ISSN 1330-3651 (Print), ISSN 1848-6339 (Online)
https://doi.org/10.17559/TV-20220609025146
Original scientific paper

An Optimization Approach for pricing of Discrete European Call options Based on the Preference of Investors

Weiwei WANG*, Guoqing GU, Fameng SUN, Xiaoping HU

Abstract: Firstly, a method for measuring the risk aversion of investors was proposed based on the prospect theory. Secondly, under a sole hypothetical condition in which the risk aversion degree for different assets is the same in a market, the pricing of discrete European options was given based on the objective probability. Thirdly, it was proven that the European option price obtained was a non-arbitrate price. And then, both for the binomial tree, which is a complete market, and for the trinomial tree, which is an incomplete market, pricing European options were discussed by implementing the method provided in this paper. Lastly, an illustration is used to demonstrate how to estimate preference parameters from market data and how to calculate options prices. The result states that the method in this paper is the same as the traditional risk-neutral methods in a complete market, but it is different from the traditional risk-neutral methods in an incomplete market, and more, the price obtained in this paper is affected by the objective probability and also contains the risk attitude of the investors.

Keywords: discrete European option; pricing; risk aversion; objective probability

1 INTRODUCTION

The expression was generated by Black & Scholes (1973) [1] using Dynamic replication and the Delta hedge method, which is equivalent to the Risk-neutral pricing method. Risk-neutral was a basic assumption in studying pricing options. Moriggia & Muzzioli (2009) [2] considering the positions of points in a tree and the relationships between dividends and the risk-free rate of return, provided the option implied tree method which is based on the inductive method and could exclude negative probability, resulting to improve the accuracy of the option pricing significantly. A more flexible parametric generalized stochastic volatility model given by Panayiotis & Chris et. al (2010) [3] could characterize risk assets variance well when describing the short maturity of risk assets and could be consistent with the implied volatility well when pricing options. Wang & Wu [4] studied the options pricing problems in which the return of risk assets is subject to an auto-regressive moving average model, and the European option pricing formula which is similar to the B-S formula was gotten. Chiu et al. [5] discussed the European option pricing method and gave the form of a semi-analytical solution to the problem for underlying assets with multiscale stochastic volatility mean reversion. An efficient numerical method with a secondary convergence was proposed by William & Hsu (2011) [6] who studied European Asian option pricing problems possessing fixed strike price and discrete-time observations. Bao & Li et al. (2012) [7] studied the VXX option pricing problem with default risk and positive skewness fluctuations, comparing the proposed model to other models in many aspects thus concluding that the text model is the best. Su & Wang et al. (2012) [8] studied the risk of random fluctuations amendment jump Markov model to minimize the spread option pricing problem and obtained European option price and the optimal hedging strategies by using Radon-Nikodym derivative to get minimum martingale measure and solving partial differential equations. Kwai (2013) [9] studied the option pricing problem of floating option strike price review in the Heston stochastic volatility model and gave the analytic form of the pricing formula. Han Yan, Cui Shu Min (2010) [10] established the renminbi American index futures options pricing model and proposed solving method based on quadratic approximation. Application of the model examines the change of RMB index futures options theory of American values. It found that when the option is in the real value of the state, ahead of the implementation of the premium has a significant impact on option pricing. Li, Qu & Huang [11] portray the correlation structure between the fragile Warrant and counterparty default probability by Fréchet Copula and Related measure Kendall τ, giving fragile European call option price closed-form expression. Then fragile European call option price has been calculated and sensitivity analyzed by Kendall τ and different values of the underlying asset price and the strike price ratio. However, whether the underlying risk assets or derivatives market, investors are not risk-neutral. Byung & Tong (2006) [12] studied how to use the risk-neutral probability density function implied in option data to generate the risk aversion function of investors. It is found that the more flexibility the utility function has, the stronger the predictability of the subjective probability density function is, and the measure of relative risk aversion is not zero significantly. Option pricing risk-neutral law aside from objective probability of risk assets, based on the closing price of risk assets, found a so-called risk-neutral probability equivalent to objective probability. In the risk-neutral probability measure, the expected rate of return of assets of all transactions is a risk-free rate. There exists an infinite number of risk-neutral probability measures in incomplete markets, and such risk-neutral probability often separates the links with the objective probability of market transactions.

