}{p}\right\}\left\{\varrho\left(c_2\mathrm{e}^{-pX/\alpha} - c_1\mathrm{e}^{pX/\alpha}\right)\right\}^{-1}$$

$$B = \mathrm{e}^{-pX/\alpha}\mu\left\{\frac{1}{p} - \frac{1}{\tau p^2} + \frac{1}{\tau p^2}\mathrm{e}^{-p\tau}\right\}\frac{1}{p}\left\{\varrho\left(c_1\mathrm{e}^{pX/\alpha} - c_2\mathrm{e}^{-pX/\alpha}\right)\right\}^{-1}$$

where

$$\frac{c_2}{c_1} = \frac{\alpha_1^2 Y_1 Z_1 - \alpha^2 Y Z}{\alpha_1^2 Y_1 Z_1 + \alpha^2 Y Z} = c_0 \quad \text{say} \tag{19}$$

and

$$\mu = \frac{hZ}{\beta} V_0 + \left(\lambda + \frac{h\nu}{\beta}\right) \frac{Y^2 Z}{K} \Phi_0 \, .$$

Substituting the values of $A$ and $B$ in Eq. (14) we obtain

$$(\overline{\Psi})_0 = -\frac{\mu}{c_1 \varrho} \frac{1}{p} \left\{ \frac{1}{p} - \frac{1}{\tau p^2} + \frac{1}{\tau p^2} \mathrm{e}^{-p\tau} \right\} (1 - c_0 \, \mathrm{e}^{-2pX/\alpha})^{-1} (1 - \mathrm{e}^{-2pX/\alpha}) \, . \tag{20}$$

The inverse transform of Eq. (20) is given by

$$(\Psi)_0 = -\frac{\mu}{c_1 \varrho} \frac{2X}{\alpha} (1 + c_0) \left\{ 1 - \frac{t}{\tau} + \frac{1}{\tau}(t - \tau) H(t - \tau) \right\} +$$
$$+ \frac{\mu}{c_1 \varrho} c_0 \frac{4X^2}{\alpha^2} \left\{ -\frac{1}{\tau} + \frac{1}{\tau} H(t - \tau) \right\} \, . \tag{21}$$

### *3.2. Displacement for the case (ii) (the input signals varying exponentially with time acting for a finite interval)*

Solving Eqs. (6), (12), (14) and (15) we obtain (after taking the inverse transformation)

$$(\Psi)_0 = -\frac{\mu}{c_1 \varrho} \frac{2X}{\alpha} (1 + c_0) \left\{ \mathrm{e}^{\omega t} - \mathrm{e}^{\omega t} H(t - \tau) \right\} +$$
$$+ \frac{\mu}{c_1 \varrho} c_0 \frac{4X^2}{\alpha^2} \left\{ \omega \mathrm{e}^{\omega t} - \mathrm{e}^{\omega t} \, \delta(t - \tau) - \omega \mathrm{e}^{\omega t} H(t - \tau) \right\} \, . \tag{22}$$

### *3.3. Displacement for the case (iii) (the input signals varying periodically with time acting for a finite interval)*

Solving Eqs. (6), (13), (14) and (15) we obtain (after taking the inverse transformation)

$$(\Psi)_0 = -\frac{\mu}{c_1 \varrho} \frac{2X}{\alpha} (1 + c_0) \{ \sin \omega t - \cos \omega (t - \tau) H(t - \tau) \sin \omega \tau -$$
$$- H(t - \tau) \sin \omega (t - \tau) \cos \omega \tau \} + \frac{\mu}{c_1 \varrho} c_0 \frac{4X^2}{\alpha^2} [\omega \cos \omega t - \sin \omega \tau \{ \delta(t - \tau) -$$
$$- \omega \sin \omega (t - \tau) H(t - \tau) \} - \cos \omega \tau \{ \omega \cos \omega (t - \tau) H(t - \tau) \} ] \, . \tag{23}$$

Eqs. (21), (22), (23) show the response of the bar under three different time-dependent input signals.

For numerical computations, the standard values of the material constants have been taken from Cady [3], Gibbs [10], Jin-Feng Wang et al. [11], Sasaki [12] and Imano [13], while values like $Y$, $Z$, $X$, $\omega$, $V_0$, $\Phi_0$, $\tau$ have been chosen suitably to facilitate the numerical computations as follows:

$\Phi_0 = 4.18 \cdot 10^3$ J  $Y = Z = 0.01$ m
$V_0 = 300$ V  $X = 0.1$ m
$\omega = 1.57$ rad/s  $\tau = 1$ s.

