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ON SOME PROPERTIES OF (2 + 1)–DIMENSIONAL PERTURBED KdV EQUATION

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Explicit solitary-wave solutions of the (2 + 1)–dimensional perturbed KdV equation obtained by Ma et al. are obtained by using the Backlund transformation. Next, we obtain the two-soliton solution in a form which implies a form of superposition. Lastly, the corresponding modified equations are obtained by a new form of the Miura map.

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1. Introduction

In a recent communication, Ma and Fuchssteiner [1] have shown that a simple scaling of the space-time coordinates and a perturbation of the nonlinear field itself can lead to a (2 + 1)-dimensional form of perturbed KdV equation. It is also possible to derive an extended form of the Lax form for this new equation. In this communication, we have solved the Backlund transformation to obtain the explicit soliton solution of this (2 + 1)-dimensional problem. Later, a technique due to Hirota et al. [2] is used to construct the two-soliton solution which exhibits a superposition-like character. Lastly, the extended Miura transformation [3] is utilised to construct the modified version of this (2 + 1)-dimensional equation [4].

2. Formulation

Let us review the salient features of the (2 + 1)–dimensional perturbed KdV equation. Consider the KdV equation

$$u_t + u_{xxx} + 12uu_x = 0\,, \tag{1}$$

and consider the change of variables

$$(x,y) \to (x,y,t), \quad y = \epsilon x\,, \quad \text{and} \quad u \to u + \epsilon v\,, \tag{2}$$

ϵ being a small parameter. Then to the first order in ϵ, it was demonstrated by Ma and Fuchssteiner [1] that one gets the following set of coupled equations

$$\begin{aligned} u_t + u_{xxx} + 12uu_x = 0\,, \\ v_t + v_{xxx} + 3(u_{xx} + 2u^2)_y + 12(uv)_x = 0\,, \end{aligned} \tag{3}$$

which is called the (2 + 1)–dimensional perturbed KdV problem. Ma et al. proved many important properties of (1), such as the existence of the recursion operator, hereditary symmetry and so on. On the other hand, in a recent communication, Sakovich [5] showed how from the Lax pair of the original KdV equations one can deduce a Lax operator for the new equation applying the procedure. But a new feature of this new Lax operator is the existence of two spectral parameters (α, β) which are not constant, but obey the relations

$$\alpha_t = \beta_t = 0, \quad \text{and} \quad \alpha_y = -\beta_x\,. \tag{4}$$

Let us now recapitulate the usual Backlund transformation (BT) for the KdV equation. If $\bar{u}$ and u are two solutions, then BT is a relation of the following form

$$\bar{u} = -u - Y^2 + \lambda\,, \tag{5}$$

where λ is the constant spectral parameter occuring in the equation, satisfied by the pseudopotential Y,

$$\begin{aligned} Y_x = -2u - Y^2 + \lambda\,, \\ Y_t = 4\{(u+\lambda)(2u + Y^2 - \lambda) + \tfrac{1}{2}u_{xx} - u_x Y\}\,. \end{aligned} \tag{6}$$

Let us now in Eq. (6) make the change of variables (2), along with

$$\lambda \to \alpha + \epsilon\beta\,, \tag{7}$$

assuming (4) to be valid. Furthermore, for the pseudopotential we set

$$Y \to Y_1 + \epsilon Y_2\,, \tag{8}$$

which leads to

$$\begin{aligned} Y_{1x} = -2u - Y_1^2 + \alpha\,, \\ Y_{1y} + Y_{2x} = -2v - 2Y_1Y_2 + \beta\,. \end{aligned} \tag{9}$$

The corresponding time part leads to

$$
\begin{aligned}
Y_{1t} &= 4(u+\alpha)(2u+Y_1^2-\alpha)+2u_{xx}-4u_xY_1\,,\\
Y_{2t} &= 4(u+\alpha)(2u+2Y_1Y_2-\beta)+4(v+\beta)(2u+Y_1^2-\alpha)\\
&\quad +4(\tfrac{1}{2}v_{xx}+u_{yx}-u_xY_2-(v_x+u_y)Y_1)\,.
\end{aligned}
\tag{10}
$$

On the other hand, the Backlund transformation (5) itself decomposes into

$$
\bar{u} = -u - Y_1^2 + \alpha, \quad \bar{v} = -v - 2Y_1Y_2 + \beta\,.
$$

To start with, we consider the trivial solution $u = 0, v = 0$ and rewrite Eqs. (9) and (10) as follows

$$
Y_{1x} = -Y_1^2 + \alpha, \qquad Y_{1y} + Y_{2x} = -2Y_1Y_2 + \beta\,, \tag{12a}
$$

$$
Y_{1t} = 4\alpha(Y_1^2 - \alpha), \quad Y_{2t} = 4\alpha(2Y_1Y_2 - \beta) + 4\beta(Y_1^2 - \alpha)\,. \tag{12b}
$$