Prospect theory assumes that losses and gains are valued differently, and thus individuals make decisions based on perceived gains instead of perceived losses. Also known as the "loss-aversion" theory, the general concept is that if two choices are put before an individual, both equal, with one presented in terms of potential gains and the other in terms of possible losses, the former option will be chosen. Wang (2021) [13] proposed a novel model to incorporate prospect theory into the consumption-based asset pricing model, where habit formation of consumption is employed to determine endogenously the reference point. If investors evaluate their excess or shortage amounts in consumption relative to their habit consumption levels based on prospect theory, the equity premium puzzle can be resolved. Barberis (2021) [14] presented a new model of asset prices in which investors evaluate risk according to prospect theory and examine its ability to explain 23 prominent stock market anomalies. The model incorporates all of the elements of prospect theory, accounts for investors' prior gains and losses, and makes quantitative predictions about an asset's average return based on empirical estimates of the asset's volatility, skewness, and past capital gain. Shirvani (2021) [15] explained the main concepts of prospect theory and cumulative prospect theory within the rational dynamic asset pricing framework. They derived option pricing formulas when asset returns were altered by a generalized prospect theory value function or a modified Prelec's weighting probability function. However, the authors do not give whether there is arbitrage in the option prices obtained under prospect theory, nor do they indicate the relationship with traditional risk-neutral pricing.

The innovation of this paper is to describe the proposed measure of investor risk aversion based on the Prospect theory. Based on the underlying objective probability of risk assets, the no-arbitrage pricing formula containing the objective probability of European options is gotten under the only assumption that the same degree of risk aversion of investors to the market all assets. The remaining section of this article is organized as follows: the second part describes the method described in investor risk aversion as well as based on the objective probability of the European Option Pricing Method. The third part demonstrates the proposed method in the complete market - under the circumstances of a binomial tree. The fourth part demonstrates the similarities and differences between this method and the general risk-neutral method under an incomplete market. An illustration is given in the fifth section. The sixth part is the conclusion and prospect.

2 RISK AVERSION AND PRICING OPTIONS

Market is composed of a risk-free asset, a risk asset, and a European Option built on risk assets. The rate of return of a risk-free asset is constant $r > 0$ (assign $R = 1 + r$). The initial price of the risky asset is S. The number of terminal prices states is N. The terminal prices in i-th ($i = 1, 2, \ldots, N$) state are SU_i, and $U_i < U_{i+1}$ for $i = 1, 2, \ldots, N - 1$). There is a j with $U_j \le R$, $U_{j+1} \ge R$. The objective probability is $p_i > 0$ for the i-th state. So,

$$\sum_{i=1}^{\mathrm{N}} p_i = 1 \tag{1}$$

The payment Function of a European call option is $(S_T - K)^{+1}$, whose $SU_1 < K < SU_N$ is a European call option exercise price. It assumes that investors had different attitudes toward losses and gains. We required that

$$\sum_{i=1}^{N} p_i \left(U_i - R\right)^+ - \lambda \sum_{i=1}^{N} p_i \left(R - U_i\right)^+ = 0 \tag{2}$$

$\sum_{i=1}^{N} p_i \left(U_i - R\right)^+$ is the expectation of excess return. $\sum_{i=1}^{N} p_i \left(R - U_i\right)^+$ is the expectation of relatively risk-free revenue loss. $\lambda > 0$ is the risk appetite of investors. The risk for investors is better if $0 < \lambda < 1$. The risk for investors is neutral while $\lambda = 1$. The risk for investors is disgust while $\lambda > 1$. Wang (2000) [17], Mandan (2017) [18], and Yao (2019) [19] used different distortion operators to transform the risk-neutral probabilities and applied them to option pricing, but the form and parameters of these distortion twist operators are difficult to choose and difficult to estimate. Assume risk assets and risk preferences of the European call option λ are the same for investors, then in objective probability p_i, there was:

$$\begin{aligned} &\sum_{i=1}^{N} p_i \left(\left(SU_i - K\right)^+ - CR\right)^+ - \\ &\lambda \sum_{i=1}^{N} p_i \left(CR - \left(SU_i - K\right)^+\right)^+ = 0 \end{aligned} \tag{3}$$