The mechanical response of a piezoquartz bar for various input signals corresponding to a time interval from $t = 0$ to $t = 1$ s have been shown in Table 1.

Table 1. Variation of the mechanical responses of a piezoquartz bar for three different inputs with time.

| Time (s) | For linear input signal $(\Psi_0) \times 10^{-7}$ m | For exponential input signal $(\Psi_0) \times 10^{-6}$ m | For periodic input signal $(\Psi_0) \times 10^{-8}$ m |
|---|---|---|---|
| $t = 0$ | –5.96 | –0.59 | 0.0 |
| $t = 0.1$ | –5.36 | –0.69 | –0.16 |
| $t = 0.2$ | –4.76 | –0.81 | –0.32 |
| $t = 0.3$ | –4.17 | –0.95 | –0.48 |
| $t = 0.4$ | –3.57 | –1.11 | –0.65 |
| $t = 0.5$ | –2.98 | –1.30 | –0.81 |
| $t = 0.6$ | –2.39 | –1.52 | –0.97 |
| $t = 0.7$ | –1.78 | –1.78 | –1.14 |
| $t = 0.8$ | –1.19 | –2.09 | –1.30 |
| $t = 0.9$ | –0.59 | –2.44 | –1.40 |
| $t = 1.0$ | –0.0 | –2.86 | –1.63 |

## 4. *Discussion*

The response given by Eqs. (21), (22) and (23) have been illustrated in Figs. 1, 2 and 3, respectively, for a piezoquartz bar. It is observed that for thermal and electrical input signals which vary linearly with time, the disturbance is linear (Fig. 1). Again, for input signals which vary exponentially with time the response shows an arc of a parabola (Fig. 2). Further, under periodic input signals, the disturbance gives out a linear relationship (Fig. 3).

It is interesting to note that in Figs. 1 and 2 some disturbances persist at $t = 0$ while in Fig. 3 the disturbance ceases at $t = 0$. This treatment is valid only within investigated range of time.

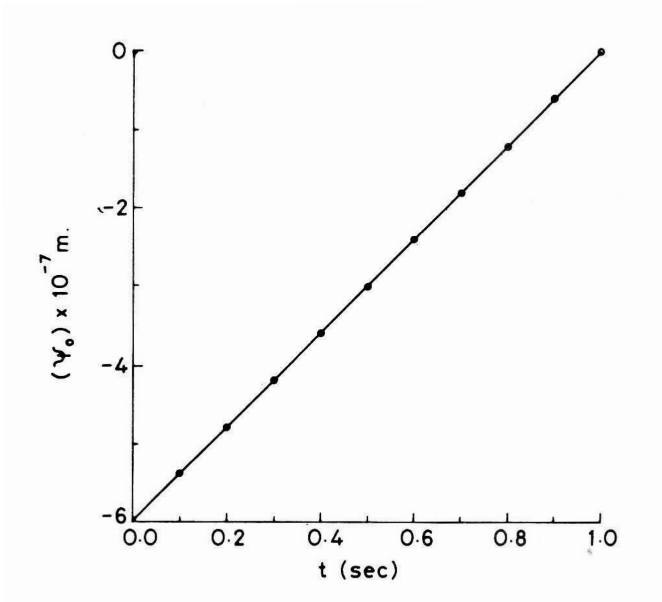


*Fig. 1. The response of a piezoquartz bar under input signal varying linearly with time and acting for a finite interval.*

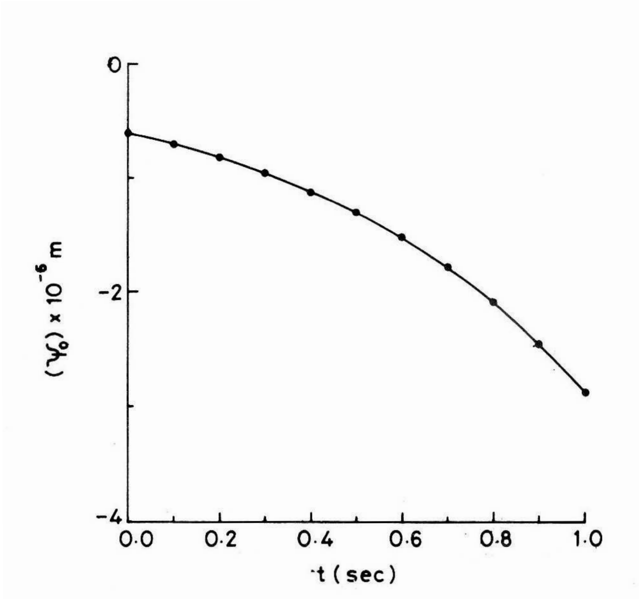


*Fig. 2. The response of a piezoquartz bar under input signal varying exponentially with time and acting for a finite interval.*

