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ON THE SUBSTITUTIVITY OF VARIABLES - THE ROOTS OF UNDECIDABILITY

Abstract

This paper examines the constraints for the decidability of a logical system by analyzing the theories of undecidability in formal logic and recursive functions. The proof of the diagonal lemma requires the implicit premise that all formulas are closed under a *variable composition*, i.e., the composition of variables in the formulas signatures. A variable composition is representable in first-order logic if $\forall\psi\forall k(\vdash \psi(\overline{k}) \leftrightarrow \exists y(\psi \wedge y{\equiv}\overline{k}))$ can be derived, e.g., from a semantic definition of the quantifiers. A further condition is the assumption of an interpretation providing the existence of what will be defined as an *indeterminate variable signature* with a *constant interpretation*.
A recursion theorem for formulas in analogy to the recursion theorem of Kleene (1943, pp.52-53) will be proved which covers the diagonal lemma as a special case. The chosen notation and reasoning intend to make the necessary conditions for its provability explicit. It will be proved that the representability of variable composition and the existence of an indeterminate variable signature to represent a constant interpretation are consequences of the recursion theorem, i.e., equivalent to the existence of fixed points $\forall\psi\exists\varphi(\vdash \varphi \leftrightarrow \psi(\ulcorner\overline{\varphi}\urcorner))$.
These results give a reason why the negation of the diagonal lemma can be proved for a predicative logic that contradicts these premises but holds the explicit condition of the diagonal lemma, i.e., a language of this logic is capable of representing all computable functions, as has been shown as a non-expectable result in Solte 2020. The paper concludes with an outline of a decidable structure of computable functions. I.e., it is possible to provide an interpretation of a predicative logic without undecidability in this structure.

Keywords: diagonal lemma; recursion theorem; decidable structure

ÜBER DIE SUBSTITUIERBARKEIT VON VARIABLEN - DIE WURZELN DER UNENTSCHEIDBARKEIT

Zusammenfassung

In diesem Beitrag werden die Bedingungen für die Entscheidbarkeit eines logischen Systems untersucht, indem Theorien der Unentscheidbarkeit, innerhalb der formalen Logik und der rekursiven Funktionen, analysiert werden. Eine notwendige und implizite Voraussetzung für den Beweis des Fixpunkttheorems (Diagonalisierungslemma) ist die Annahme der Abgeschlossenheit aller Formeln unter einer *Variablen-Komposition*, d.h., der Verknüpfung von Variablen in den Signaturen von Formeln. Eine Variablen-Komposition von Formeln ist in einer Prädikatenlogik repräsentierbar, wenn der Satz $\forall\psi\forall k\big(\vdash \psi(\overline{k}) \leftrightarrow \exists y(\psi \wedge y{\equiv}\overline{k})\big)$ hergeleitet werden kann, etwa durch eine geeignete semantische Definition der Quantoren. Als weitere Bedingung wird im Beitrag herausgearbeitet, dass bei dem Beweis eine Interpretation zugrundeliegen muss, aus der sich die Existenz einer in diesem Text definierten *variablen Signatur in einer Unbestimmten* mit einer *konstanten Interpretation* herleiten lässt.
Es wird ein Rekursionssatz für Formeln in Analogie zum Rekursionssatz von Kleene (1943, S. 52-53) bewiesen, der das Diagonalisierungslemma als Spezialfall abdeckt. Die hier gewählte Notation und Beweisführung soll die Bedingungen für seine Beweisbarkeit deutlich machen. Es wird zudem bewiesen, dass die Annahmen der Repräsentierbarkeit einer Variablen-Komposition und der mittels einer variablen Signatur in einer Unbestimmten darstellbaren konstanten Interpretation aus der Behauptung des Rekursionssatzes folgen, also äquivalent sind zur Existenz von Fixpunkten $\forall\psi\exists\varphi\big(\vdash \varphi \leftrightarrow \psi(\overline{\ulcorner\varphi\urcorner})\big)$.
Mit diesen Resultaten kann begründet werden, warum in einem prädikativen logischen System die Negation des Diagonalisierungslemmas bewiesen werden kann, in dem diese Prämissen nicht gelten aber dennoch die explizite Voraussetzung des Diagonalisierungslemmas erfüllt ist. D.h. eine Sprache der prädikativen Logik ist mächtig genug, alle berechenbaren Funktionen zu repräsentieren, so wie Solte 2020 es als unerwartetes Ergebnis aufgezeigt hat. Als Ausblick auf ein derartiges logisches System wird eine entscheidbare Struktur berechenbarer Funktionen skizziert. Das bedeutet, in dieser Struktur kann eine prädikative Logik interpretiert werden, in der es keine unentscheidbaren Sätze gibt.

Schlüsselwörter: Fixpunkttheorem; Rekursionssatz; Entscheidbare Struktur

Introduction

The diagonal lemma (fixed-point theorem) of logical systems (Gödel 1931) is fundamental and widely known. In section 2, the diagonal lemma is analyzed to identify the roots of undecidability. The necessary conditions for its provability will be explicitly defined. Similar to the diagonal lemma is the recursion theorem for partial recursive functions of Kleene (1943, pp.52-53). It will be proved in section 3 that both theorems are equivalent to an implicit assumption that formulas are closed under an *indeterminate variable composition*, i.e., that a variable of a formula can actually or syntactically be replaced by an *indeterminate variable function signature*. E.g., if $h_{\llbracket y \rrbracket}(\vec{x})$ denotes a variable function signature up to an indeterminate y then $f(h_{\llbracket y \rrbracket}(\vec{x}), x_1, \dots x_{n-1})$ denotes the indeterminate variable composition of $f(x_0, \dots x_{n-1})$ with a variable signature up to y in the variable x_0 intended to be interpreted as a function with $1{+}n$ variables. The term *indeterminate variable signature* refers to functions, predicates, or formulas. It will be explained that it depends on the definition of an interpretation whether formulas denoting the code of indeterminate variable signatures represent computable functions or result in an infinite recursive interpretation.

As above $h_{\llbracket y \rrbracket}(\vec{x})$, some rather unusual notations are introduced to make all necessary premises providing the provability of the diagonal lemma and recursion theorem explicit. The intended interpretation of $h_{\llbracket y \rrbracket}(\vec{x})$ is an n-ary function determined by a Gödel code substituting the indeterminate y and $\llbracket y \rrbracket$ denoting a formula in y such that $\llbracket \ulcorner \phi \urcorner \rrbracket$ represents a canonic representation of a formula ϕ with Gödel code $\ulcorner \phi \urcorner$. As much as possible, the invented notations aim to adopt the usual intended interpretations in the context of formal logic and computability theory. E.g., in def. 3.2 the notation $[\,\psi \mid \{\langle \phi \rangle\} /\!\!/ x_i]$ is introduced to represent a composition of a formula ψ with a function view on ϕ in a variable x_i. This adopts the notation $\{e\}(\vec{x}) \simeq \mathsf{y}$ used by van Dalen (2013, pp.218) for formulas ϕ represented by recursive functions with an index $\langle \phi \rangle$ substituting e. The first section gives an explanatory overview of the used notations and their intended interpretation.

The definition of relativized quantification (c.f. van Dalen 2013, p.75) is a usual notation that should be denoted *variable composition*. $(\exists x)(P(x) \wedge \varphi)$ is interpreted to represent a composition of the variable signature $P(x)$ up to P with φ up to φ in the variable x, i.e., the symbols P and φ are treated as indeterminates[1] to be replaced by the signature of a concrete

[1] Formally the whole notation $P(x)$ has to be treated as an indeterminate intended to be substituted by signatures of unary formulas only.

formula instance. To illustrate the interpretation of $(\exists x)(P(x) \wedge \varphi)$ as a syntactical replacement, let $f_P(x)$ and $f_\varphi(\vec{x})$ denote the functions in variable x resp. in the variables $\vec{x}$ represented as formulas P resp. φ and assuming an arity $1{+}n$ of φ. Then $P\Big(\overline{f_\varphi(\vec{x})}\Big) \leftrightarrow (\exists x)\Big(P(x) \wedge x{\equiv}\overline{f_\varphi(\vec{x})}\Big)$ expresses this syntactical representation of the variable composition, i.e., the composition of $P(x)$ with $f_\varphi(\vec{x})$ in the variable x. In the structure of partial recursive functions, $P\Big(\overline{f_\varphi(\vec{x})}\Big)$ is interpreted as[2] $P(\chi_\sigma(f_\varphi(\vec{x}), f_P(x)))$.

In section 4, a decidable structure beyond partial recursive functions and relations is explained, providing to define an interpretation in equivalence classes of functions and formulas. This avoids undecidability as the basis of a decidable logical system capable of representing all computable functions in terms of this structure.

1. Some remarks to the used notation

In the following, we first explain some details of the used notation, mainly adopting the notation of van Dalen 2013. The notations to represent the different kinds of substitution and their intended interpretations are formally defined in section 3, def. 3.2 - def. 3.5.

Let $\mathcal{S}_{PA}$ denote a signature of a first order language $L^{\mathcal{S}_{PA}}$ capable of representing all computable functions.

Let $\vdash$ denote the provability-relation in $L^{\mathcal{S}_{PA}}$.

Let $=\!\!|\!\models$ and $\Longleftrightarrow$ denote both the usual representations of semantical equivalence.

Let $\varpi_{Y^*} :\cong \varpi_{X^*}$ denote an isomorphism of notations, i.e., an interpretation of the notation denoted by ϖ_{X^*} exists, resulting in the interpretation of the notation denoted by ϖ_{Y^*}.

Let $\mathfrak{A}$ and $\mathfrak{X}$ denote structures. Let $\mathfrak{N}$ denote the structure $(\mathbb{N}, +, \cdot, \dot{-}, =)$, with $\dot{-}$ intended to be interpreted as $m \geq n \Longleftrightarrow n \dot{-} m = 0$.

Let $\equiv$ denote the representation of the relation $=$ in a formal language.

The definition of *representability* in van Dalen (2013, p.242) (c.f. def. 2.1 below) does not have any notion for variables denoting elements of the domain of discourse.

[2] Let $\chi_\sigma(f_\varphi(\vec{x}), f_P(x)) = f_\varphi(\vec{x})$. Here we intentionally use the symbol χ which Gödel uses to denote a dedicated variable function up to an implicit indeterminate to prove the recursiveness of a relation $(\exists x)(R(x, \vec{y}) \wedge x{\leq}f(\vec{x}))$, in his notation $(Ex)\,[x{\leq}\varphi(\mathfrak{x})\ \&\ R(x, \mathfrak{y})]$ by constructing $R[\,\chi(\varphi(\mathfrak{x}), \mathfrak{y}), \mathfrak{y}]$; c.f., Gödel (1931, p.181).