C is the Opening price of a European call option. Eq. (3) could be changed to:

$$\begin{aligned} &\sum_{i=j+1}^{N} p_i \left(\left(SU_i - K\right)^+ - CR\right) - \\ &\lambda \sum_{i=1}^{j} p_i \left(CR - \left(SU_i - K\right)^+\right) = 0 \end{aligned} \tag{4}$$

We may get a European call option price:

$$C = \frac{1}{R} \frac{\lambda \sum_{i=1}^{j} p_i \left(SU_i - K\right)^+ + \sum_{i=j+1}^{N} p_i \left(SU_i - K\right)^+}{\lambda \sum_{i=1}^{j} p_i + \sum_{i=j+1}^{N} p_i} \tag{5}$$

We would find that the European call option price from Eq. (5) is decided by both Investor risk appetite λ and objective probability p_i.

Defining a new discrete probability measure Q,

$$\begin{aligned} &\frac{\lambda p_1}{\lambda \sum_{i=1}^{j} p_i + \sum_{i=j+1}^{N} p_i}, \frac{\lambda p_2}{\lambda \sum_{i=1}^{j} p_i + \sum_{i=j+1}^{N} p_i}, \ldots, \frac{\lambda p_j}{\lambda \sum_{i=1}^{j} p_i + \sum_{i=j+1}^{N} p_i}, \\ &\frac{p_{j+1}}{\lambda \sum_{i=1}^{j} p_i + \sum_{i=j+1}^{N} p_i}, \ldots \frac{p_N}{\lambda \sum_{i=1}^{j} p_i + \sum_{i=j+1}^{N} p_i} \end{aligned} \tag{6}$$

[1]Define $(x)^+ = \max(x, 0)$

A discrete probability measure is equivalent to objective probability while $\lambda > 0$. Eq. (5) could be rewritten in the discrete probability measure Q to

$$C = \frac{1}{R} E\left[\left(S\right)\right] \tag{7}$$

This indicates that the European call expected rate of return is equal to the risk-free rate of return in the discrete probability measure. We had the following from Eq. (2) and definition j.

$$\left(\sum_{i=1}^{j} \lambda p_i U_i + \sum_{i=j+1}^{N} p_i U_i\right) \Big/ \left(\lambda \sum_{i=1}^{j} p_i + \sum_{i=j+1}^{N} p_i\right) = R \tag{8}$$

The expected rate of return on risky assets is also equal to the risk-free rate of return in a discrete probability measure Q. Q is a Risk-neutral probability measure equal to objective probability. Therefore, the European call option price described from Eq. (5) is no arbitrage price.

In the complete market (Complete Market, such as a binomial market), the risk-neutral probability measure is the only one. Eq. (5) is consistent with the existing risk-neutral pricing method. In the imperfect market, there are infinitely many risk-neutral probability measures. Eq. (5) is equal to risk-neutral pricing and it could be linked with objective probability and investor risk appetite.

3 PRICING EUROPEAN OPTIONS FOR BINOMIAL TREE

Assume the risk of asset price determined by the binary tree, opening price S, closing pricing SU, or SD whose probability of occurrence is p, $1 - p(0 < p < 1)$, and $U > R > D$. Other mentioned above. We could get the following from Eq. (2)

$$\lambda = \frac{p}{1-p} \frac{U-R}{R-D} \tag{9}$$

European call option price could be got from Eq. (5):

$$C = \frac{1}{R} \frac{p\left(SU - K\right)}{p + \lambda\left(1-p\right)} \tag{10}$$

Put Eq. (9) into Eq. (10), and we have

$$C = \frac{1}{R} \frac{R-D}{U-D}\left(SU - K\right) \tag{11}$$

Eq. (11) is also the European call option price under the existing risk-neutral pricing.

4 PRICING EUROPEAN OPTIONS FOR TRINOMIAL TREE

Assume that the risky asset prices are decided by the Triple Tree, the beginning price is S and the ending prices of risky assets are SU, S, SD of which the probabilities are $p, q, 1-p-q$ ($0 < p, q < 1$). $U > R > 1 > D$, and others mentioned above. From Eq. (2) shows

$$\lambda = \frac{p\left(U-R\right)}{q\left(R-1\right) + \left(1-p-q\right)\left(R-D\right)} \tag{12}$$

We can get the European call option price from Eq. (5)

$$C = \frac{1}{R} \frac{p\left(SU-K\right)^{+} + \lambda q\left(S-K\right)^{+}}{p + \lambda q + \lambda\left(1-p-q\right)} \tag{13}$$

