ISSN 1330–0008
CODEN FIZAE4

A RIEMANN–HILBERT APPROACH TO POLARIZED SOLITONS IN A THREE–LEVEL MEDIUM

SUKANYA BURMAN and ASESH ROY CHOWDHURY

High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta 700 032, India

Received 18 November 1996

UDC 538.958

PACS 42.50.-p

Solitons due to the interaction of polarized waves in optically nonlinear media have been analysed using the Riemann–Hilbert transform. We consider the propagation and interaction of differently polarized two–frequency ultrashort optical pulses in a resonant medium consisting of three–level particles. The exact form of the solitons is obtained explicitly.

1. Introduction

Nonlinear optics is one of the most important fields of physics where solitons play a dominant role. Such solitons have been observed experimentally. Elaborate papers have been published by several authors: Basharov and Maimistov [1], Steudel [2] and Kaup [3], who analysed the problem through inverse scattering technique. Later, Roy Chowdhury and De [4,5] showed how the same problem or its generalization can be more elegantly treated using the Riemann–Hilbert approach [6]. In general, there are two classes of problems which deserve attention. One is the propagation of optical pulses in a fibre – the fibre optics. This is of prime importance in the present day scenario due to the introduction of new modes in communication technology. In general, the propagation of a nonlinear wave in an optical fibre is described by the so called unstable nonlinear Schrödinger equation. Such an equation was originally deduced by Wadati et al. [7] and subsequently studied by many authors, e.g. Mukhopadhyay and Roy Chowdhury [8], Fuchssteiner and Konopelochen [9] and others. On the other hand, a separate class of problems involves the study of the interaction of such an optical pulse during its propagation in a liquid medium. When an ultrashort optical pulse propagates in a liquid or gaseous medium, some important phenomena take place such as self induced transparency [10], simulated Raman scattering [11], simulated Brillouin scattering [12], etc. Our motivation was to analyse the latter problem. Since the experiments are always performed within a finite boundary, it is interesting and encouraging to dispense with the idea of scattering [13]. In this communication, we treat the situation when a polarized two–frequency ultrashort optical pulse interacts with a three–level medium. The degeneracy of the medium is taken into account, which is typical for real media such as gases, and is necesarry for the description of the real situation. The equations under consideration were formulated by Basharov and Maimistov [14].

2. Formulation

The generalized Maxwell–Bloch equations, including arbitrary polarization, are considered in the case of equal resonance absorption lengths $\Lambda(E_b > E_c > E_a)$ and $\nu(E_c > E_a > E_b)$ at the two frequencies, where E_b is the three–fold degenerate level and non–degenerate levels are E_a and E_c. We consider the collinear propagation of two–frequency ultrashort pulses with electric fields;

$$\overline{E}_j = E_j \exp\left(\mathrm{i}(k_j z - \omega_j t)\right) + c.c., \qquad j = 1,2$$

under the condition of double resonance

$$\omega_1 \approx \frac{|E_b - E_a|}{\hbar}, \qquad \omega_2 \approx \frac{|E_b - E_c|}{\hbar}.$$

The states of the atom in the medium are described by the density matrices $\varphi^a_{mm'}$, $\varphi^b_{uu'}$ and $\varphi^c_{vv'}$ which characterize the states of the atom in the corresponding energy levels. The equations can then be written as;

$$\frac{\partial \varepsilon^q_j}{\partial \varphi} = -\mathrm{i} p^q_j,$$

$$\frac{\partial \eta_j}{\partial \tau} = -\mathrm{i}\left(\sum_q \varepsilon^q_j p^{q^*}_j - \varepsilon^{q^*}_j p^q_j\right), \qquad j = 1,2,$$

$$\left(\frac{\partial}{\partial \tau} - \mathrm{i}\Delta\right) p^q_1 = -\mathrm{i}\left(\sum_{q'} \varepsilon^{q'}_1 m_{q'q} - \varepsilon^q_1 n_1 - \varepsilon^q_2 r\right),$$

$$\left(\frac{\partial}{\partial \tau} - \mathrm{i}\Delta\right) p^q_2 = -\mathrm{i}\sum_{q'} (\varepsilon^{q'}_2 m_{q'q} - \varepsilon^q_2 n_2 - \varepsilon^q_1 r^*),$$

