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CURVATURE SYSTEMATICS IN GENERAL RELATIVITY

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A clear physical description of a variety of spacetime geometries is provided in terms of the various connection coefficients and curvature-related tensors. The affine connection coefficients, the Riemann curvature tensor, Ricci tensor, scalar curvature and Einstein tensor, and associated discussion is provided for flat spacetime, the Schwarzschild geometry, the Morris-Thorne wormhole geometry, the Friedmann-Robertson-Walker geometry, and a static spherical geometry.

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1. *Introduction*

General relativity textbooks universally present the various tensors and connection coefficients associated with spacetime geometry. The textbooks are excellent in terms of their description of the theory, but are often written from a perspective of the student that will eventually do research in general relativity.

General relativity is no longer a field limited to a narrow subset of physics, and it has entered the commercial world through the expanded use of global positioning systems. The concepts of general relativity are also increasingly important in space exploration and more students are taking courses that include at least elements of general relativity. This expanded usefulness of general relativity and numerous student comments motivate this paper.

Physics students, including advanced undergraduates and non-theorists, attempting their initial course in general relativity are often overwhelmed by the mathematics associated with various tensors and connection coefficients. As such, some students focus on the mathematical machinery and not the underlying physical principles, and miss the beauty of the underlying spacetime geometry. This tendency is particularly true for students who will only take a single general relativity course.

One commonly mentioned student issue is the lack of a compilation of the various tensors and connection coefficients. Such a compilation serves to minimize the frustration in calculating these quantities, facilitates physical insight into their meaning, and enhances understanding of various spacetime geometries.

A number of these geometries also admit structures (e.g. wormholes) that have been popularized in science fiction. Wormholes and their relationship to the various spacetime geometries is another misconception of a number of students participating in general relativity courses. As such a clarification of the wormhole concept in the context of spacetime geometry is warranted and will enhance a student's understanding of general relativity.

This paper has been written to address the aforementioned student issues. The author has found that a unified set of basic connection coefficients and tensors for a number of representative geometries enhances student comprehension of the physical meaning of these spacetime geometries. In addition, a consistent treatment of these geometries corrects misconceptions regarding wormhole phenomena and their physical meaning. Herein, we present information that the author has found to improve the comprehension and general understanding of general relativity to a broad audience of students.

2. *Basic curvature quantities*

There are a number of quantities that can be used to describe spacetime geometries. These include the metric tensor, inverse metric tensor, affine connection coefficients or Christoffel symbols, the Riemann curvature tensor, Ricci tensor, scalar curvature, and the Einstein tensor [1−3]. Each of these is well defined once the spacetime geometry is specified. For each geometry, a specific coordinate system is provided. The various tensors and connection coefficients are defined in terms of these coordinates.

The metric tensor $g_{\mu\nu}$ is provided in terms of the specified coordinates. From a given metric $g_{\mu\nu}$, we compute the components of the following: the inverse metric, the Christoffel symbols or affine connection coefficients, the Riemann curvature tensor, the Ricci tensor, the scalar curvature, and the Einstein tensor.

The Christoffel symbols are defined in terms of the inverse metric tensor and partial derivatives of the metric tensor,

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \,, \tag{1}$$

where ∂_α stands for the partial derivative $\partial/\partial x^\alpha$, and repeated indices are summed. An examination of Eq. (1) reveals that the Christoffel symbols are symmetric in the lower two indices,

$$\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu} \,. \tag{2}$$

The Christoffel symbols are uniquely related to the equation for timelike geodesics,

$$\frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}\tau^2} + \Gamma^{\lambda}{}_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \,, \tag{3}$$

where x^λ are the coordinates in our 4-dimensional basis and τ is the proper time. As noted by Misner et al. [1] in describing geodesic motion on the earth: " ... the connection coefficients serve as 'turning coefficients' to tell how fast to 'turn' the components of a vector in order to keep that vector constant (against the turning influence of the base vectors)".

The Christoffel symbols are also an important ingredient of the equation of geodesic deviation,

$$R^\lambda{}_{\mu\nu\sigma} = \partial_\nu \Gamma^\lambda{}_{\mu\sigma} - \partial_\sigma \Gamma^\lambda{}_{\mu\nu} + \Gamma^\eta{}_{\mu\sigma}\Gamma^\lambda{}_{\eta\nu} - \Gamma^\eta{}_{\mu\nu}\Gamma^\lambda{}_{\eta\sigma}\,. \tag{4}$$

The quantity $R^\lambda{}_{\mu\nu\sigma}$ is a rank-four tensor called the Riemann curvature tensor or Riemann curvature. It represents a measure of spacetime curvature.