Risk-neutral probability by the Eq. (6) is described in the following measure formula

$$\frac{p}{p+\lambda\left(1-p\right)}, \frac{\lambda q}{p+\lambda\left(1-p\right)}, \frac{\lambda\left(1-p-q\right)}{p+\lambda\left(1-p\right)} \tag{14}$$

From Eqs. (13), (14) it can be seen whether it is a European call option price or the corresponding risk-neutral probability measure expression, and ultimately contain objective probability, and this risk-neutral probability measure is uniquely determined by the objective probability and it contains a market risk appetite, resulting arbitrage-free price for the option price.

5 ILLUSTRATION

The financial market consists of a risk-free asset and a risky asset (stock). The annual interest rate on the risk-free asset is a constant r. At the beginning of the period, the price of the stock is S_0. At the end of the period, the price of the stock follows a random distribution. Assume that this random distribution is discrete (if it is discrete, you can discretize the continuous distribution to obtain a discrete distribution). Because there are a so-called tick size and up-down trading halt mechanism in the actual stock market, this discrete distribution must have finitely many states.

Based on the Black-Scholes (1973) approach, the distribution of the collected historical data on stock annual returns was fitted with a continuous normal distribution, and the parameters corresponding to the normal distribution were obtained as μ, σ. In the equivalent risk-neutral measure transformation, the objective probability distribution (physical probability distribution) is transformed flat, and the risk-neutral probability distribution due to stock returns is $N(r, \sigma)$.

The risk preference parameter λ can be estimated from market data based on prospect theory (see [15]). Let $\phi(.)$ be the probability density function of the standard normal distribution, and $x_1 < x_2 < \ldots < x_{N-1} < x_N$ be the N -state discretization of the stock return. The risk-neutral probability of the i-th state is

$$\frac{\phi\left(\frac{x_i - r}{\sigma}\right)}{\sum_{j=1}^{N} \phi\left(\frac{x_j - r}{\sigma}\right)} \tag{15}$$

The probability corresponding to the i-th state under risk neutrality considering the distortion of risk preferences is

$$p_i = \begin{cases} \dfrac{\lambda\phi\left(\dfrac{x_i - r}{\sigma}\right)}{\lambda\sum\limits_{x_j \le r}\phi\left(\dfrac{x_j - r}{\sigma}\right) + \sum\limits_{x_j > r}\phi\left(\dfrac{x_j - r}{\sigma}\right)}, x_i \le r \\ \dfrac{\phi\left(\dfrac{x_i - r}{\sigma}\right)}{\lambda\sum\limits_{x_j \le r}\phi\left(\dfrac{x_j - r}{\sigma}\right) + \sum\limits_{x_j > r}\phi\left(\dfrac{x_j - r}{\sigma}\right)}, x_i > r \end{cases} \tag{16}$$

The price of a European call option with a one-year expiration and a strike price of K is

$$C = \mathrm{e}^{-r}\sum_{i=1}^{N} p_i \max\left(S_0 \exp\left(x_i\right) - K, 0\right) \tag{17}$$

On May 9, 2022, Microsoft Corporation's stock has 180-day historical volatility σ = 0.3263 and implied volatility $\tilde{\sigma}$ = 0.3682[2]. On May 9, 2022, the previous day's closing price of the one-year T-bills yield was 1.930%[3]. S_0 = 100, T = 0.5, λ = 0.98, N = 1000, do the following discretization.

$$x_i = rT + \sigma\left(-3.6 + \frac{7.2}{N}i\right), i = 0,1,2,...,N \tag{18}$$

Based on the pricing method for the European call option given above, the relationship between the European call option price and the Exercise price when historical volatility is used is shown in Fig. 1.

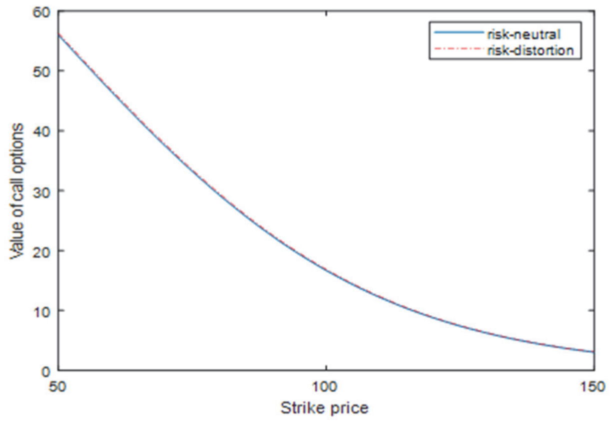

Figure 1 European call option price to strike price relationship with historical volatility

When using implied volatility, the relationship between the European call option price and the strike price is shown in Fig. 2.