The condition $\alpha_x = -\beta_y$ can be easily satisfied if we take $\alpha = 2y^2$ and $\beta = -4xy$. With these simplifications of α and β, we proceed to solve (12a) to (12b). Let us set

$$
Y_1 = \sqrt{2}\,y \tanh\Theta(x,y,t) \tag{13}
$$

in (12a). Whence,

$$
\frac{\partial\Theta}{\partial x} = \sqrt{2}\,y \quad \text{or} \quad \Theta = \sqrt{2}\,xy + f(yt)\,.
$$

Utilising this in the time part, that is (12b), we get

$$
\Theta = \sqrt{2}\,xy - 8\sqrt{2}\,y^3t + \Theta_0, \qquad \Theta_0 \text{ being arbitrary constant}\,. \tag{14}
$$

Whence,

$$
Y_1 = \sqrt{2}\,y \tanh(\sqrt{2}\,xy - 8\sqrt{2}\,y^3t + \Theta_0)\,.
$$

Plugging in this solution for Y_1, we get the following two equations for Y_2

$$
Y_{2x} = -2\sqrt{2}\,y\tanh\Theta Y_2 - 4xy - \sqrt{2}\tanh\Theta - (2xy - 48y^3t)\,\mathrm{sceh}^2\Theta\,, \tag{15}
$$

$$
Y_{2t} = 16\sqrt{2}\,y^3\tanh\Theta\,Y_2 + 32\,xy^3\mathrm{sech}^2\Theta + 32\,xy^3\,. \tag{16}
$$

To solve for Y_2, we put

$$
Y_2 = A\tanh\Theta + B\,\mathrm{sech}^2\Theta + C
$$

in Eqs. (15) and (16) with A, B and C as functions of (x, y, t). This leads to

$$A_x = -\sqrt{2} - 2\sqrt{2}\, yC, \quad B_x = \sqrt{2}\, yA - 2xy + 48y^3 t, \quad C_x = -2\sqrt{2}\, yA - 4xy\,, \tag{17}$$

and

$$A_t = 16\sqrt{2}\, Cy^3, \qquad B_t = -8\sqrt{2}\, Ay^3 + 32xy^3, \qquad C_t = 16\sqrt{2}\, Ay^3 + 32xy^3\,. \tag{18}$$

Solving (17) and (18), we get

$$A = -\sqrt{2}\, x, \qquad B = -2x^2 y + 48xy^3 t, \qquad \text{and} \quad C = 0\,. \tag{18a}$$

Finally, the solution for Y_2 turns out to be

$$Y_2 = -\sqrt{2}\, x \tanh\Theta + (-2x^2 y + 48xy^3 t)\, \mathrm{sech}^2\Theta\,, \tag{19}$$

where

$$\Theta = \sqrt{2}\, xy - 8\sqrt{2}\, y^3 t + \Theta_0\,.$$

Using solutions (14) and (19) for Y_1 and Y_2, we finally obtain the solution for the BT, Eq. (11),

$$\bar{u} = -2y^2 \tanh^2(\sqrt{2}\, xy - 8\sqrt{2}\, y^3 t + \Theta_0) + 2y^2\,, \tag{20}$$

and

$$\begin{aligned} \bar{v} = &-(4\sqrt{2}\, x^2 y^2 - 96\sqrt{2}\, xy^4 t) \tanh^3(\sqrt{2}\, xy - 8\sqrt{2}\, y^3 t + \Theta_0) \\ &+4xy \tanh^2(\sqrt{2}\, xy - 8\sqrt{2}\, y^3 t + \Theta_0) \\ &+(4\sqrt{2}\, x^2 y^2 - 96\sqrt{2}\, xy^4 t) \tanh(\sqrt{2}\, xy - 8\sqrt{2}\, y^3 t + \Theta_0) - 4xy\,. \end{aligned} \tag{21}$$

So, $(\bar{u}, \bar{y})$ are a set of nontrivial solutions of Eq. (3).

3. *The two-soliton solution*

From the previous result, we ascertain that the two solutions $(\bar{u}, \bar{y})$ can be written as

$$\bar{u} = m^2 y^2 \mathrm{sech}^2\Theta, \quad \text{and} \quad \bar{v} = -x\bar{u}_y\,. \tag{22}$$

To construct the two-soliton solution, we follow Hirota et al. [2] and set

$$u' = \bar{u}_1 + \bar{u}_2, \quad \text{and} \quad v' = -x(\bar{u}_{1y} + \bar{u}_{2y})\,, \tag{23}$$

where

$$\begin{aligned} \bar{u}_1 &= m_1^2 y^2 H(\Theta_2)\,\mathrm{sech}^2(\Theta_1 + G(\Theta_2))\,, \\ \bar{u}_2 &= m_2^2 y^2 H(\Theta_1)\,\mathrm{sech}^2(\Theta_2 + G(\Theta_1))\,, \end{aligned} \tag{24}$$

where

$$\Theta_1 = m_1 xy - 4m_1^3 y^3 t, \qquad \Theta_2 = m_2 xy - 4m_2^3 y^3 t\,, \tag{25}$$

where the solution (22) is obtained if one starts with $\alpha = m^2 y^2$ and $\beta = -2m^2 xy$, of which (14) is a special case for $m = \sqrt{2}$. This validity of the solution can be checked by a direct substitution or from the observation that the first equation of our set can be linearised in the Hirota form.