We define:

Let x_i be the representation of an $\mathfrak{A}$-*variable* $\widetilde{i}$.

Let $\mathfrak{x}$ denote a list of $x \dot{-} 1$ variables $x_1, \ldots x_{x-1}$.

Let $\vec{x} :\cong x_0, \mathfrak{x}$ denote a list of x variables $x_0, x_1 \ldots x_{x-1}$.

Let $x_{\mathfrak{x}}$ denote a variable not in the list of variables $x_0, x_1 \ldots x_{x-1}$.

As usual, small letters v, x, y and z also denote $\mathcal{S}_{PA}$-variables representing $\mathfrak{A}$-variables $\widetilde{v}$, $\widetilde{x}$, $\widetilde{y}$ and $\widetilde{z}$.

Let $\ulcorner\varphi\urcorner$ represent a Gödel code of an $\mathcal{S}_{PA}$-formula φ.

Let the Gödel code be an element k of the interpreting structure $\mathfrak{A}$.

Let $\overline{\ulcorner\varphi\urcorner}$ denote the representation of a Gödel code in $\mathfrak{X}$.

Let $q\,[\,k/\widetilde{x}]$ denote the substitution of $\widetilde{x}$ by k in the signature of a predicate q.

Let $\phi\left[\,\overline{k}/x\right]$ denote the substitution of x by $\overline{k}$ in the formula ϕ.

In section 2, notations for interpretations of $L^{\mathcal{S}_{PA}}$ expressions with notational elements to represent substitution are invented:

$\mathfrak{S}_{\text{PA}}\left(\overline{f_{\mathcal{I}}(\ulcorner\phi\urcorner)}\right)$ denotes the $L^{\mathcal{S}_{PA}}$ expression *equivalent* to ϕ where all substitution elements in ϕ have been resolved. $f_{\mathcal{I}}$ denotes the function resulting in the Gödel code of the resolved formula. This notation aims to associate with the symbol $\mathfrak{S}_{\text{PA}}$ its interpretation as a logical function, mapping the representation of a Gödel code onto the corresponding formula.

$\mathfrak{S}_{\chi}(x_i)[\,x_0,\ldots x_{n-1}, y]$ denotes what will be defined as the representation of an *indeterminate variable signature up to* x_i. The intention behind this chosen notation is to associate with it an interpretation similar to $\mathfrak{S}_{\text{PA}}(x_i)$ as a logical function representing a special interpretation of a formula with a Gödel code $\ulcorner\phi\urcorner$ if x_i is substituted by $\overline{\ulcorner\phi\urcorner}$. The intended special interpretation of this logical function is substitution. The variables $x_0,\ldots x_{n-1}, y$ are set in square brackets to express syntactically that the signature of the formula represented through $\mathfrak{S}_{\chi}(x_i)[\,x_0,\ldots x_{n-1}, y]$ depends on the substitution of x_i. The definition of a *constant interpretation* makes explicit that the intended semantic attached to the indeterminate variable signature is only substitution. With such a definition, the notation of a constant interpretation is representing a reference notation for a formula where a specific substitution has been resolved.

$\llbracket\,\ulcorner\phi\urcorner\rrbracket_{\text{PA}}$ denotes the $L^{\mathcal{S}_{PA}}$ expression representing a *canonical* formula equivalent to ϕ where all substitution elements in ϕ have been resolved.

In section 3, the chosen notation intended to be interpreted in the structure of partial recursive functions adopts van Dalen (2013, pp. 218) with some more details which it is necessary to express:

Let $\mathfrak{S}_\mu$ denote the structure of partial recursive functions and relations.

Let $x_\square$ denote an *indeterminate placeholder* different from all variables x_ϖ. ϖ denotes any notation to specify a variable, e.g., i, 0, $x-1$, etc.

Let $[\,\{x_\square\}\simeq\text{y}]$ denote an *indeterminate signature up to* $x_\square$.

Let $\langle\phi\rangle$ and e_ϕ denote an interpretation of a formula ϕ named *index*.

Let $\vec{e_x}$ denote a list of x indices $e_0, e_1 \ldots e_{x-1}$.

Let $\left[\psi \,\middle|\, \overrightarrow{\langle\phi_{\mathfrak{x}}\rangle} \,/\!\!/\, \vec{\mathfrak{x}}\right]$ denote the substitution of variables by notations representing indices.

Let $\left[\psi \,\middle|\, \overrightarrow{\{\langle\phi_{\mathfrak{x}}\rangle\}} \,/\!\!/\, \vec{\mathfrak{x}}\right]$ denote the replacement of variables by notations intended to be interpreted as functions.

Let $\Xi(\langle\phi\rangle)$ represent a *variable interpretation* resolving notational substitutions and compositions in a formula ϕ. C.f. def. 3.3.

Let $[\![\langle\phi\rangle]\!]_\mu$ represent an *equivalence interpretation* in a theory μ denoting a reference notation or normal form of a formula. In this paper, μ refers exemplary to the theory of μ-recursive functions.

Let $\vec{\mathfrak{x}}$ denote any list of variables. If none of the above notations are used to make details explicit, let $\vec{\mathfrak{x}}$ denote the list of all variables occurring in a formula.

2. Identifying the roots of undecidability

With the notational modifications mentioned in the first section, the definition of representability and a usual formulation of the diagonal lemma and its proof will be adopted from van Dalen (2013, p.242 and p.250) into a second-order notation.

Definition 2.1. *Representability.*

Let k and k_i be elements of a domain of discourse $\mathbb{N}$.

Let $\overline{k}$ denote the representation of a constant function $f_{\mathfrak{C}}(k) = k$ as a term.

Let ϕ denote an $\mathcal{S}_{PA}$-formula and y and $x_0,\ldots x_{n-1}$ substitutable variables.

(a) A formula $\phi(x_0,\ldots x_{n-1}, y)$ represents an n-ary function f if for all $\overline{k}_0,\ldots \overline{k}_{n-1}$
$f(k_0,\ldots k_{n-1}) = k \Rightarrow\ \vdash \forall y\big(\phi(\overline{k}_0,\ldots \overline{k}_{n-1}, y) \leftrightarrow y \equiv \overline{k}\,\big)$.

(b) A formula $\phi(x_0,\ldots x_{n-1})$ represents an n-ary predicate P if for all $k_0,\ldots k_{n-1}$
$P(k_0,\ldots k_{n-1}) \Rightarrow\ \vdash \phi(\overline{k}_0,\ldots \overline{k}_{n-1})$ *and*
$\neg P(k_0,\ldots k_{n-1}) \Rightarrow\ \vdash \neg\phi(\overline{k}_0,\ldots \overline{k}_{n-1})$.

(c) A term $t(x_0,\ldots x_{n-1})$ represents an n-ary function f if for all $k_0,\ldots k_{n-1}$
$f(k_0,\ldots k_{n-1}) = k \Rightarrow\ \vdash t(\overline{k}_0,\ldots \overline{k}_{n-1}) \equiv \overline{k}$.

Definition 2.2. *Indeterminate variable signature and constant interpretation in $\mathcal{S}_{PA}$.*

Let $f_{\mathcal{I}}(\ulcorner\phi\urcorner)$ denote an interpretation resolving alternative and short notations in ϕ and those notations representing substitutions and compositions such that

$f_{\mathcal{I}}(\ulcorner\phi\urcorner) = \ulcorner\psi\urcorner \Rightarrow \vdash \phi \leftrightarrow \psi$.

Let $\mathfrak{S}_{PA}\left(\overline{f_{\mathcal{I}}(\ulcorner\phi\urcorner)}\right)$ denote the resolved $\mathcal{S}_{PA}$-formula ψ equivalent to ϕ.

Let $f_{\llbracket\rrbracket_{PA}}(f_{\mathcal{I}}(\ulcorner\phi\urcorner))$ denote an equivalence interpretation resulting in the code of a reference notation or normal form of a formula such that

$f_{\llbracket\rrbracket_{PA}}(f_{\mathcal{I}}(\ulcorner\phi\urcorner)) = \ulcorner\gamma\urcorner \Rightarrow \vdash \phi \leftrightarrow \gamma$.

Let $\llbracket\ulcorner\phi\urcorner\rrbracket_{PA}$ denote the canonical $\mathcal{S}_{PA}$-formula γ equivalent to ϕ.

Let $\mathfrak{S}_{\chi}(x_i)[x_0,\dots x_{n-1},y]$ denote a formula such that

$\mathfrak{S}_{PA}\left(\overline{f_{\mathcal{I}}\left(\ulcorner\mathfrak{S}_{\chi}(x_i)[x_0,\dots x_{n-1},y][\overline{k}/x_i]\urcorner\right)}\right) :\cong \mathfrak{S}_{\chi}(\overline{k})[x_0,\dots x_{i-1},\overline{k},x_{i+1},\dots x_{n-1},y]$.

$\mathfrak{S}_{\chi}(x_i)[x_0,\dots x_{n-1},y]$ is named an indeterminate variable signature up to x_i iff from

$\llbracket\overline{\ulcorner\mathfrak{S}_{\chi}(x_i)[x_0,\dots x_{n-1},y][\overline{\ulcorner\phi\urcorner}/x_i]\urcorner}\rrbracket_{PA} :\cong \mathfrak{S}_{PA}\left(\overline{f_{\mathcal{I}}\left(\ulcorner\mathfrak{S}_{PA}\left(\overline{f_{\mathcal{I}}(\overline{\ulcorner\phi\urcorner})}\right)[\overline{\ulcorner\phi\urcorner}/x_i]\urcorner\right)}\right)$

follows that $f_{\llbracket\rrbracket_{PA}}\left(f_{\mathcal{I}}\left(\ulcorner\mathfrak{S}_{\chi}(x_i)[x_0,\dots x_{n-1},y][\overline{\ulcorner\phi\urcorner}/x_i]\urcorner\right)\right) = f_{\mathcal{I}}\left(\ulcorner\phi[\overline{\ulcorner\phi\urcorner}/x_i]\urcorner\right)$.