As can be seen in Figs. 1 and 2, the distortion of risk appetite gives a higher weight to returns at different strike prices (λ < 1), leading to higher call option prices considering risk appetite than those obtained from traditional risk neutrality, a result consistent with the findings of the literature [16].

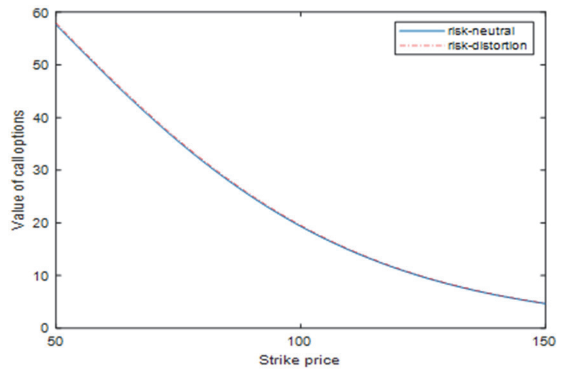

Figure 2 European call option price to strike price relationship with implied volatility

Due to the current impact of the new crown epidemic, the Russian-Ukrainian war, and other factors, the uncertainty of the market's future has increased, resulting in implied volatility being higher than historical volatility, thus the option price obtained when pricing options using implied volatility is always higher than the option price obtained using historical volatility.

6 CONCLUSION

We studied the difference between the European call option pricing problem and the existing risk-neutral pricing method. First, consider the risk preferences of investors. Then European call option formula is given using objective probability to prove that this is no option price arbitrage price. In this paper, we demonstrate the option pricing method, and compare the similarities and differences between the proposed method and the existing risk-neutral pricing method, respectively, for the complete market of binary and trinomial tree incomplete markets. Under the binary case, we find that the objective probability did not enter the option pricing formula, and the proposed method is consistent with the existing risk-neutral pricing conclusion. However, in the case of a trinomial tree, the objective probabilities enter the option pricing formula finally. The reason is that the risk-neutral probability measure is unique for the binary description of a complete market, while the trigeminal tree market is incomplete, existing with an infinite number of risk-neutral probability measures. This method in this paper is equivalent to the use of one of the risk-neutral probabilities.

In this paper, the single European call of multinomial-tree of pricing was studied, but pricing methods and conclusions of any multinomial-tree of European Options alone are true. However, in our paper, the market is a single-period market; we hope to extend the study to option pricing problems in multi-period and dynamic markets. In future research, it is expected to further expand to more of the tree, pricing American options, Asian options and other exotic options to be considered extended to continuous-time option pricing problem.

[2]https://www.alphaquery.com/stock/MSFT/volatility-option-statistics/180-day/historical-volatility

[3]https://www.marketwatch.com/investing/bond/tmubmusd01y?countrycode=bx

Acknowledgments

This research is partially supported by The National Social Science Fund (18BGL004) and High-quality Development Research Institute in Huaihai Economic Zone, Xuzhou University of Technology, China.

7 REFERENCE

[1] Black, F. & Scholes, M. S. (1973). The pricing of option and corporate liabilities. *Journal of Political Economy*, *81*(1), 637-654. https://doi.org/10.1086/260062

[2] Kwai, S. L. (2013). Ananalytic pricing formula for lookback options under stochastic volatility. *Applied Mathematics Letters*, *26*(1), 145-149. https://doi.org/10.1016/j.aml.2012.07.008

[3] Chiu, M. C., Lo Y. W., & Wong, H. Y. (2011). Asymptotic expansion for pricing options for a mean-reverting asset with multi-scale stochastic volatility. *Operations Research Letters*, *39*(1), 289-295. https://doi.org/10.1016/j.orl.2011.06.002

[4] Hsu, W. W. Y. & Lyuu Y. D. (2011). Efficient pricing of discrete Asian options. *Applied Mathematics and Computation*, *217*(1), 9875-9894. https://doi.org/10.1016/j.amc.2011.01.015