An examination of Eq. (4) reveals the antisymmetry of the Riemann curvature tensor under the exchange of the first two indices and the last two indices,

$$R^\lambda{}_{\mu\nu\sigma} = -R_\mu{}^\lambda{}_{\nu\sigma}\,, \tag{5}$$

$$R^\lambda{}_{\mu\nu\sigma} = -R^\lambda{}_{\mu\sigma\nu}\,. \tag{6}$$

By summing the first and third indices of the Riemann curvature tensor, the rank-two Ricci tensor $R_{\mu\nu}$ is obtained,

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}\,. \tag{7}$$

The Ricci tensor can be expressed in terms of the Christoffel symbols,

$$R_{\mu\nu} = \frac{\partial \Gamma^\gamma{}_{\mu\nu}}{\partial x^\gamma} - \frac{\partial \Gamma^\gamma{}_{\mu\gamma}}{\partial x^\nu} + \Gamma^\gamma{}_{\mu\nu}\Gamma^\delta{}_{\gamma\delta} - \Gamma^\gamma{}_{\mu\delta}\Gamma^\delta{}_{\nu\gamma}\,. \tag{8}$$

An inspection of Eq. (8) reveals that the Ricci tensor is symmetric in μ and ν.

The scalar curvature (R) is defined in terms of the inverse metric and the Ricci tensor,

$$R = g^{\mu\nu} R_{\mu\nu}\,. \tag{9}$$

Finally, the Einstein curvature tensor ($G^{\mu\nu}$) is defined in terms of the Ricci tensor, metric tensor, and the scalar curvature,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\,. \tag{10}$$

For completeness, we note that the Einstein curvature tensor, describing the spacetime geometry, is related to the stress-energy tensor $T_{\mu\nu}$,

$$G_{\mu\nu} = 8\pi T_{\mu\nu}\,, \tag{11}$$

where $T_{\mu\nu}$ is the measure of the matter-energy density.

In Section 3, a summary of connection coefficients and tensors for common spacetime geometries is provided. Only non-zero components are presented and symmetry properties are utilized to minimize the number of components presented.

3. Tensors and connection coefficients

A number of commonly encountered spacetime geometries are investigated to illustrate their impact on their associated derived quantities. For each, we provide all nonzero Christoffel symbols, the scalar curvature, and nonzero elements of the Riemann curvature tensor, the Ricci tensor, and the Einstein tensor. These quantities are provided for flat spacetime, the Schwarzschild geometry [4, 5], the Morris-Thorne (MT) wormhole geometry [6 − 8], the Friedmann-Robertson-Walker (FRW) geometry [1, 3, 9], and a static spherical geometry [10]. In the subsequent discussion, spherical coordinates $\{r, \theta, \phi, t\}$ are utilized in the description of all spacetime geometries. The ranges of these coordinates are: $0 \leq r \leq \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq t \leq \infty$.

The use of spherical coordinates provides internal consistency between the various metrics utilized in this paper. It is worth noting that a specific orthonormal basis could provide a simpler expression for a highly symmetric metric or one that is not singular at a coordinate singularity. However, these bases would depend on the specific spacetime geometry. An orthonormal basis would also simplify the solution the Einstein equation, but solutions of the Einstein equation are beyond the scope of this paper.

Geometrized units [1, 3] are used in the subsequent discussion. These units are convenient for general relativity, and utilize a system in which mass, length, and time all have units of length. In these units, the speed of light and the gravitational constant have unit value.

3.1. Flat spacetime geometry

The coordinates used to define the flat spacetime geometry are $\{r, \theta, \phi, t\}$. The metric tensor ($g_{\mu\nu}$) is given by

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{12}$$

and the inverse metric tensor ($g^{\mu\nu}$) by

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/r^2 & 0 & 0 \\ 0 & 0 & \mathrm{cosec}^2\theta/r^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{13}$$

For the flat spacetime, it is expected that the scalar curvature will be zero. Zero curvature also suggests the tensors associated with its definition (e.g., the Ricci tensor and the Riemann curvature tensor) will also have few, if any, non-zero elements. In a similar fashion, the Einstein curvature in flat spacetime is expected to have few, if any, non-zero elements. This qualitative argument is supported by calculation of the elements of these tensors.

The affine coefficients are not expected to be all zero. An inspection of the flat spacetime metric suggests the Christoffel symbols involving t as an index are zero since no metric coefficients are time dependent. Since there is an interrelationship between r, θ and ϕ, it is expected that some of the Christoffel symbols having these elements will be non-zero. This is in fact the case. A listing of these flat-spacetime connection coefficients and tensors follows.