A formula $\sigma_{\chi}(\vec{x},y)$ represents a constant interpretation up to x_i if

$\vdash \forall y\left(\sigma_{\chi}(\vec{x},y) \leftrightarrow y \equiv \overline{f_{\llbracket\rrbracket_{PA}}(f_{\mathcal{I}}(\ulcorner\mathfrak{S}_{\chi}(x_i)[x_0,\dots x_{n-1},y]\urcorner))}\right)$

and for all $\mathcal{S}_{PA}$-formulas ϕ

$\vdash \forall y\left(\sigma_{\chi}(\vec{x},y)[\overline{\ulcorner\phi\urcorner}/x_i] \leftrightarrow y \equiv \overline{f_{\llbracket\rrbracket_{PA}}\left(f_{\mathcal{I}}\left(\ulcorner\mathfrak{S}_{\chi}(x_i)[x_0,\dots x_{n-1},y][\overline{\ulcorner\phi\urcorner}/x_i]\urcorner\right)\right)}\right)$.

Remark: In the above def. 2.2, an indeterminate variable signature has been *defined* to be computable through an equation stipulating that a canonical formula is equivalent to a resolved formula which should not necessarily be a canonical formula. A definition

$\llbracket\overline{\ulcorner\mathfrak{S}_{\chi}(x_i)[x_0,\dots x_{n-1},y][\overline{\ulcorner\phi\urcorner}/x_i]\urcorner}\rrbracket_{PA} :\cong \llbracket\overline{\ulcorner\llbracket\overline{\ulcorner\phi\urcorner}\rrbracket_{PA}[\overline{\ulcorner\phi\urcorner}/x_i]\urcorner}\rrbracket_{PA}$

would eliminate this root of undecidability because $\llbracket\overline{\ulcorner\rho[\overline{\ulcorner\rho\urcorner}/x_i]\urcorner}\rrbracket_{PA}$ would not exist since $f_{\llbracket\rrbracket_{PA}}\left(f_{\mathcal{I}}\left(\ulcorner\rho[\overline{\ulcorner\rho\urcorner}/x_i]\urcorner\right)\right)$ would result in an infinite recursion and thus would not be computable for $\rho :\cong \mathfrak{S}_{\chi}(x_i)[x_0,\dots x_{n-1},y]$ iff $i < n$.

Lemma 1. *Fixed-point theorem of FOL arithmetic.*

Let $\xi_{PA}(x)$ denote an $\mathcal{S}_{PA}$-formula representing the function $f_{\llbracket\rrbracket_{PA}}(f_{\mathcal{I}}(\widetilde{x}))$.

Let ψ be a variable denoting $\mathcal{S}_{PA}$-formulas with one variable y and let φ be a variable denoting closed $\mathcal{S}_{PA}$-formulas.

Proposition: $\forall\psi\exists\varphi\big(\vdash\varphi\leftrightarrow\psi\big(\overline{\ulcorner\varphi\urcorner}\big)\big)$.

Proof. Let $\sigma_\chi(x,y)$ represent a constant interpretation up to x, i.e.,
$\sigma_\chi(x,y) :\cong \xi_{\text{PA}}\Big(\overline{\ulcorner\mathfrak{S}_\chi(x)[x,y]\urcorner}\Big)$ and
$\sigma_\chi\big(\overline{\ulcorner\phi\urcorner},y\big) :\cong \xi_{\text{PA}}\Big(\overline{\ulcorner\mathfrak{S}_\chi(x)[x,y]\big[\overline{\ulcorner\phi\urcorner}/x\big]\urcorner}\Big)$.
(1) $\sigma_\chi(x,y)$ and $\sigma_\chi\big(\overline{\ulcorner\phi\urcorner},y\big)$ represent ***obviously*** computable functions!
(2) Let $\rho(x)$ be the formula $\exists y(\psi\wedge\sigma_\chi(x,y))$ and
let φ be the formula $\rho\Big(\overline{\ulcorner\rho(x)\urcorner}\Big)$.
(3) From (2) follows $\varphi\leftrightarrow\exists y\Big(\psi\wedge\sigma_\chi\Big(\overline{\ulcorner\rho(x)\urcorner},y\Big)\Big)$.
With def. 2.2, it can be concluded
$\vdash\forall y\left(\sigma_\chi\Big(\overline{\ulcorner\rho(x)\urcorner},y\Big)\leftrightarrow y\equiv\overline{f_{\mathcal{I}}\left(\ulcorner\rho\Big[\overline{\ulcorner\rho(x)\urcorner}/x\Big]\urcorner\right)}\right)$ resulting in
(4) $\vdash\forall y\Big(\sigma_\chi\Big(\overline{\ulcorner\rho(x)\urcorner},y\Big)\leftrightarrow y\equiv\overline{\ulcorner\varphi\urcorner}\Big)$.
Substituting identities in (3) with (4) results in $\varphi\leftrightarrow\exists y\big(\psi\wedge y\equiv\overline{\ulcorner\varphi\urcorner}\big)$.
(5) Let $\exists y\big(\psi\wedge y\equiv\overline{\ulcorner\varphi\urcorner}\big)\leftrightarrow\psi\big(\overline{\ulcorner\varphi\urcorner}\big)$.
Under this interpretation, it can be concluded $\vdash\varphi\leftrightarrow\psi\big(\overline{\ulcorner\varphi\urcorner}\big)$. ⊣

Remark: Alternatively (c.f. Ebbinghaus, Flum and Thomas 2007) the diagonal lemma can be proved by defining $\rho(x)$ in (2) as $\forall y(\sigma_\chi(x,y)\to\psi)$ and concluding (5) with the argument $\forall\psi\forall k\big(\vdash\psi\big(\overline{k}\big)\leftrightarrow\forall y\big(y\equiv\overline{k}\to\psi\big)\big)$.

The argument $\forall\psi\forall k\big(\vdash\psi\big(\overline{k}\big)\leftrightarrow\exists y\big(\psi\wedge y\equiv\overline{k}\big)\big)$ in lemma 1 (5) does not follow from the definition of representability in def. 2.1. As briefly figured out in the introduction, it is a consequence of an implicit premise about the semantics of quantification, which is equivalent to the representability of a variable composition.

Under the interpretation $\ulcorner\varphi\urcorner\neq\ulcorner\psi(\overline{\ulcorner\varphi\urcorner})\urcorner$, the closed formula φ representing a predicate $p_\psi(k)$ has to be constructed as the representation of $p_\psi(\widetilde{y})$ composed with a function $g(k_0,\dots k_{n-1})$. This composition has to be representable as an $\mathcal{S}_{PA}$-formula with a formula $\phi\big(\overline{k}_0,\dots\overline{k}_{n-1}\big)$ representing $g(k_0,\dots k_{n-1})$ since a representation of $g(k_0,\dots k_{n-1})$ as a term can not be assured. By *defining* a semantic interpretation of *relativized quantifiers* (c.f. van Dalen 2013, p.75) the variable composition is syntactically representable as $\exists y(\psi\wedge\phi)$ and the next theorem can be proved[3].

[3] I am grateful to Christoph Kreitz who provided the semi-formal proof of theorem 1 to me.

Theorem 1. $\forall\psi\forall k\big(\vdash \psi(\overline{k}) \leftrightarrow \exists y(\psi \wedge y{\equiv}\overline{k})\big)$.

Let ψ represent a predicate $p(\widetilde{y})$ and let ϕ represent a predicate q.

Proposition: $\forall\psi\forall k\big(\vdash \psi(\overline{k}) \leftrightarrow \exists y(\psi \wedge y{\equiv}\overline{k})\big)$.

Proof.
1. Assume $\psi(\overline{k})$.
Under the interpretation that $\exists y(\psi \wedge \phi)$ represents the semantic:
k exists such that $\models p(k)$ and $\models q[k/\widetilde{x}]$
it follows $\vdash \big(\psi(y) \wedge y{\equiv}\overline{k}\big)\big[\overline{k}/y\big] \Rightarrow \exists y\big(\psi \wedge y{\equiv}\overline{k}\big)$.
2. Assume $\exists y\big(\psi \wedge y{\equiv}\overline{k}\big)$.
Under the interpretation as seen above, it follows:
an a exists such that $\models p(a)$ and $\models a = k$.
Let the semantic of identity be defined by
$a = k \;\;{=}\!\!|\!\!\models\;\; q(a) \Rightarrow q(k)$ for all q.
Then we have $\models p(a)$ and $\models p(a) \Rightarrow p(k)$
and with $\models p(k)$ it can be concluded $\exists y\big(\psi \wedge y{\equiv}\overline{k}\big) \Rightarrow \psi(\overline{k})$. $\dashv$

For the following theorem, let $[\,R(x,\vec{y})\,|\,\chi(f(\vec{x}), f_R(x,\vec{y}))\,/\!\!/\,x]$ represent a variable composition of $R(x,\vec{y})$ with the function $\chi(f(\vec{x}), f_R(x,\vec{y}))$ in x.

Considering recursive functions and relations Gödel (1931, p.181) proved the recursiveness of a relation $(\exists x)(R(x,\vec{y}) \wedge x{\leq}f(\vec{x}))$, in his notation $(Ex)\,[x{\leq}\varphi(\mathfrak{x})\;\&\;R(x,\mathfrak{y})]$ by constructing $R[\,\chi(\varphi(\mathfrak{x}),\mathfrak{y}),\mathfrak{y}]$:

Theorem 2. *Existence of a recursive definition of* $(\exists x)(R(x,\vec{y}) \wedge x{\leq}f(\vec{x}))$.

Proposition: For all recursive relations $R(x,\vec{y})$ and recursive functions $f(\vec{x})$, the relation $[\,R(x,\vec{y})\,|\,\chi(f(\vec{x}), f_R(x,\vec{y}))\,/\!\!/\,x]$ is recursive.

Proof.
Let $y_{\mathfrak{x}}$ be a variable different from x and not in the list of variables $\vec{x}$ and $\vec{y}$.
Let $f_\alpha(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ else} \end{cases}$ be a recursive function interpretable as negation.
Then a recursive function $\chi(y_{\mathfrak{x}}, f_R(x,\vec{y}))$ can be defined as follows:

1. $\chi(0, f_R(x,\vec{y})) = 0,$
2. $a :\cong f_\alpha(f_\alpha(f_R(0,\vec{y})) + f_R(x,\vec{y}) + \chi_R(n, f_R(x,\vec{y}))),$
3. $\chi(n{+}1, f_R(x,\vec{y})) = a\cdot(n{+}1) + f_\alpha(a)\cdot f_R(x,\vec{y}).$

With $\chi(y_{\mathfrak{x}}, f_R(x,\vec{y}))$ being a recursive function, the variable composition $[\,R(x,\vec{y})\,|\,\chi(f(\vec{x}), f_R(x,\vec{y}))\,/\!\!/\,x]$ is a recursive relation intended to be interpreted as equivalent to $(\exists x)(R(x,\vec{y}) \wedge x{\leq}f(\vec{x}))$ $\dashv$

The proof of theorem 2 implicitly assumes all recursive functions and relations to be closed under variable composition. Any proof of the recursion theorem has to be constructive since it is possible to consider a non-standard interpretation of a predicative logic avoiding undecidability (c.f. Solte 2020). The structure explained in section 4 provides the definition of an interpretation resulting in a canonical representation of functions and predicates. A predicate $\not{e}(\widetilde{x})$ as defined in the next def. 2.3 is representable.