We set

$$f = 1 + \exp(2\Theta_1) + \exp(2\Theta_2) + A\exp(2\Theta_1 + 2\Theta_2)\,, \tag{26}$$

whence

$$\begin{aligned} ff_{xx} - f_x^2 &= 4m_1^2 y^2 p(\Theta_2)\,\exp(2\Theta_1) + 4m_2^2 y^2 p(\Theta_1)\,\exp(2\Theta_2)\,, \\ p(\Theta_2) &= 1 + \beta_2\,\exp(2\Theta_2) + A\,\exp(4\Theta_2)\,, \\ p(\Theta_1) &= 1 + \beta_1\,\exp(2\Theta_1) + A\,\exp(4\Theta_1)\,, \\ m_2^2\beta_1 + m_1^2\beta_2 &= 2(m_1 - m_2)\,. \end{aligned} \tag{27}$$

Therefore, with $H(\Theta_i) = p(\Theta_i)/q(\Theta_i), i = 1, 2$, we can write

$$\frac{ff_{xx} - f_x^2}{f^2} = m_1^2 y^2 H(\Theta_2)\,\mathrm{sech}^2(\Theta_1 + G(\Theta_2)) + m_2^2 y^2 H(\Theta_1)\,\mathrm{sech}^2(\Theta_2 + G(\Theta_1))\,, \tag{28}$$

where

$$\begin{aligned} q(\Theta_1) &= 1 + (1 + A)\,\exp(2\Theta_1) + A\,\exp(4\Theta_1)\,, \\ q(\Theta_2) &= 1 + (1 + A)\,\exp(2\Theta_2) + A\,\exp(4\Theta_2)\,. \end{aligned} \tag{29}$$

In the above expression, it should be noted that f^2 can be written as

$$f^2 = 4q(\Theta_1)\,\exp(2\Theta_2)\cosh^2(\Theta_2 + G(\Theta_1))$$

$$= 4q(\Theta_2)\,\exp(2\Theta_1)\cosh^2(\Theta_1 + G(\Theta_2))\,.$$

Explicitely, we can write

$$H(\Theta_2) = \frac{p(\Theta_2)}{q(\Theta_2)} = \frac{1 + \beta_2\,\exp(2\Theta_2) + A\,\exp(4\Theta_2)}{1 + (1 + A)\,\exp(2\Theta_2) + A\,\exp(4\Theta_2)}\,,$$

$$H(\Theta_1) = \frac{p(\Theta_1)}{q(\Theta_1)} = \frac{1 + \beta_1 \, \exp(2\Theta_1) + A \, \exp(4\Theta_1)}{1 + (1 + A) \, \exp(2\Theta_1) + A \, \exp(4\Theta_1)} \, ,$$

$$G(\Theta_1) = \frac{1}{2} \ln \frac{1 + A \, \mathrm{e}^{2\Theta_1}}{1 + \mathrm{e}^{2\Theta_1}}, \quad G(\Theta_2) = \frac{1}{2} \ln \frac{1 + A \, \mathrm{e}^{2\Theta_2}}{1 + \mathrm{e}^{2\Theta_2}}, \quad A = \left[\frac{m_1 - m_2}{m_1 + m_2} \right]^2 . \quad (30)$$

So, the two-soliton solution for v' can be written as

$$v' = \{ -m_1^2 x y^2 H(\Theta_2) \, \mathrm{sech}^2(\Theta_1 + G(\Theta_2)) - m_2^2 x y^2 H(\Theta_1) \, \mathrm{sech}^2(\Theta_2 + G(\Theta_1)) \}_y \, . \quad (31)$$

4. The modified equation

It is well known that the KdV equation is connected to the mKdV problem via the Miura map

$$u = -\frac{1}{2}\omega^2 - \frac{1}{2}\omega_x \, ,$$

where ω satisfies the equation

$$\omega_t - 6\omega^2 \omega_x + \omega_{xxx} = 0 \, . \quad (32)$$

We apply the transformation of Eq. (2) to the equation and get the following coupled Miura map

$$u = -\frac{1}{2}\omega_1^2 - \frac{1}{2}\omega_{1x}, \qquad v = -\omega_1 \omega_2 - \frac{1}{2}\omega_{1y} - \frac{1}{2}\omega_{2x} \, , \quad (33)$$

where ω_1 and ω_2 satisfy relations

$$\begin{gathered} \omega_{1t} - 6\omega_1^2 \omega_{1x} + \omega_{1xxx} = 0 \, , \\ \omega_{2t} - 6\omega_1^2 \omega_{2x} - 6\omega_1^2 \omega_{1y} - 12\omega_1 \omega_{1x} \omega_2 + 3\omega_{1xxy} + \omega_{2xxy} = 0 \, , \end{gathered} \quad (34)$$

which are the new coupled (modified) system.