[5] Panayiotis C., Chris C., & Spiros H. M. (2010). Generalized parameter functions for option pricing. *Journal of Banking & Finance*, *34*(10), 633-646. https://doi.org/10.1016/j.jbankfin.2009.08.027

[6] Wang, C. W., Wu, C. W., & Tzang, S. W. (2012). Implementing option pricing models when asset returns follow an auto-regressive moving average process. *International Review of Economics and Finance*, *24*(1), 8-25. https://doi.org/10.1016/j.iref.2011.12.003

[7] Bao, Q. F. , Li, S . H., & Gong, D. (2012). Pricing VXX option with default risk and positive volatility skew. *European Journal of Operational Research*, *223*(1), 246-255. https://doi.org/10.1016/j.ejor.2012.06.006

[8] Su, X. , Wang, W., & Hwang, K. (2012). Risk-minimizing option pricing under a Markov-modulated jump-diffusion model with stochastic volatility. *Statistics and Probability Letters*, *82*(1), 1777-1785. https://doi.org/10.1016/j.spl.2012.05.026

[9] Moriggia, V., Muzzioli, S., & Torricelli, C.(2009). On the no-arbitrage condition in option implied trees. *European Journal of Operational Research*, *193*(1), 212-221. https://doi.org/10.1016/j.ejor.2007.10.017

[10] Byung, J. K. & Tong, S. K. (2006). Option-implied risk preferences: An extension to wider classes of utility functions. *Journal of Financial Markets*, *9*(1), 180-198. https://doi.org/10.1016/j.finmar.2005.12.004

[11] Han, L. & Cu, I. (2010). Study on pricing of Chinese Yuan index American futures options. *Journal of Management Sciences in China*, *13*(3), 50-63. https://doi.org/10.1097/ICO.0b013e3182000add

[12] Ping, L., Bo, Q., & Guang-Dong, H. (2012). Pricing of European vulnerable options based on Fréchet Copula. *Journal of Management Sciences in China*, *15*(4), 23-29.

[13] Wang, J., Wu, C., & Zhong, X. (2021). Prospect theory and stock returns: Evidence from foreign share markets. *Pacific-Basin Finance Journal*, *69*, 101644. https://doi.org/10.1016/j.pacfin.2021.101644

[14] Barberis, N., Jin, L. J., & Wang, B. (2021). Prospect theory and stock market anomalies. *The Journal of Finance*, *76*(5), 2639-2687. https://doi.org/10.2139/SSRN.3477463

[15] Shirvani, A., Fabozzi, F. J., Racheva-Lotova B. et al. (2021). Option Pricing with Greed and Fear Factor: The Rational Finance Approach. *The Journal of Derivatives*, *29*(2), 77-119. https://doi.org/10.48550/arXiv.1709.08134

[16] Yacine, A., Wang, Y., & Francis Y. (2001). Do option markets correctly price the probabilities of movement of the underlying asset? *Journal of Econometrics*, *102*, 67-110. https://doi.org/10.1016/S0304-4076(00)00091-9

[17] Wang, A & Shaun, S. (2000). A class of distortion operators for pricing financial and insurance risks. *Journal of risk and insurance*, *1*, 15-36.

[18] Madan Dilip, B. & Schoutens, W. (2017). Conic option pricing. *The Journal of Derivatives*, *25*(1), 10-36.

[19] Yao, L. & Yang, G. (2019). Option Pricing by Probability Distortion Operator Based on the Quantile Function. *Mathematical Problems in Engineering*, *2019*, 5831569. https://doi.org/10.1155/2019/5831569

Contact information:

Weiwei WANG
(Corresponding author)
School of Applied Technology,
Nanjing University of Information Science and Technology,
Nanjing 210044, PR China
E-mail: 002095@nuist.edu.cn

Guoqing GU
Finance Office, Jiangsu Open University,
Nanjing 210036, PR China
E-mail: Gugq@jsou.edu.cn.

Fameng SUN
Sinopec Chemical Sales Co., Ltd. Jiangsu Branch,
Nanjing 210002PR China
E-mail: mengfa2012@163.com

Xiao Ping HU
School of Economics and Management, Southeast University,
Nanjing 210096PR China
E-mail: hxpnj@seu.edu.cn