Definition 2.3. *Predicative quantification.*
Let $\mathsf{P}(x)$ *represent a predicate* $\mathrm{p}(\widetilde{x})$.
Then let $\notin(x)$ *represent a predicate* $\not{e}(\widetilde{x})$ *such that*
$\models \neg \not{e}(\widetilde{x})$ *and* $\models \notin\left(\overline{\ulcorner \mathsf{P}(x) \urcorner}\right) \Dashv\models\ \models \neg p(k)$ *for all k.*

Remark: The negation symbol in the definition $\models \neg \not{e}(\widetilde{x})$ has intentionally been used.

Corollary 1. $\vdash \varphi \leftrightarrow \exists x(\varphi)$ *for all closed formulas* φ

Proof.
(1) Since φ is a closed formula, it is $\forall x \forall \overline{k}\left(\varphi \leftrightarrow \varphi\left[\,\overline{k}/x\right]\right)$.
(2) Assuming $\models \varphi$, it follows $\forall x(\varphi)$ and thus $\exists x(\varphi)$.
(3) Assuming $\models \exists x(\varphi)$, it follows $\exists \overline{k}\left(\varphi\left[\,\overline{k}/x\right]\right)$ and
with (1) it can be concluded $\models \varphi$.
(4) Assuming $\models \neg\varphi$, it follows $\forall x(\neg\varphi)$ and thus $\neg\exists x(\varphi)$.
(5) Assuming $\models \neg\exists x(\varphi)$, it follows $\forall \overline{k}\left(\neg\varphi\left[\,\overline{k}/x\right]\right)$ and
with (1) it can be concluded $\models \neg\varphi$.
(2) - (5) results in $\varphi \leftrightarrow \exists x(\varphi)$. $\dashv$

Lemma 2. $\nexists\psi\forall\varphi\left(\vdash \nexists x(\varphi) \leftrightarrow \psi\left(\overline{\ulcorner\varphi\urcorner}\right)\right) \Dashv\models\ \forall\psi\exists\varphi\left(\vdash \varphi \leftrightarrow \psi\left(\overline{\ulcorner\varphi\urcorner}\right)\right)$.

Proof.
$\nexists\psi\forall\varphi\left(\vdash \nexists x(\varphi) \leftrightarrow \psi\left(\overline{\ulcorner\varphi\urcorner}\right)\right) \Dashv\models\ \forall\psi\exists\varphi\left(\vdash \nexists x(\varphi) \not\leftrightarrow \psi\left(\overline{\ulcorner\varphi\urcorner}\right)\right)$
$\Dashv\models\ \forall\psi\exists\varphi\left(\vdash \exists x(\varphi) \leftrightarrow \psi\left(\overline{\ulcorner\varphi\urcorner}\right)\right)$ and with $\vdash \varphi \leftrightarrow \exists x(\varphi)$
$\Dashv\models\ \forall\psi\exists\varphi\left(\vdash \varphi \leftrightarrow \psi\left(\overline{\ulcorner\varphi\urcorner}\right)\right)$. $\dashv$

From lemma 2, it follows that the diagonal lemma cannot be proved by contradiction if a representation of $\not{e}(\widetilde{x})$ exists since the interpretation of $\notin(\overline{\ulcorner\varphi\urcorner})$ corresponds to $\nexists x(\varphi)$.

Summarizing the analysis, the root of undecidability is the stipulation that

- a constant interpretation of an indeterminate variable signature exists,
- all formulas are closed under indeterminate variable composition,

- an indeterminate variable signature is interpreted to represent a reference notation of a resolved formula, i.e., representations of substitutions and compositions in the resulting formula are not interpreted.

In the following section, this will be proved to be equivalent to the existence of fixed points.

3. Indeterminate variable composition and undecidability

The above proof of lemma 1 is very much similar to the proof of the recursion theorem of Kleene (cf., e.g., van Dalen 2013, pp.222-223) applied to unary functions with index e. To explain the similarity with respect to a substitutivity of variables, we formulate a recursion theorem of $\mathfrak{S}_\mu$-formulas adopting the notation in van Dalen (2013) with some modifications aiming to make explicit the different forms of substitution and including the concept of a canonical normal form. The defined terminology will be close to the terminology of the structure of recursive functions and relations. Nevertheless, the definitions are intentionally independent of any concrete structure that is represented in a formal language.

Definition 3.1. *$\mathfrak{S}_\mu$-formulas and indices*

A syntactical representation ϖ of a relation which will be notationally illustrated[4] *as* $\left[\ \lambda y.f_{[\![\langle\varpi\rangle]\!]_\mu}(\vec{\mathfrak{x}})\simeq y\right]$ *is named a* *$\mathfrak{S}_\mu$-formula.*

Let y denote an abstracting indeterminate in the interpretation of $\mathfrak{S}_\mu$-formulas. The lambda notation is used if an abstraction has to necessarily be considered. Usually, the explicit notation of the y-abstraction is omitted.

Let $\varphi_i^n(\vec{x})$ denote initial $\mathfrak{S}_\mu$-formulas $[\ f_i^n(\vec{x})\simeq y]$ variable in n parameters.

An index $e_\varpi :\cong \langle\varpi\rangle$ denotes an interpretable code of a $\mathfrak{S}_\mu$-formula ϖ.

$\langle\cdot\rangle$ is intended to represent a function that translates a notation into an interpreting structure.

Let $\overline{e}_\varpi$ denote the interpretation $f_{\mathfrak{C}}(\langle\varpi\rangle)$ of a dedicated formula $[\ f_{\mathfrak{C}}(\langle\varpi\rangle)\simeq y]$ as a constant.

Let x_ϖ denote the interpretation $f_{\mathfrak{X}}(\langle\varpi\rangle)$ of a dedicated formula $[\ f_{\mathfrak{X}}(\langle\varpi\rangle)\simeq y]$ as a variable. As usual, we use notations like $x_0 \ldots x_{x-1}$, etc. to denote variables.

Let $x_\square$ represent an indeterminate different from all variables x_ϖ.

Lemma 2 gives a reason that the recursion theorem cannot be proved by contradiction but has to be constructive. To make explicit that any proof of

[4] $\{\langle\varpi\rangle\}$ and $f_{[\![\langle\varpi\rangle]\!]_\mu}(\vec{\mathfrak{x}})$ are alternatively used notations to denote functions.

the recursion theorem requires the representability of an *indeterminate variable signature* and that all formulas have to be closed under variable composition, we formally define a representation of substitution and composition.

Definition 3.2. *Variable substitution and composition.*
Let $\left[\psi \,\middle|\, \overrightarrow{\langle\phi_{\mathfrak{x}}\rangle} \mathbin{/\!\!/} \vec{\mathfrak{x}}\right]$ *denote a variable substitution representing the replacement of variables* x_i *in the list of variables* $\vec{\mathfrak{x}}$ *in* ψ *by notations* $\langle\phi_i\rangle$.
Let $\left[\psi \,\middle|\, \overrightarrow{\{\langle\phi_{\mathfrak{x}}\rangle\}} \mathbin{/\!\!/} \vec{\mathfrak{x}}\right]$ *denote a variable composition representing the replacement of variables* x_i *in the list of variables* $\vec{\mathfrak{x}}$ *in* ψ *by notations* $\{\langle\phi_{\mathfrak{x}}\rangle\}$ *representing the variable interpretation of* $\langle\phi_i\rangle$ *as a function* f_{ϕ_i} *as explained in the next def. 3.3.*

Definition 3.3. *Variable interpretation of* $\mathfrak{S}_\mu$*-formulas.*
Let $[\{x_\square\}\simeq \mathsf{y}]$ *denote an indeterminate signature up to* $x_\square$ *representing a variable interpretation* $i_{\mathcal{S}}(x_\square)$ *resolving notations representing substitutions and compositions. Let* $[i_{\mathcal{S}}(x_\square)]$ *denote the representation of the result of* $i_{\mathcal{S}}(x_\square)$.
Let $f_{\Xi}(\langle\phi\rangle)$ *denote the index of the formula* $\left[\lambda \mathsf{y}.f_{[i_s(\langle\phi\rangle)]}(\vec{\mathfrak{x}})\simeq \mathsf{y}\right]$.
I.e., Ξ *denotes a* $\mathfrak{S}_\mu$*-formula* $[f_{\Xi}(x_\square)\simeq \mathsf{y}]$ *with index* e_Ξ *representing the variable interpretation resolving substitutions and compositions in a* $\mathfrak{S}_\mu$*-formula, i.e.,*
$f_{\Xi}(\langle\phi\rangle) :\cong \langle[i_{\mathcal{S}}(\langle\phi\rangle)]\rangle$.
Let $\mathfrak{S}_\mu(f_{\Xi}(\langle\phi\rangle))$ *denote the resolved formula.*
Let $\Xi(\langle\phi\rangle)$ *be named the variable interpretation of a formula.*
Let the variable interpretation Ξ *be defined inductively:*
For all $\mathfrak{S}_\mu$*-formulas* ψ, ϕ, *variables* r, v *and* $\mathfrak{s} :\cong \langle\phi\rangle$ *or* $\mathfrak{s} :\cong \{\langle\phi\rangle\}$ *let*