Christoffel symbols:

$$\begin{aligned} \Gamma^r{}_{\theta\theta} &= -r\,, & \Gamma^r{}_{\phi\phi} &= -r\sin^2\theta\,, \\ \Gamma^\theta{}_{\theta r} &= \frac{1}{r}\,, & \Gamma^\theta{}_{\phi\phi} &= -\cos\theta\sin\theta\,, \\ \Gamma^\phi{}_{\phi r} &= \frac{1}{r}\,, & \Gamma^\phi{}_{\phi\theta} &= \cot\theta\,. \end{aligned} \tag{14}$$

Riemann curvature tensor. All elements of the Riemann curvature tensor are zero within the flat spacetime geometry,

$$R^\lambda{}_{\mu\nu\sigma} = 0\,. \tag{15}$$

Ricci tensor. All elements of the Ricci tensor are zero within the flat spacetime geometry.

$$R_{\mu\nu} = 0\,. \tag{16}$$

Scalar curvature. The scalar curvature is zero within the flat spacetime geometry.

$$R = 0\,. \tag{17}$$

Einstein tensor. All elements of the Einstein tensor are zero within the flat spacetime geometry.

$$G_{\mu\nu} = 0\,. \tag{18}$$

3.2. Schwarzschild geometry

The simplest curved spacetimes of general relativity are those that are the most symmetric. One of the most useful spacetime geometries is the Schwarzschild geometry that describes empty space outside a spherically symmetric source of curvature (e.g., a spherical star). In addition, the Schwarzschild geometry is a solution of the vacuum Einstein equation or the equation describing spacetime devoid of matter [1, 3].

The coordinates used to define the Schwarzschild metric are $\{r, \theta, \phi, t\}$, and the metric tensor is given by

$$g_{\mu\nu} = \begin{bmatrix} \dfrac{1}{1-2m/r} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -1+2m/r \end{bmatrix} \tag{19}$$

and the inverse metric tensor is

$$g^{\mu\nu} = \begin{bmatrix} 1-2m/r & 0 & 0 & 0 \\ 0 & 1/r^2 & 0 & 0 \\ 0 & 0 & \mathrm{cosec}^2\theta/r^2 & 0 \\ 0 & 0 & 0 & r/(2m-r) \end{bmatrix} . \tag{20}$$

The Schwarzschild metric has the following properties [1, 3]:

* The metric is independent of time.

* The metric is spherically symmetric. The geometry of a surface of constant t and constant r has the symmetries of a sphere of radius r with respect to exchange of the angles θ and ϕ.

* The coordinate r is not the distance from any center. It is related to the area (A) of a two-dimensional sphere of fixed r and t, $r = (A/4\pi)^{1/2}$.

* The constant m can be identified as the total mass of the source of curvature.

* The geometry becomes interesting at $r = 0$ and $r = 2m$. The $r = 2m$ value is called the Schwarzschild radius that is the characteristic length scale for curvature in the Schwarzschild geometry. The surface of a static star (i.e., a star not undergoing gravitational collapse) lies well outside $r = 0$ and $r = 2m$.

* At large r ($r \gg 2m$), the Schwarzschild spacetime approaches flat spacetime.

* For small m ($m \to 0$), the Schwarzschild spacetime approaches flat spacetime.

An examination of Eqs. (19) and (20) indicates that the Schwarzschild geometry is identical to the flat spacetime in the limit that $m \to 0$ or $r \gg 2m$ (Eqs. (12) and (13)). This limit serves as a natural check on the affine connection coefficients and curvature tensors presented below.

Christoffel symbols:

$$\begin{aligned} &\Gamma^r{}_{rr} = m/(2mr - r^2)\,, && \Gamma^r{}_{\theta\theta} = 2m - r\,, && \Gamma^r{}_{\phi\phi} = (2m-r)\sin^2\theta\,, \\ &\Gamma^r{}_{tt} = m(-2m+r)/r^3\,, && \Gamma^\theta{}_{\theta r} = 1/r\,, && \Gamma^\theta{}_{\phi\phi} = -\cos\theta\sin\theta\,, \\ &\Gamma^\phi{}_{\phi r} = 1/r\,, && \Gamma^\phi{}_{\phi\theta} = \cot\theta\,, && \Gamma^t{}_{tr} = m/(-2mr + r^2)\,. \end{aligned} \tag{21}$$

Examination of Eqs. (21) supports the requirement that the Christoffel symbols derived from the Schwarzschild geometry reduce to those derived from flat spacetime in the $m \to 0$ limit or the $r \gg 2m$ limit.