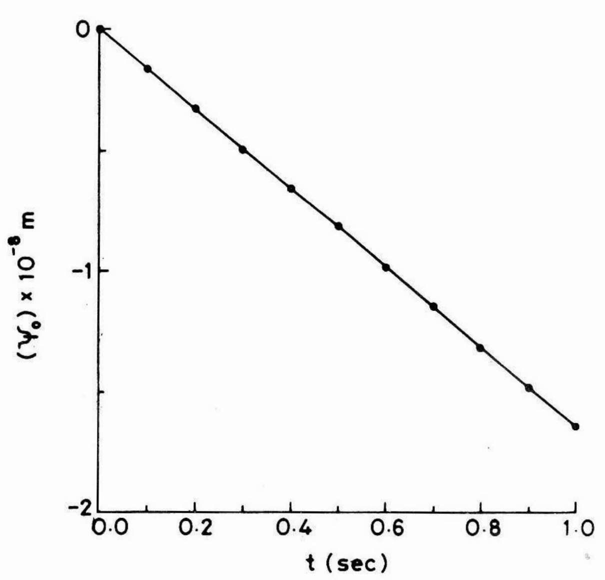


*Fig. 3. The response of a piezoquartz bar under input signal varying periodically with time and acting for a finite interval.*

The mechanical response of a piezoquartz bar for various input signals corresponding to a time interval from $t = 0$ to $t = 1$ s have been shown in Table 1.

References

1) W. P. Mason, *Piezoelectric Crystals and Their Applications to Ultrasonics*, D. Van Nostrand Co. Inc. New York **84** (1950);

2) F. J. Haskins and L. J. Walsh, J. Acoust. Soc. Am. **29** (1951) 729;

3) W. G. Cady, *Piezoelectricity*, Mc Graw-Hill, New York, (1959);

4) M. Redwood, J. Acoust. Soc. Am. **33** (1961) 527;

5) G. Paria, Jour. Sc. Eng. Res. **4** (1960) 381;

6) R. D. Mindlin, *On the Equation of Motion of Piezoelectric Crystals – Problems on Continuum Mechanics* (Society for Industrial and Applied Mathematics, Pensylvania) **282** (1961);

7) K. K. Kundu, Indian J. Tech. **19** (1981) 504;

8) V. B. Samoilov and L. V. Shchedrina, Ferroelectrics (UK) **118** (1991) 225;

9) H. S. Paul and G. V. Raman, J. Acoust. Soc. Am. (USA) **90** (1991) 1729;

10) D. F. Gibbs, Ferroelectrics (UK) **82** (1988) 133;

11) Jin-Feng Wang, Chen Chen, Lei Zhang and Zi-Kai Qin, Phys. Rev. B. **39** (1989) 12888;

12) Y. Sasaki and Y. Takeuchi, J. Acoust. Soc. Jpn. (E) Japan **10** (1989) 181;

13) K. Imano and D. Okuyama, Trans. Inst. Electron. Inf. Commun. Eng. A (Japan) J **74** A (1991) 797.

# MEHANIČKI ODZIV PIEZOKVARCNOG ŠTAPA S TOPLINSKIM DOTOKOM NA JEDNOM KRAJU

TAPAS K. MUNSHI

*Department of Physics, Kharagpur College, P.O. Kharagpur, Dist. Midnapore, West Bengal, India*

KARTIK K. KUNDU

*Department of Physics, City College, Calcutta 700009, India*

i

RANGIT K. MAHALANABIS

*Department of Mathematics, Jadavpur University, Calcuta 700032, India*

UDK 573.226.32

Originalni znanstveni rad

Analitički su istražena mehanička svojstva piezoelektričnog kremenog štapa koji nije u električnom krugu i kojemu se jedan kraj drži pod stalnim uvjetima a drugi je podvrgnut izvjesnoj električnoj i toplinskoj uzbudi. Primjenjena je metoda operacijske analize, a numerički rezultati su predočeni grafički. Za vremenske intervale do 1 s nalaze se pomaci od $10^{-6}$ do $10^{-8}$ m. Važno je istaći da neki učinci uzbude ostaju u $t = 0$ za linearne i eksponencijalne ulazne signale, dok pomaci nestaju u $t = 0$ u slučaju periodičkih ulaznih signala. Nalazi se parabolička ovisnost učinaka u slučaju eksponencijalnog ulaznog signala, a linearna ovisnost u slučaju periodičkih i linearnih ulaznih signala.