1. $\Xi(\langle\Xi(\langle[\phi \mid \mathfrak{s} \mathbin{/\!\!/} \mathsf{r}]\rangle)\rangle) :\cong \Xi(\langle[\phi \mid \mathfrak{s} \mathbin{/\!\!/} \mathsf{r}]\rangle)$,
2. $\Xi(\langle[\Xi(\langle\phi\rangle) \mid \mathfrak{s} \mathbin{/\!\!/} \mathsf{r}]\rangle) :\cong \Xi(\langle[\phi \mid \mathfrak{s} \mathbin{/\!\!/} \mathsf{r}]\rangle)$,
3. $\Xi(\langle[[\psi \mid f_{\mathfrak{C}}(\langle\varpi\rangle) \mathbin{/\!\!/} \mathsf{v}] \mid \mathfrak{s} \mathbin{/\!\!/} \mathsf{r}]\rangle) :\cong \Xi(\langle[\psi \mid f_{\mathfrak{C}}(\langle\varpi\rangle) \mathbin{/\!\!/} \mathsf{v}]\rangle)$,
4. $\Xi(\langle[[\psi \mid f_{\mathfrak{X}}(\langle\varpi\rangle) \mathbin{/\!\!/} \mathsf{v}] \mid \mathfrak{s} \mathbin{/\!\!/} \mathsf{r}]\rangle) :\cong \begin{cases} \Xi(\langle[\psi \mid \mathfrak{s} \mathbin{/\!\!/} \mathsf{v}]\rangle) \text{ if } \mathsf{r} :\cong f_{\mathfrak{X}}(\langle\varpi\rangle), \\ \Xi(\langle[\psi \mid f_{\mathfrak{X}}(\langle\varpi\rangle) \mathbin{/\!\!/} \mathsf{v}]\rangle) \text{ else}, \end{cases}$
5. $\Xi\left(\left\langle\left[\left[\psi \,\middle|\, \overrightarrow{\langle\phi_{\mathfrak{x}}\rangle} \mathbin{/\!\!/} \vec{\mathfrak{x}}\right] \,\middle|\, \mathfrak{s} \mathbin{/\!\!/} \mathsf{r}\right]\right\rangle\right) :\cong \Xi\left(\left\langle\left[\psi \,\middle|\, \overrightarrow{\langle\Xi(\langle[\phi_{\mathfrak{x}} \mid \mathfrak{s} \mathbin{/\!\!/} \mathsf{r}]\rangle)\rangle} \mathbin{/\!\!/} \vec{\mathfrak{x}}\right]\right\rangle\right)$,
6. $\Xi\left(\left\langle\left[\left[\psi \,\middle|\, \overrightarrow{\{\langle\phi_{\mathfrak{x}}\rangle\}} \mathbin{/\!\!/} \vec{\mathfrak{x}}\right] \,\middle|\, \mathfrak{s} \mathbin{/\!\!/} \mathsf{r}\right]\right\rangle\right) :\cong \Xi\left(\left\langle\left[\psi \,\middle|\, \overrightarrow{\{\langle\Xi(\langle[\phi_{\mathfrak{x}} \mid \mathfrak{s} \mathbin{/\!\!/} \mathsf{r}]\rangle)\rangle\}} \mathbin{/\!\!/} \vec{\mathfrak{x}}\right]\right\rangle\right)$.

Definition 3.4. *Equivalence interpretation of* $\mathfrak{S}_\mu$*-formulas.*
Let $[\![f_{\Xi}(\langle\phi\rangle)]\!]_\mu$ *denote an equivalence interpretation resulting in a reference notation or normal form of a formula.*

We refer to Kleene (1943, pp.51-52) for the definition of a normal form of recursive predicates.

In def. 3.3 above, we have focused on the main aspects of the variable interpretation relevant for the following explanations and refrain from documenting all other details of an inductive definition to resolve substitution and composition of initial formulas, alternative and short notations, e.g., as defined in the next def. 3.5.

Definition 3.5. *Replacements and substitutions.*
Defs. 3.1 - 3.3 allow us to define alternative and short notations:
For all variables r
let $[\,\psi \mid f_\phi(\vec{\mathfrak{x}}) \,/\!\!/\, \mathsf{r}] :\cong [\,\psi \mid \{\langle\Xi(\langle\phi\rangle)\rangle\} \,/\!\!/\, \mathsf{r}]$ *denote the representation of* ψ *composed with the resolved formula* ϕ *in variable* r*, i.e.,* $f_\phi(\vec{\mathfrak{x}}) :\cong \{\langle\Xi(\langle\phi\rangle)\rangle\}$,
let $[\,\psi \mid x_\varpi \,/\!\!/\, \mathsf{r}] :\cong [\,\psi \mid f_{\mathfrak{x}}(\langle\varpi\rangle) \,/\!\!/\, \mathsf{r}]$ *denote the representation of the replacement of variable* r *by the interpretation of* $\langle\varpi\rangle$ *as a variable* x_ϖ *and*
let $[\,\psi \mid \overline{e}_\varpi / \mathsf{r}] :\cong [\,\psi \mid f_{\mathfrak{e}}(\langle\varpi\rangle) \,/\!\!/\, \mathsf{r}]$ *denote the representation of the replacement of variable* r *by the interpretation of* $\langle\varpi\rangle$ *as a constant* $\overline{e}_\varpi$.
Let $\left[\,\psi \mid \vec{\overline{e}}_x / \vec{x}\right] :\cong \Xi\left(\left\langle\left[\,\psi \,\middle|\, \overrightarrow{\overline{f_{\mathfrak{e}}(\langle\phi_x\rangle)}} \,/\!\!/\, \vec{x}\right]\right\rangle\right)$ *denote the representation of a formula to be interpreted as* <u>*closed*</u> *for* ψ *being variable in* $\vec{x}$.
Let $\left[\,\psi \,\middle|\, \overline{f_\phi(\vec{\overline{e}}_x)} / \mathsf{r}\right] :\cong [\,\psi \mid f_{\mathfrak{e}}(\{\langle[\![\langle[\,\phi \mid \vec{\overline{e}}_x/\vec{x}]\rangle]\!]_\mu\rangle\}) \,/\!\!/\, \mathsf{r}]$ *denote the representation of the replacement of variable* r *by the interpretation of* $f_\phi\left(\vec{\overline{e}}_x\right)$ *as a constant.*

Definition 3.6. *Indeterminate variable* $\mathfrak{S}_\mu$*-signature and constant interpretation.*
Let $\mathfrak{S}_\chi(x_{\mathfrak{x}})[\,\vec{\mathfrak{x}}\,]$ *be a notation interpreted to be variable in* $x_{\mathfrak{x}}$ *and* $\vec{\mathfrak{x}}$.
If for all formulas ρ, ϕ*, variables* v *and* $\mathfrak{s} :\cong \langle\rho\rangle$ *or* $\mathfrak{s} :\cong \{\langle\rho\rangle\}$
$\Xi(\langle[\,\mathfrak{S}_\chi(x_{\mathfrak{x}})[\,\vec{\mathfrak{x}}\,] \mid \mathsf{v} \,/\!\!/\, x_{\mathfrak{x}}]\rangle) :\cong \mathfrak{S}_\chi(\mathsf{v})[\,\vec{\mathfrak{x}}\,]$
and for all variables r *such that* $\mathsf{r} :\not\cong \mathsf{v}$ *it is*
$\Xi(\langle\, [\![f_\equiv(\langle[\,[\,\mathfrak{S}_\chi(\mathsf{v})[\,\vec{\mathfrak{x}}\,] \mid \mathfrak{s} \,/\!\!/\, \mathsf{r}] \mid e_\rho \,/\!\!/\, \mathsf{v}]\rangle)]\!]_\mu\rangle) :\cong \Xi(\langle[\,[\,\mathfrak{S}_\chi(e_\rho)[\,\vec{\mathfrak{x}}\,] \mid \mathfrak{s} \,/\!\!/\, \mathsf{r}] \mid e_\rho \,/\!\!/\, \mathsf{v}]\rangle)$,
then $\mathfrak{S}_\chi(x_{\mathfrak{x}})[\,\vec{\mathfrak{x}}\,]$ *is named an* <u>*indeterminate variable* $\mathfrak{S}_\mu$*-signature*</u> *up to* $x_{\mathfrak{x}}$.
An indeterminate variable $\mathfrak{S}_\mu$*-signature* $\Xi_\mu(x_{\mathfrak{x}})[\,\vec{\mathfrak{x}}\,]$ *such that* $\Xi_\mu(e_\rho)[\,\vec{\mathfrak{x}}\,] :\cong \Xi(e_\rho)$ *is named* <u>*constant interpretation*</u> *up to* $x_{\mathfrak{x}}$.

Remark: Def. 3.3 (2.) with def. 3.6 provides the interpretation that a notation $\Xi_\mu(e_\rho)[\,\vec{\mathfrak{x}}\,]$ in $\Xi(\langle[\,[\,\Xi_\mu(e_\rho)[\,\vec{\mathfrak{x}}\,] \mid \mathfrak{s} \,/\!\!/\, \mathsf{r}] \mid e_\rho \,/\!\!/\, \mathsf{v}]\rangle)$ can be substituted by $\Xi(e_\rho)$. Assuming the existence of a constant interpretation is intended to be interpreted as a stipulation of an equivalence interpretation that attaches this semantic to a notation as, e.g., has been defined in def. 3.6.

A structure is briefly explained in section 4 that allows us to define an equivalence interpretation $[\![f_\equiv(\langle\phi\rangle)]\!]$ as the representation of a composition of a *canonic interpretation* $i_{[]}(\widetilde{x})$ and a *variable interpretation* $i_S(x_\Box)$ such that neither an indeterminate variable signature nor a constant interpretation exists. Consider that variable substitution and composition have been defined only for variables and not for the placeholder $x_{\square}$. $x_{\square}$ is intended to be interpreted by $i_{\mathcal{S}}(x_{\square})$ as the unique indeterminate of the interpreting formulas Ξ, $\mathfrak{c}$, $\mathfrak{x}$ and $\llbracket\rrbracket$.

Under the interpretation[5]: for all $\psi \models f_{\mathfrak{c}}(\langle\varphi\rangle) :\ncong f_{\mathfrak{c}}(\langle[\,\psi \mid f_{\mathfrak{c}}(\langle\varphi\rangle) \mathbin{/\!\!/} \mathsf{r}]\rangle)$, no formula φ exists being syntactically equivalent to $[\,\psi \mid \langle\varphi\rangle \mathbin{/\!\!/} \mathsf{r}]$, i.e., such that φ is the result of a variable interpretation of $[\,\psi \mid \langle\varphi\rangle \mathbin{/\!\!/} \mathsf{r}]$ resolving notations representing substitution or composition.

Thus it is $\varphi :\ncong \Xi(\langle[\,\psi \mid \langle\varphi\rangle \mathbin{/\!\!/} \mathsf{r}]\rangle)$ and a fixed-point φ has to be constructed indirectly by a $\mathfrak{S}_{\mu}$-formula θ such that an equivalence interpretation $\llbracket f_{\Xi}(\langle\theta\rangle)\rrbracket_{\mu}$ is equivalent to the variable interpretation of a formula representing the variable composition of ψ with θ in r. As a consequence, it can be concluded

$\vdash \varphi \leftrightarrow \psi(\overline{\ulcorner\varphi\urcorner}) \Longleftrightarrow \models \Xi\big(\big\langle \llbracket f_{\Xi}(\langle\theta\rangle)\rrbracket_{\mu}\big\rangle\big) :\cong \Xi(\langle[\,\psi \mid \{\langle\theta\rangle\} \mathbin{/\!\!/} \mathsf{r}]\rangle)$.