5. Conclusion

In our above analysis, we have shown that by a special choice of the spectral parameters (α, β), introduced for the (2 + 1)–dimensinal KdV equation, one can effectively use a new Backlund transformation to generate two-soliton solution and study all properties of this new, integrable system. The forms of this new type of solutions are depicted in Fig. 1.

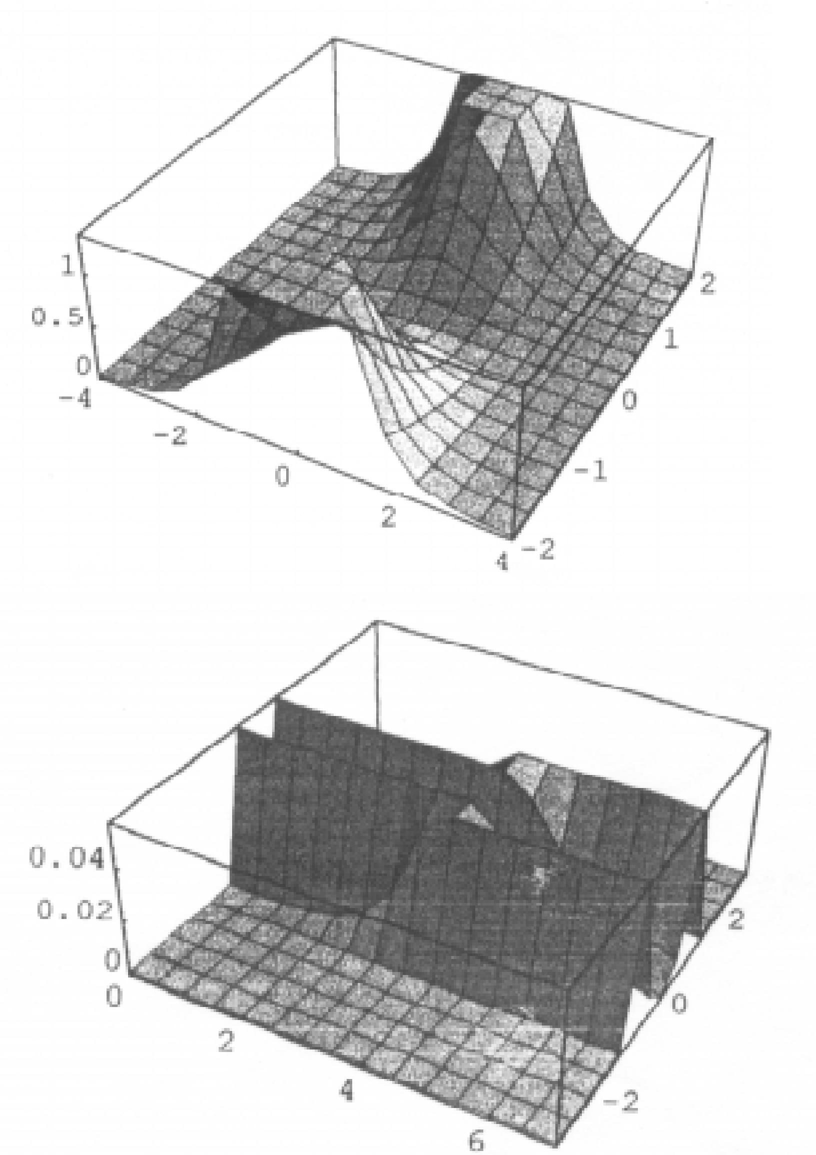

Fig. 1. New types of solutions of the (2 + 1)–dimensional KdV equation obtained by special choices of the spectral parameters α and β.

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NEKA SVOJSTVA (2 + 1)–DIMENZIJSKE KdV JEDNADŽBE SA SMETNJOM

Primjenom Backlundove transformacije dobili smo eksplicitna rješenja (2 + 1)–dimenzijske KdV jednadžbe sa smetnjom koju su izveli Ma i sur. Zatim izvodimo dvosolitonska rješenja u obliku koji podrazumijeva dodavanje rješenja. Konačno, odgovarajuće promijenjene jednadžbe izvodimo novim vidom Miurinog preslikavanja.