$$\frac{\partial}{\partial\tau} m_{qq'} = -\mathrm{i} \sum_{j=1,2} (\varepsilon_j^{q^*} p_j^{q'} - p_j^{q^*} \varepsilon_j^{q'}),$$

$$\frac{\partial}{\partial\tau} r = -\mathrm{i} \sum_q (\varepsilon_1^q p_2^{q^*} - \varepsilon_2^{q^*} p_1^q), \qquad (1)$$

which describe both the $v-$ and $\Lambda-$configuration defined by the following relations:

v–configuration

$$\varepsilon_1 = \frac{E_1 t_0 d_{ab}}{\sqrt{3}\hbar},$$

$$\varepsilon_2 = \frac{E_2 t_0 d_{cb}}{\sqrt{3}\hbar}.$$

ε_1 and ε_2 are the two components of the electric field, and

$$p_1^q = \frac{\varphi_{0q}^{(ab)}}{N_0}, \quad p_2^q = \frac{\tilde{\varphi}_{0q}^{(cb)}}{N_0}, \quad r = \frac{\tilde{\varphi}_{00}^{(ab)}}{N_0},$$

$$n_1 = \frac{\varphi_{00}^{(a)}}{N_0}, \quad n_2 = \frac{\varphi_{00}^{(c)}}{N_0}, \quad m_{qq'} = \frac{\varphi_{qq'}^{(b)}}{N_0}, \quad N_0 = \frac{N_b}{3}.$$

Λ–configuration

$$\varepsilon_1 = \frac{E_1 t_0 d_{ba}}{\sqrt{3}\hbar}, \quad \varepsilon_2 = \frac{E_2 t_0 d_{bc}}{\sqrt{3}\hbar},$$

$$p_1^q = \frac{\tilde{\varphi}_{-q0}^{(ba)}}{N_0}, \quad p_2^q = \frac{\tilde{\varphi}_{-q0}^{(bc)}}{N_0}, \quad r = -\frac{\tilde{\varphi}_{00}^{(ca)}}{N_0},$$

$$n_1 = -\frac{\varphi_{00}^{(a)}}{N_0}, \quad n_2 = -\frac{\varphi_{00}^{(c)}}{N_0}, \quad m_{qq'} = -\frac{\varphi_{-q'-q}^{(b)}}{N_0}, \quad N_0 = N_a.$$

Here $t_0 = 2\pi\omega_1 (d_{ab})^2 N_0/3\hbar)^{-1/2}$ is a constant with the dimension of time, N_a and N_b are the densities of atoms populating the lower level in the $v(\Lambda)$-configuration, indices q and

q' can have values $\neq 1$, and d_{ab} represent dipole moments. The Lax pair of equations for the system (1) can be written as:

$$\frac{\partial}{\partial\tau}(\psi) = \begin{pmatrix} -\mathrm{i}\lambda & 0 & -\mathrm{i}\varepsilon_1^{-1} & -\mathrm{i}\varepsilon_2^{-1} \\ 0 & -\mathrm{i}\lambda & -\mathrm{i}\varepsilon_1^{1} & -\mathrm{i}\varepsilon_2^{1} \\ -\mathrm{i}\varepsilon_1^{-1*} & -\mathrm{i}\varepsilon_1^{1*} & \mathrm{i}\lambda & 0 \\ -\mathrm{i}\varepsilon_2^{-1*} & -\mathrm{i}\varepsilon_2^{1*} & 0 & \mathrm{i}\lambda \end{pmatrix}(\psi), \tag{2a}$$

$$\frac{\partial}{\partial\varphi}(\psi) = \frac{\mathrm{i}}{2\lambda+\Delta}\begin{pmatrix} -m_{-1-1} & -m_{1-1} & p_1^{-1} & p_2^{-1} \\ -m_{-11} & -m_{11} & p_1^{1} & p_2^{1} \\ p_1^{-1*} & p_1^{1*} & -n_1 & -r^* \\ p_2^{-1*} & p_2^{1*} & -r & -n_2 \end{pmatrix}(\psi) \tag{2b}$$

where ψ is the Lax eigenfunction, a column vector with four components. In the following we will develop the Riemann–Hilbert approach to explore the structure of the nonlinear fields p_1, q_1, ε_1 etc.