The following recursion theorem for $\mathfrak{S}_{\mu}$-formulas can be proved if the existence of indeterminate variable signatures and a constant interpretation $\sigma_{\mu}(x_{\mathfrak{x}}) :\cong [\,\Xi_{\mu}(x_{\mathfrak{x}})[\,\vec{\mathfrak{x}}\,] \mid x_{\mathfrak{x}} \mathbin{/\!\!/} \mathsf{r}]$ up to $x_{\mathfrak{x}}$ closed under variable substitution in $x_{\mathfrak{x}}$ are assumed. $[\,\Xi_{\mu}(x_{\mathfrak{x}})[\,\vec{\mathfrak{x}}\,] \mid x_{\mathfrak{x}} \mathbin{/\!\!/} \mathsf{r}]$ intends to assure that $\sigma_{\mu}(x_{\mathfrak{x}})$ does not have the variable r in its signature.

THEOREM 3. *Recursion theorem for $\mathfrak{S}_{\mu}$-formulas.*
Let ψ denote a $\mathfrak{S}_{\mu}$-formula with $\mathfrak{n}$ variables $\vec{x}$ different from $x_{\mathfrak{x}}$ such that[6]
for all ϕ it is $[\,[\,\psi \mid \{\langle\phi\rangle\} \mathbin{/\!\!/} \mathsf{r}] \mid e_{\varpi} \mathbin{/\!\!/} x_{\mathfrak{x}}] :\cong [\,\psi \mid \{\langle[\,\phi \mid e_{\varpi} \mathbin{/\!\!/} x_{\mathfrak{x}}]\rangle\} \mathbin{/\!\!/} \mathsf{r}]$.
Let a constant interpretation $\sigma_{\mu}(x_{\mathfrak{x}}) :\cong [\,\Xi_{\mu}(x_{\mathfrak{x}})[\,\vec{\mathfrak{x}}\,] \mid x_{\mathfrak{x}} \mathbin{/\!\!/} \mathsf{r}]$ up to $x_{\mathfrak{x}}$ closed under variable substitution in $x_{\mathfrak{x}}$ exist.
Proposition: For all ψ it exists an index $\langle\theta\rangle$ such that
$\Xi\big(\big\langle \llbracket f_{\Xi}(\langle\theta\rangle)\rrbracket_{\mu}\big\rangle\big) :\cong \Xi(\langle[\,\psi \mid \{\langle\theta\rangle\} \mathbin{/\!\!/} \mathsf{r}]\rangle)$.

PROOF.
Put $e_{\rho} :\cong \langle[\,\psi \mid \{\langle\sigma_{\mu}(x_{\mathfrak{x}})\rangle\} \mathbin{/\!\!/} \mathsf{r}]\rangle$ and $\theta :\cong \Xi(\langle[\,\sigma_{\mu}(x_{\mathfrak{x}}) \mid e_{\rho} \mathbin{/\!\!/} x_{\mathfrak{x}}]\rangle)$ such that
$\Xi\big(\big\langle \llbracket f_{\Xi}(\langle\theta\rangle)\rrbracket_{\mu}\big\rangle\big) :\cong \Xi\big(\big\langle \llbracket f_{\Xi}(\langle\Xi(\langle[\,\sigma_{\mu}(x_{\mathfrak{x}}) \mid e_{\rho} \mathbin{/\!\!/} x_{\mathfrak{x}}]\rangle)\rangle)\rrbracket_{\mu}\big\rangle\big)$.
Due to def. 3.3 (1.) it can be concluded
$\Xi\big(\big\langle \llbracket f_{\Xi}(\langle\Xi(\langle[\,\sigma_{\mu}(x_{\mathfrak{x}}) \mid e_{\rho} \mathbin{/\!\!/} x_{\mathfrak{x}}]\rangle)\rangle)\rrbracket_{\mu}\big\rangle\big) :\cong \Xi\big(\big\langle \llbracket f_{\Xi}(\langle[\,\sigma_{\mu}(x_{\mathfrak{x}}) \mid e_{\rho} \mathbin{/\!\!/} x_{\mathfrak{x}}]\rangle)\rrbracket_{\mu}\big\rangle\big)$.
Under the interpretation due to defs. 3.3 and 3.6 it is
$\Xi\big(\big\langle \llbracket f_{\Xi}(\langle[\,\sigma_{\mu}(x_{\mathfrak{x}}) \mid e_{\rho} \mathbin{/\!\!/} x_{\mathfrak{x}}]\rangle)\rrbracket_{\mu}\big\rangle\big) :\cong \Xi(\langle[\,\Xi(\langle[\,\psi \mid \{\langle\sigma_{\mu}(x_{\mathfrak{x}})\rangle\} \mathbin{/\!\!/} \mathsf{r}]\rangle) \mid e_{\rho} \mathbin{/\!\!/} x_{\mathfrak{x}}]\rangle)$.

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[5] this interpretation is comparable to $\ulcorner\varphi\urcorner \neq \ulcorner\psi(\overline{\ulcorner\varphi\urcorner})\urcorner$ for any substitution of ψ and φ.
[6] It is up to the definition of the variable interpretation (c.f. defs. 3.2 and 3.3) whether
$[\,[\,\psi \mid \{\langle\phi\rangle\} \mathbin{/\!\!/} \mathsf{r}] \mid e_{\varpi} \mathbin{/\!\!/} x_{\mathfrak{x}}] :\cong [\,\psi \mid \{\langle[\,\phi \mid e_{\varpi} \mathbin{/\!\!/} x_{\mathfrak{x}}]\rangle\} \mathbin{/\!\!/} \mathsf{r}]$ as it has been defined above or
$[\,[\,\psi \mid \{\langle\phi\rangle\} \mathbin{/\!\!/} \mathsf{r}] \mid e_{\varpi} \mathbin{/\!\!/} x_{\mathfrak{x}}] :\cong [\,[\,\psi \mid e_{\varpi} \mathbin{/\!\!/} x_{\mathfrak{x}}] \mid \{\langle[\,\phi \mid e_{\varpi} \mathbin{/\!\!/} x_{\mathfrak{x}}]\rangle\} \mathbin{/\!\!/} \mathsf{r}]$

From def. 3.3 (2.) it follows
$\Xi(\langle[\,\Xi(\langle[\,\psi\,|\,\{\langle\sigma_\mu(x_\mathfrak{x})\rangle\}\,/\!\!/\,\mathsf{r}]\rangle)\,|\,e_\rho\,/\!\!/\,x_\mathfrak{x}]\rangle) :\cong \Xi(\langle[\,[\,\psi\,|\,\{\langle\sigma_\mu(x_\mathfrak{x})\rangle\}\,/\!\!/\,\mathsf{r}]\,|\,e_\rho\,/\!\!/\,x_\mathfrak{x}]\rangle)$.
From def. 3.3 (6.) it follows
$\Xi(\langle[\,[\,\psi\,|\,\{\langle\sigma_\mu(x_\mathfrak{x})\rangle\}\,/\!\!/\,\mathsf{r}]\,|\,e_\rho\,/\!\!/\,x_\mathfrak{x}]\rangle) :\cong \Xi(\langle[\,\psi\,|\,\{\langle\Xi(\langle[\,\sigma_\mu(x_\mathfrak{x})\,|\,e_\rho\,/\!\!/\,x_\mathfrak{x}]\rangle)\rangle\}\,/\!\!/\,\mathsf{r}]\rangle)$.
Due to the definition of θ it follows
$\Xi(\langle[\,\psi\,|\,\{\langle\Xi(\langle[\,\sigma_\mu(x_\mathfrak{x})\,|\,e_\rho\,/\!\!/\,x_\mathfrak{x}]\rangle)\rangle\}\,/\!\!/\,\mathsf{r}]\rangle) :\cong \Xi(\langle[\,\psi\,|\,\{\langle\theta\rangle\}\,/\!\!/\,\mathsf{r}]\rangle)$ concluding
$\Xi\big(\big\langle\,\llbracket f_\Xi(\langle\theta\rangle)\rrbracket_\mu\big\rangle\big) :\cong \Xi(\langle[\,\psi\,|\,\{\langle\theta\rangle\}\,/\!\!/\,\mathsf{r}]\rangle)$. ⊣

Finally, it will be proved that the representability of a constant interpretation $[\,\Xi_\mu(x_\mathfrak{x})[\,\vec{\mathfrak{x}}\,]\,|\,x_\mathfrak{x}\,/\!\!/\,\mathsf{r}]$ follows from the proposition of theorem 3.

Theorem 4. *Existence of a constant interpretation.*
Let ψ denote a $\mathfrak{S}_\mu$-formula with n variables $\vec{x}$ different from $x_\mathfrak{x}$.
Let for all ψ an index $\langle\theta\rangle$ exist such that $\Xi\big(\big\langle\,\llbracket f_\Xi(\langle\theta\rangle)\rrbracket_\mu\big\rangle\big) :\cong \Xi\big(\big\langle[\,\psi\,|\,\{\langle\theta\rangle\}\,/\!\!/\,\mathsf{r}]\big\rangle\big)$
Proposition: A constant interpretation $[\,\Xi_\mu(x_\mathfrak{x})[\,\vec{\mathfrak{x}}\,]\,|\,x_\mathfrak{x}\,/\!\!/\,\mathsf{r}]$ closed under variable substitution in $x_\mathfrak{x}$ is representable.

Proof.
In the most general case, θ can be considered to be a $\mathfrak{S}_\mu$-formula σ_μ with n variables $\vec{x}$ with a variable, e.g., r substituted by an e_ϱ, i.e., $\theta :\cong \Xi(\langle[\,\sigma_\mu\,|\,e_\varrho\,/\!\!/\,\mathsf{r}]\rangle)$.
Renaming variable r into $x_\mathfrak{x}$ avoids conflicts resolving the substitution of r in ψ. Thus let $\sigma_\mu(x_\mathfrak{x}) :\cong \Xi(\langle[\,\sigma_\mu\,|\,x_\mathfrak{x}\,/\!\!/\,\mathsf{r}]\rangle)$ and consider $\langle\theta\rangle :\cong \langle\Xi(\langle[\,\sigma_\mu(x_\mathfrak{x})\,|\,e_\varrho\,/\!\!/\,x_\mathfrak{x}]\rangle)\rangle$ with $f_{\sigma_\mu}(e_\varrho) :\cong \{\langle\theta\rangle\}$.
Due to def. 3.3 (1.) it is $\Xi(\langle\Xi(\langle\phi\rangle)\rangle) :\cong \Xi(\langle\phi\rangle)$.
Substituting $\langle\theta\rangle$ it can be deduced[7]
$\Xi\big(\big\langle\,\llbracket f_\Xi(\langle\theta\rangle)\rrbracket_\mu\big\rangle\big) :\cong \Xi(\langle[\,\psi\,|\,\{\langle\theta\rangle\}\,/\!\!/\,\mathsf{r}]\rangle)$
$\iff$

$$\Xi\big(\big\langle\,\llbracket f_\Xi([\,\sigma_\mu(x_\mathfrak{x})\,|\,e_\varrho\,/\!\!/\,x_\mathfrak{x}])\rrbracket_\mu\big\rangle\big) :\cong \Xi(\langle[\,\psi\,|\,\{\langle\Xi(\langle[\,\sigma_\mu(x_\mathfrak{x})\,|\,e_\varrho\,/\!\!/\,x_\mathfrak{x}]\rangle)\rangle\}\,/\!\!/\,\mathsf{r}]\rangle). \quad \text{(I)}$$

Hence it has to be $\llbracket f_\Xi(\langle[\,\sigma_\mu(x_\mathfrak{x})\,|\,e_\varrho\,/\!\!/\,x_\mathfrak{x}]\rangle)\rrbracket_\mu \overset{!}{:\cong} \Xi(\langle[\,\sigma_\mu(x_\mathfrak{x})\,|\,e_\varrho\,/\!\!/\,x_\mathfrak{x}]\rangle)$
and thus substituting $\llbracket f_\Xi(\langle[\,\sigma_\mu(x_\mathfrak{x})\,|\,e_\varrho\,/\!\!/\,x_\mathfrak{x}]\rangle)\rrbracket_\mu$ in (I) it has to be
$\Xi(\langle\Xi(\langle[\,\sigma_\mu(x_\mathfrak{x})\,|\,e_\varrho\,/\!\!/\,x_\mathfrak{x}]\rangle)\rangle) \overset{!}{:\cong} \Xi(\langle[\,\psi\,|\,\{\langle\Xi(\langle[\,\sigma_\mu(x_\mathfrak{x})\,|\,e_\varrho\,/\!\!/\,x_\mathfrak{x}]\rangle)\rangle\}\,/\!\!/\,\mathsf{r}]\rangle)$
Due to def. 3.3 (1.), (6.) and (2.) this is equivalent to
$\Xi(\langle[\,\sigma_\mu(x_\mathfrak{x})\,|\,e_\varrho\,/\!\!/\,x_\mathfrak{x}]\rangle) \overset{!}{:\cong} \Xi(\langle[\,\Xi(\langle[\,\psi\,|\,\{\langle\sigma_\mu(x_\mathfrak{x})\rangle\}\,/\!\!/\,\mathsf{r}]\rangle)\,|\,e_\varrho\,/\!\!/\,x_\mathfrak{x}]\rangle)$
This represents the semantic that
the variable interpretation of $[\,\sigma_\mu(x_\mathfrak{x})\,|\,e_\varrho\,/\!\!/\,x_\mathfrak{x}]$
has to be equivalent to
the variable interpretation of $[\,\Xi(\langle[\,\psi\,|\,\{\langle\sigma_\mu(x_\mathfrak{x})\rangle\}\,/\!\!/\,\mathsf{r}]\rangle)\,|\,e_\varrho\,/\!\!/\,x_\mathfrak{x}]$.

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[7] Let $\varpi_{Y^*} \overset{!}{:\cong} \varpi_{X^*}$ denote ”there has to be an isomorphism of notations ϖ_{Y^*} and ϖ_{X^*}”

The only option to achieve this equivalence is to define
$e_\varrho :\cong \langle[\,\psi \mid \{\langle\sigma_\mu(x_{\mathfrak{x}})\rangle\} \sslash \mathsf{r}]\rangle$ considering a representability of
$[\,\sigma_\mu(x_{\mathfrak{x}}) \mid e_\varrho \sslash x_{\mathfrak{x}}] :\cong [\,\Xi(e_\varrho) \mid e_\varrho \sslash x_{\mathfrak{x}}]$ which is equivalent to the existence of a constant interpretation $[\,\Xi_\mu(x_{\mathfrak{x}})[\,\vec{\mathfrak{x}}\,] \mid x_{\mathfrak{x}} \sslash \mathsf{r}]$ up to $x_{\mathfrak{x}}$ such that from

$[\,\Xi(e_\varrho) \mid e_\varrho \sslash x_{\mathfrak{x}}] :\cong [\,[\,\Xi_\mu(x_{\mathfrak{x}})[\,\vec{\mathfrak{x}}\,] \mid x_{\mathfrak{x}} \sslash \mathsf{r}] \mid e_\varrho \sslash x_{\mathfrak{x}}]$

it follows

$\Xi(\langle\, [\![f_{\Xi}(\langle[\,\sigma_\mu(x_{\mathfrak{x}}) \mid e_\varrho \sslash x_{\mathfrak{x}}]\rangle)]\!]_\mu \rangle) :\cong \Xi(\langle[\,\psi \mid \{\langle\theta\rangle\} \sslash \mathsf{r}]\rangle)$

which is equivalent to the conclusion that it has to be

$\sigma_\mu(x_{\mathfrak{x}}) \overset{!}{:\cong} [\,\Xi_\mu(x_{\mathfrak{x}})[\,\vec{\mathfrak{x}}\,] \mid x_{\mathfrak{x}} \sslash \mathsf{r}]$ and $e_\varrho \overset{!}{:\cong} \langle[\,\psi \mid \{\langle\sigma_\mu(x_{\mathfrak{x}})\rangle\} \sslash \mathsf{r}]\rangle$

and thus the representation $\sigma_\mu(x_{\mathfrak{x}})$ of a constant interpretation has to exist. $\dashv$

4. Computable relations beyond recursion

As explained in the previous section, the existence of a representation of an indeterminate variable signature, a constant interpretation and all formulas to be closed under variable substitution and composition, is equivalent to the fixed-point theorems. The recursion theorem for $\mathfrak{S}_\mu$-formulas holds for structures providing a constant interpretation of an indeterminate variable signature. This raises the question of how to define a structure that avoids undecidability but includes the class of predicates definable as the roots of partial recursive functions. An answer is given next by explaining

- a basal structure $(\mathcal{D}, \Updownarrow, \oplus, \otimes, \ominus)$ of elements named *binerals* and
- a conceptual interpretation of binerals.

We illustrate the structure $(\mathcal{D}, \Updownarrow, \oplus, \otimes, \ominus)$ by referencing a usual notation of a language $L^{\mathcal{S}_{PA}}$ itbi (abbreviation for "intended to be interpreted") with a standard interpretation $\mathfrak{I}_{\mathbb{N}}$ in the structure $(\mathbb{N}, +, \cdot, -, =)$.

Definition 4.1. *Binerals and the function* $\Updownarrow$.

Binerals $\mathfrak{d}_{\mathfrak{n}}$ *are the elements of a universe* $\mathcal{D}$ *itbi as a set of disjunct representations* $\dot{\mathfrak{n}}$ *and* $\overline{\mathfrak{n}}$ *of* $n \in \mathbb{N}$*, i.e.,*

- $n \in \mathbb{N} \mathrel{=\!\!|\!\!\models} \dot{\mathfrak{n}} \in \dot{\mathbb{N}}$,
- $\dot{\mathfrak{n}} \in \dot{\mathbb{N}} \mathrel{=\!\!|\!\!\models} \overline{\mathfrak{n}} \in \overline{\mathbb{N}}$,
- $\dot{\mathfrak{n}} \in \dot{\mathbb{N}} \mathrel{=\!\!|\!\!\models} \dot{\mathfrak{n}} \notin \overline{\mathbb{N}}$.

To illustrate the intended interpretation of $\mathfrak{d}_{\mathfrak{n}} \Updownarrow \mathfrak{d}_{\mathfrak{m}}$ *consider a signed number representation* $(\mathcal{I}_S(\mathfrak{d}_{\mathfrak{n}}), \not{b}_{\mathfrak{d}_{\mathfrak{n}}}) :\cong |I(\mathfrak{n})|_{\mathfrak{d}_{\mathfrak{n}}} \ldots 2_{\mathfrak{d}_{\mathfrak{n}}} 1_{\mathfrak{d}_{\mathfrak{n}}} \not{b}_{\mathfrak{d}_{\mathfrak{n}}}$ *of a bineral* $\mathfrak{d}_{\mathfrak{n}}$ *as an infinite binary band with a default symbol* $\not{b}_{\mathfrak{d}_{\mathfrak{n}}}$ *on the left end.*

Let $\left(\underline{\mathcal{I}_S(\mathfrak{d}_{\mathfrak{n}})}, \widetilde{\not{b}_{\mathfrak{d}_{\mathfrak{n}}}}\right)$ *denote the ones' complement of* $(\mathcal{I}_S(\mathfrak{d}_{\mathfrak{n}}), \not{b}_{\mathfrak{d}_{\mathfrak{n}}})$.

Let $I(\mathfrak{n})$ denote a set[8] *to reference the elements*[9] $s_{d_\mathfrak{n}} \in \{\varnothing, \circ\}$ *at* *position* $s \in I(\mathfrak{n})$.

Let $|I(\mathfrak{n})|$ denote the cardinality of $I(\mathfrak{n})$ itbi as the number of binary digits to represent a number n.

Consider $d_\mathfrak{n} \Uparrow d_\mathfrak{m} :\cong [\![(\ldots ƀ_{d_\mathfrak{n}} \mathcal{I}_S(d_\mathfrak{n}) \Uparrow \ldots ƀ_{d_\mathfrak{m}} \mathcal{I}_S(d_\mathfrak{m}))(ƀ_{d_\mathfrak{n}} \Uparrow ƀ_{d_\mathfrak{m}})]\!]$ *resulting in the bitwise connection of binaries* $B_X \Uparrow B_Y$, *skipping symbols at the left end equal to* $(ƀ_{d_\mathfrak{n}} \Uparrow ƀ_{d_\mathfrak{m}})$.