3. Riemann–Hilbert approach

In the Riemann–Hilbert approach [6], one starts from a seed solution of the inverse problem, which is actually a trivial solution of the given nonlinear problem. One then assumes that ψ is an analytic function in the complex λ–plane with boundary values ψ_+ and ψ_-, respectively, in the inside and outside domain of a contour in the same λ–plane, satisfying the relation:

$$\psi_+\psi_- = G(\lambda), \tag{3}$$

where G is a given matrix function. The problem becomes further simplified if G is taken to be an unit matrix. With the assumption that ψ undergoes a transformation of the type

$$\psi = \chi\psi_0, \tag{4}$$

where ψ_0 is the solution of the Lax equation corresponding to seed solution, we get

$$u = \chi\sigma\chi^{-1} + \chi u_0\chi^{-1}, \tag{5}$$

u standing for the matrix on the right–hand side of Eq. (2a) and u_0 its value for the seed solution. One now assumes that χ has a simple pole

$$\chi = 1 + \frac{R}{\lambda-\lambda_1}, \tag{6}$$

$$\chi^{-1} = 1 + \frac{S}{\lambda-\mu_1}.$$

$\chi\chi^{-1} = 1$ then entails

$$R = -S = (\lambda_1 - \mu_1)P; \qquad P^2 = P. \tag{6a}$$

Using this form of χ in Eq. (5), and equating residue at the pole, we get

$$u = u_0 + [S, A], \tag{7}$$

where

$$A = \begin{pmatrix} -\mathrm{i}\lambda & 0 & 0 & 0 \\ 0 & -\mathrm{i}\lambda & 0 & 0 \\ 0 & 0 & \mathrm{i}\lambda & 0 \\ 0 & 0 & 0 & \mathrm{i}\lambda \end{pmatrix}.$$

Equation (7) clearly shows that, starting from u_0, we can reconstruct the general u and hence the nonlinear field as function of x and t. In the particular case under consideration, we choose the seed solution to be:

$$\varepsilon_1^{\pm 1} = \varepsilon_2^{\pm 1} = \varepsilon_1^{\pm 1*} = \varepsilon_2^{\pm 1*} = 0,$$

and

$$m_{-1-1} = -\alpha,\ m_{11} = -\beta,\ n_1 = -\gamma,\ n_2 = -\delta,$$

α, β, γ and δ) being constants, and the rest of the fields being zero. For this simple set of values, it is possible to solve the Lax pair and obtain the solution,

$$|\psi_0 >= g_0, \tag{8}$$

where g_0 is the column vector,

$$g_0 = \left(\mathrm{e}^{-\mathrm{i}x_1(\lambda)}, \mathrm{e}^{-\mathrm{i}x_2(\lambda)}, \mathrm{e}^{-\mathrm{i}y_1(\lambda)}, \mathrm{e}^{-\mathrm{i}y_2(\lambda)}\right)^{\mathrm{t}}.$$

Here t stands for transpose, and

$$\begin{aligned} x_1(\lambda) &= \lambda\tau + \frac{\alpha\sigma}{2\lambda + \Delta}, \\ x_2(\lambda) &= \lambda\tau + \frac{\beta\sigma}{2\lambda + \Delta}, \\ y_1(\lambda) &= \lambda\tau - \frac{\gamma\sigma}{2\lambda + \Delta}, \\ y_2(\lambda) &= \lambda\tau - \frac{\delta\sigma}{2\lambda + \Delta}. \end{aligned} \tag{9}$$

The projection operator P in Eq. (6a) can be constructed with $|\psi_0>$ as:

$$P = \frac{|\psi_0><\psi_0|}{<\psi_0|\psi_0>}\ , \tag{10}$$

which in turn yields S, and when this is used in Eq. (7), we are led to the new nonlinear field. Since the algebraic expressions become very complicated in the general case, we have considered the special situation when $\beta = \alpha$, $\delta = \gamma$, ($\mu = -\lambda$, $\alpha = -\gamma$). For example, we obtain

$$p_1^{-1} = \frac{1}{2}\exp\left(\frac{2\mathrm{i}\gamma\sigma\Delta}{\Delta^2 - 4\lambda^2}\right)\mathrm{sech}\left(2\mathrm{i}\lambda\tau + \frac{4\mathrm{i}\gamma\sigma\lambda}{\Delta^2 - 4\lambda^2}\right), \tag{11}$$

which is the form of a one–soliton solution. It is a complex function since the p_1^{-1} field is complex.

4. Two–soliton case

To obtain multisoliton solution, we set

$$\chi = 1 + \frac{R_1}{\lambda - \lambda_1} + \frac{R_2}{\lambda - \lambda_2},$$

$$\chi^{-1} = 1 + \frac{S_1}{\lambda - \mu_1} + \frac{S_2}{\lambda - \mu_2}. \tag{12}$$

The condition $\chi\chi^{-1} = 1$ leads to

$$S_1 + \frac{S_1R_1}{\lambda_1 - \mu_1} + \frac{S_1R_2}{\lambda_1 - \mu_2} = 0,$$

$$R_1 - \frac{S_1R_1}{\lambda_1 - \mu_1} - \frac{S_2R_1}{\lambda_2 - \mu_1} = 0,$$

$$S_2 + \frac{S_2R_1}{\lambda_2 - \mu_1} + \frac{S_2R_2}{\lambda_2 - \mu_2} = 0, \tag{13}$$

$$R_2 - \frac{S_1R_2}{\lambda_1 - \mu_2} - \frac{S_2R_2}{\lambda_2 - \mu_2} = 0.$$

To obtain a solution, we assume that each of the matrices S_1, R_1, etc. can be written as the product of two four–component vectors

$$(S_1)_{ij} = p_i q_j, \quad (S_2)_{ij} = r_i t_j,$$

$$(R_1)_{ij} = x_i y_j, \quad (R_2)_{ij} = k_i z_j, \tag{14}$$

where the vectors x_i, q_i, k_i and t_i are constructed with the seed solution $\psi_0(\lambda)$ and some constant vectors a_i, $\overline{a}_i$, b_i, $\overline{b}_i$ as follows

$$x_i = \psi_0(\mu_1)a_i, \quad k_i = \psi_0(\mu_2)b_i,$$

$$q_j = \overline{a}_j\overline{\psi}_0(\lambda_1), \quad t_j = \overline{b}_j\overline{\psi}_0(\lambda_2). \tag{15}$$

Then Eq. (13) can be explicitly solved for y_i, z_i, p_i and r_i

$$y_j = \frac{(\lambda-\mu_1)(\lambda_2-\mu_1)}{N}\left[(\lambda_2-\mu_2)t_j\sigma-(\lambda_1-\mu_2)q_j\beta\right],$$

$$z_j = \frac{(\lambda_2-\mu_2)(\lambda_1-\mu_2)}{\overline{N}}\left[(\lambda_2-\mu_1)t_j\alpha-(\lambda_1-\mu_1)q_j r\right],$$

$$p_i = \frac{(\lambda_1-\mu_2)(\lambda_1-\mu_1)}{M}\left[(\lambda_2-\mu_2)k_i\gamma-(\lambda_2-\mu_1)x_i\beta\right],$$

$$r_i = \frac{(\lambda_2-\mu_2)(\lambda_2-\mu_1)}{\overline{M}}\left[(\lambda_1-\mu_2)k_i\alpha-(\lambda_1-\mu_1)x_i\sigma\right], \tag{16}$$

where

$$M = (\lambda_1-\mu_1)(\lambda_2-\mu_2)\sigma\gamma-(\lambda_1-\mu_2)(\lambda_2-\mu_1)\alpha\beta,$$

$$\overline{M} = (\lambda_2-\mu_1)(\lambda_1-\mu_2)\alpha\beta-(\lambda_2-\mu_2)(\lambda_1-\mu_1)\sigma\gamma,$$

$$\overline{N} = (\lambda_2-\mu_2)(\lambda_1-\mu_2)\alpha\beta-(\lambda_1-\mu_1)(\lambda_2-\mu_2)\sigma\gamma,$$

with

$$\alpha = \sum q_k x_k, \quad \beta = \sum t_k t_k,$$

$$\gamma = \sum t_k x_k, \quad \sigma = \sum q_k k_k.$$

Going back to Eq. (5) and repeating the computation with the two pole structure, we get

$$\overline{u}_{ij} = u_{ij} + [S_1, A]_{ij} + [S_2, A]_{ij}. \tag{17}$$

For example, for the element of Eq. (13), we get

$$\overline{u}_{13} = u^0_{13} + [S_1, A]_{13} + [S_2, A]_{13}. \tag{18}$$

$$\mathrm{i}\varepsilon^{-1} = \left(\overline{\lambda}_1-\overline{\mu}_2\right)\left(\overline{\lambda}_2-\overline{\mu}_1\right)\frac{N_1}{D}-\left(\overline{\lambda}_1-\overline{\mu}_1\right)\left(\overline{\lambda}_2-\overline{\mu}_2\right)\frac{N_2}{D},$$