$\{\Uparrow\}$ *is intended to be functionally complete wrt. a boolean algebra.*

Let $\neg d_\mathfrak{n} :\cong d_\mathfrak{n} \Uparrow d_\mathfrak{n}$ *be an abbreviating notation for which* $\neg \dot{\mathfrak{n}} :\cong \overline{\mathfrak{n}}$ *and* $\neg \overline{\mathfrak{n}} :\cong \dot{\mathfrak{n}}$.

The structure $(\mathcal{D}, \oplus, \otimes, \ominus)$ will be defined in a way such that $d_\mathfrak{n} \Uparrow d_\mathfrak{m}$ is not representable in $(\mathcal{D}, \oplus, \otimes, \ominus)$ for all $d_\mathfrak{n} :\ncong d_\mathfrak{m}$.

Definition 4.2. *Functions $\oplus$, $\otimes$ and $\ominus$ on binerals.*

Let τ denote a term of $L^{S_{PA}}$ without variables itbi in $(\mathbb{N}, +, \cdot, \dot{-}, =)$.

Let $0, 1, n$ *and* m *denote decimal representations of elements of* $\mathbb{N}$.

Let $\mathfrak{I}_\mathbb{N}(|I_\chi|)$ *denote the cardinality of an indexset* I_χ.

Let $\omega :\cong \mathfrak{I}_\mathbb{N}(|I_\mathbb{N}|)$, *i.e., let ω denote the ordinal of* $\mathbb{N}$.

Let $\dot{\mathfrak{I}}_\mathbb{N}(|\tau|)$ *denote the bineral* $\dot{\mathfrak{n}}$ *for which* $n = |\mathfrak{I}_\mathbb{N}(\tau)|$ *and*
let $\overline{\mathfrak{I}}_\mathbb{N}(|\tau|)$ *denote the bineral* $\overline{\mathfrak{n}}$ *for which* $n = |\mathfrak{I}_\mathbb{N}(\tau)|$.

We illustrate the intended interpretation of the functions on $\mathcal{D}$:

$\oplus$ *is defined inductively:*

1. $\dot{\mathfrak{n}} \oplus \dot{\mathfrak{m}} :\cong \dot{\mathfrak{I}}_\mathbb{N}(|n+m|)$
2. $\overline{\mathfrak{n}} \oplus \dot{\mathfrak{m}} :\cong \begin{cases} \overline{\mathfrak{I}}_\mathbb{N}(|n \dot{-} m|) & \text{if } n \geq m \\ \dot{\mathfrak{I}}_\mathbb{N}(|m \dot{-} n \dot{-} 1|) & \text{else} \end{cases}$
3. $\overline{\mathfrak{n}} \oplus \overline{\mathfrak{m}} :\cong \overline{\mathfrak{I}}_\mathbb{N}(|n+m+1|)$

$\dot{\mathfrak{o}}$ *is named the* *additive identity.*

$\otimes$ *is defined inductively:*

1. $\dot{\mathfrak{n}} \otimes \dot{\mathfrak{m}} :\cong \dot{\mathfrak{I}}_\mathbb{N}(|n \cdot m|)$
2. $\overline{\mathfrak{n}} \otimes \dot{\mathfrak{m}} :\cong \overline{\mathfrak{o}} \otimes \left(\dot{\mathfrak{I}}_\mathbb{N}(|n+1|) \otimes \dot{\mathfrak{m}}\right)$ *if* $n > 0$
3. $\overline{\mathfrak{o}} \otimes \dot{\mathfrak{o}} :\cong \dot{\mathfrak{o}}$ — *itbi as* $(\omega \dot{-} 1) \cdot \omega$
4. $\overline{\mathfrak{o}} \otimes \dot{\mathfrak{m}} :\cong \overline{\mathfrak{I}}_\mathbb{N}(|m \dot{-} 1|)$ — *itbi as* $\omega \dot{-} m$
5. $\overline{\mathfrak{n}} \otimes \overline{\mathfrak{m}} :\cong \overline{\mathfrak{o}} \otimes \dot{\mathfrak{I}}_\mathbb{N}(|n+1|) \otimes \overline{\mathfrak{o}} \otimes \dot{\mathfrak{I}}_\mathbb{N}(|m+1|)$
6. $\overline{\mathfrak{o}} \otimes \overline{\mathfrak{o}} :\cong \mathfrak{i}$ — *itbi as* $\omega \dot{-} (\omega \dot{-} 1)$

$\mathfrak{i}$ *is named the* *multiplicative identity, i.e., the* *neutral element wrt.* $\otimes$.

[8] E.g., $I(\mathfrak{n}) :\cong \{i \mid i \leq 2^m \text{ and } 2^{m-1} < \mathfrak{n} \leq 2^m\}$. $s_{d_\mathfrak{n}}$ is itbi as the "symbol at the position currently under the head".

[9] We interpret $\circ :\cong \top$ and as a binary 1, $\varnothing :\cong \perp$ and as a binary 0.

$(\ominus d_{\mathfrak{n}}) :\cong \overline{\mathfrak{o}} \otimes d_{\mathfrak{n}}$ *defines the* <u>*inverse element wrt.* $\oplus$</u>*, i.e.,*

1. $(\ominus \dot{\mathfrak{n}}) :\cong \begin{cases} \overline{\mathfrak{I}}_{\mathbb{N}}(|n \dot{-} 1|) \text{ if } n > 0, \\ \dot{\mathfrak{o}}. \end{cases}$

2. $(\ominus \overline{\mathfrak{n}}) :\cong \dot{\mathfrak{I}}_{\mathbb{N}}(|n+1|)$

From the definition of $\oplus$ (2.), it follows $\dot{\mathfrak{I}}_{\mathbb{N}}(|m \dot{-} 1|) :\cong \overline{\mathfrak{o}} \oplus \dot{\mathfrak{m}}$.
From the definition of $\otimes$ (2.), it follows $\overline{\mathfrak{n}} \otimes \dot{\mathfrak{o}} :\cong \dot{\mathfrak{o}}$ itbi as $(\omega \dot{-} (n+1)) \cdot \omega$.
From the definition of $\ominus$, it follows $(\ominus \overline{\mathfrak{o}}) :\cong \mathtt{i}$.

The element $\overline{\mathfrak{o}}$ allows us to represent a function in the structure $(\mathcal{D}, \oplus, \otimes, \ominus)$ equivalent to $\neg d_{\mathfrak{n}}$ considered to denote a negation.

Lemma 3. *For all* $d_{\mathfrak{n}} \in \mathcal{D} : \neg d_{\mathfrak{n}} :\cong \overline{\mathfrak{o}} \otimes (d_{\mathfrak{n}} \oplus \mathtt{i})$

Proof.
Case 1: $d_{\mathfrak{n}} :\cong \dot{\mathfrak{I}}_{\mathbb{N}}(|n|)$.
Then it is $d_{\mathfrak{n}} \oplus \mathtt{i} :\cong \dot{\mathfrak{I}}_{\mathbb{N}}(||n|+1|)$
and thus $\overline{\mathfrak{o}} \otimes (d_{\mathfrak{n}} \oplus \mathtt{i}) :\cong \overline{\mathfrak{I}}_{\mathbb{N}}(||n|+1 \dot{-} 1|) :\cong \overline{\mathfrak{I}}_{\mathbb{N}}(|n|)$.
Case 2: $d_{\mathfrak{n}} :\cong \overline{\mathfrak{I}}_{\mathbb{N}}(|n|)$.

$$\text{Then it is } d_{\mathfrak{n}} \oplus \mathtt{i} :\cong \begin{cases} \overline{\mathfrak{I}}_{\mathbb{N}}(||n| \dot{-} 1|) \text{ if } |n| > 1, \\ \overline{\mathfrak{o}} \text{ if } |n| = 1, \\ \dot{\mathfrak{I}}_{\mathbb{N}}(|1 \dot{-} |n| \dot{-} 1|) \text{ if } |n| = 0 \end{cases}$$

$$\text{and thus } \overline{\mathfrak{o}} \otimes (d_{\mathfrak{n}} \oplus \mathtt{i}) :\cong \begin{cases} \overline{\mathfrak{o}} \otimes \overline{\mathfrak{o}} \otimes \dot{\mathfrak{I}}_{\mathbb{N}}(||n| \dot{-} 1 + 1|) :\cong \dot{\mathfrak{I}}_{\mathbb{N}}(|n|) \text{ if } |n| > 1, \\ \overline{\mathfrak{o}} \otimes \overline{\mathfrak{o}} :\cong \mathtt{i} \text{ if } |n| = 1, \\ \overline{\mathfrak{o}} \otimes \dot{\mathfrak{I}}_{\mathbb{N}}(|n|) :\cong \dot{\mathfrak{o}} \text{ if } |n| = 0. \end{cases}$$ ⊣

An advantage of the structure $\mathcal{D}$ is the existence of a canonical normal form of formulas ϕ interpreted in $\mathcal{D}$ such that this normal form can be interpreted as a signed element, i.e., a tuple $(sgn(\phi), abs(\phi))$. Briefly explained, consider in a first step formulas of a structure $(\mathcal{D}, \oplus, \otimes, \ominus)$ and a structure $(\mathcal{D}, \dagger)$ separately. The canonical normal form of $(\mathcal{D}, \oplus, \otimes, \ominus)$ can be derived in analogy to a polynomial normal form in a monomial order. The canonical normal form of $(\mathcal{D}, \dagger)$ can be derived in analogy to an algebraic normal form. Now consider canonical forms with the signature of $(\mathcal{D}, \dagger)$ as variables in $(\mathcal{D}, \oplus, \otimes, \ominus)$ and vice versa to derive the canonical normal form of a formula of $(\mathcal{D}, \dagger, \oplus, \otimes, \ominus)$. Under this interpretation, let $\vec{d_{\mathfrak{n}x}}$ denote an infinite list of elements of $\mathcal{D}$ and consider a formula ϕ as the representation of a relation $\left\{ \left(\vec{d_{\mathfrak{n}x}}, \mathsf{y} \right) \middle| \mathsf{y} \in \left[\phi \middle| \vec{d_{\mathfrak{n}x}} /\!\!/ \vec{x} \right] :\cong sgn\left(\left[\phi \middle| \vec{d_{\mathfrak{n}x}} /\!\!/ \vec{x} \right] \right) \right\}$ for some formula $\in$ that interprets $\left[\phi \middle| \vec{d_{\mathfrak{n}x}} /\!\!/ \vec{x} \right]$ as the representation of a characteristic function.

The definition and interpretation of a formal language in $(\mathcal{D}, \phi, \oplus, \otimes, \ominus)$ will be documented in a separate paper.

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