where

$$N_1 = \overline{a}_3\overline{b}_1 A_1 \cosh z_{22} + \overline{b}_3\overline{a}_1 \cosh z_{11},$$

$$N_2 = \overline{a}_3\overline{b}_1 B_{12} \cosh z_{12} + \overline{b}_3\overline{a}_1 \cosh z_{21},$$

$$B_{jk} = (\lambda_j - \mu_k)\exp(x_j + y_k), \tag{19}$$

$$A_l = (\overline{\lambda}_l - \overline{\mu}_l)\exp(x_l + y_l),$$

$$x_l = \lambda_l \tau + \frac{\mathrm{i}\gamma\sigma}{2\mathrm{i}\overline{\lambda}_l + \Delta}, \quad y_k = \overline{\mu}_k \tau - \frac{\mathrm{i}\alpha\sigma}{2\mathrm{i}\overline{\mu}_k + \Delta},$$

$$z_{ij} = \left(\overline{\mu}_i - \overline{\lambda}_j\right)\tau + \frac{2\alpha\sigma\left(\overline{\lambda}_j - \overline{\mu}_i\right)}{(2\mathrm{i}\overline{\lambda}_j + \Delta)(2\mathrm{i}\overline{\mu}_i + \Delta)},$$

giving the two–soliton solution. Similar expressions can also be derived for other field variables.

5. Discussion

In the present situation, the Lax pair of equations is given by 4×4 matrices. In general, it becomes very cumbersome to formulate the inverse problem for any Lax equation whose matrix is 2×2 or larger. Also, due to the importance of the finite boundary, it is not advisable to use the language of the scattering theory. In this study, we have shown that the methodology of Riemann-Hilbert problem can be used to study a 4×4 Lax problem which circumvents both these difficulties. In fact, it is possible to construct even the N-soliton solutions in a systematic manner. One may note that the previous treatment of a similar problem considered only a two-level medium, while our analysis extends to the three-level situation, and the degeneracy of the levels is also taken into account. The two soliton solutions can be used to study the interaction of solitons. Such phenomena are of utmost importance in nonlinear optics in general and have found many practical applications.

References

1) A. M. Basharov and A. I. Maimistov, Sov. Phys. JEPT **60** (1984) 913;

2) H. Steudel, Physica **D6** (1983) 155;

3) D. J. Kaup, Physica **D6** (1983) 143;

4) A. Roy Chowdhury and M. De, Inverse Problems **4** (1988) 901;

5) A. Roy Chowdhury and M. De, Aust. J. Phys. **41** (1988) 735;

6) A. Roy Chowdhury and M. De, J. Phys. Soc. (Japan) **58** (1989) 817;

7) M. Wadati, T. Iizuka and Yajima, Physica **D51** (1991) 388;

8) I. Mukhopadhyay and A. Roy Chowdhury, to be published in Chaos, Solitons, Fractals;

9) B. Fuchssteiner and Konopelochen, Padeborn preprint 1996;

10) S. L. McCall and F. L. Hahn, Phys. Rev. Letters **18** (1967) 908;

11) A. C. Newell, in *Solitons, Topisc in Current Physics*, Vol. 17, eds R. K. Bullogh and P. J. Caudrey, Springer Verlag, Berlin, p. 177;

12) G. L. Lamb, *Elements in Soliton Theory*, Wiley, New York, 1980;

13) G. L. Lamb, *Elements in Soliton Theory*, Wiley, New York, 1980;

14) S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakhorov, *Theory of Solitons*, Consultants Bureau, New York, 1984;

15) A. M. Basharov and A. I. Maimistov, Sov. Phys. JETP **67** (1988) 2426.

RIEMANN–HILBERTOV PRISTUP POLARIZIRANIM SOLITONIMA U SREDSTVU S TRI NIVOA

Uporabom Riemann–Hilbertove transformacije proučavaju se solitoni nastali međudjelovanjem polariziranih valova u optički nelinearnom sredstvu. Određena su egzaktna solitonska rješenja za širenje i međudjelovanje dvo–frekventnih ultrakratkih optičkih pulseva različitih polarizacija u rezonantnom sredstvu sa česticama na tri nivoa.