Riemann curvature tensor. The Riemann curvature tensor has a number of non-zero elements within the Schwarzschild geometry. In the $m \to 0$ limit or the $r \gg 2m$ limit, the flat spacetime results (Eq. 15) are obtained. The non-zero Schwarzschild Riemann curvature tensor elements are:

$$\begin{aligned}
&R^r{}_{\theta\theta r} = m/r\,, && R^r{}_{\phi\phi r} = m\sin^2\theta/r\,, && R^r{}_{ttr} = 2m(-2m+r)/r^4, \\
&R^\theta{}_{r\theta r} = m/\big((2m-r)r^2\big)\,, && R^\theta{}_{\phi\phi\theta} = 2m\sin^2\theta/r\,, && R^\theta{}_{tt\theta} = m(2m-r)/r^4, \\
&R^\phi{}_{r\phi r} = m/\big((2m-r)r^2\big)\,, && R^\phi{}_{\theta\phi\theta} = 2m/r\,, && R^\phi{}_{tt\phi} = m(2m-r)/r^4\,, \\
&R^t{}_{rtr} = 2m/\big((-2m+r)r^2\big)\,, && R^t{}_{\theta t\theta} = -m/r\,, && R^t{}_{\phi t\phi} = -m\sin^2\theta/r\,.
\end{aligned} \tag{22}$$

Ricci tensor. All elements of the Ricci tensor are zero within the Schwarzschild geometry.

$$R_{\mu\nu} = 0\,. \tag{23}$$

Scalar curvature. The scalar curvature is zero within the Schwarzschild geometry:

$$R = 0\,. \tag{24}$$

Einstein tensor. All elements of the Einstein tensor are zero within the Schwarzschild geometry

$$G_{\mu\nu} = 0\,. \tag{25}$$

The Schwarzschild geometry exhibits a discontinuity as $r \to 2m$. This condition may be viewed as a Schwarzschild wormhole or conduit that connects two distinct regions of a single asymptotically flat universe. Further discussion of this geometry is provided in Refs. [1 − 3].

Other spacetime geometries also exhibit wormhole characteristics. Accordingly, we will further pursue the wormhole concept in the next section of this paper.

3.3. Wormhole geometry

In order to further illustrate the wormhole concept, the Morris-Thorne wormhole geometry [6 − 8] is reviewed. The coordinates used to define the MT wormhole geometry are $\{r, \theta, \phi, t\}$, and the metric tensor is

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b^2+r^2 & 0 & 0 \\ 0 & 0 & (b^2+r^2)\sin^2\theta & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{26}$$

and the inverse metric tensor is

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/(b^2+r^2) & 0 & 0 \\ 0 & 0 & \mathrm{cosec}^2\theta/(b^2+r^2) & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{27}$$

where b is a constant having the dimensions of length. An examination of the wormhole geometry indicates that it reduces to flat spacetime (Eqs. (12) and (13)) in the limit $b \to 0$.

At the present time, the MT wormhole geometry does not represent a physically realistic spacetime. Except for the $b = 0$ metric, the geometry is not flat, but is curved. For $b \neq 0$, an embedding of the (r, ϕ) slice of the wormhole geometry produces a surface with two asymptotically flat regions connected by a region of minimum radius b. This region resembles a tunnel or wormhole connecting the two asymptotically flat regions [7, 8].

Insight into the geometry of the MT wormhole can be gained if the spherical symmetry and static nature of the metric are considered. For simplicity, the discussion is limited to the equatorial plane ($\theta = \pi/2$) at a fixed instant of time. Using the coordinate transformation [8],

$$R^2 = b^2 + r^2 \,, \tag{28}$$

the metric in the plane $\theta = \pi/2$, $t = \mathit{constant}$ is

$$\mathrm{d}\sigma^2_{2-\mathrm{surface}} = \frac{1}{1 - (b/R)^2} \mathrm{d}R^2 + R^2 \mathrm{d}\phi^2 \,. \tag{29}$$

Following Müller [8], we can imbed this two-surface in a three-dimensional Euclidean space which is represented by the cylindrical coordinates (R, ϕ, z) by identifying this surface with the surface $z = z(R)$. The metric of the surface in Euclidean space can be written as

$$\mathrm{d}\sigma^2_{\mathrm{Euclidean}} = \left[1 + \left(\frac{\mathrm{d}z}{\mathrm{d}R}^2\right)\right] \mathrm{d}R^2 + R^2 \mathrm{d}\phi^2 \,. \tag{30}$$

The comparison of Eqs. (29) and (30) and integration with respect to R leads to the shape of the embedding diagram,

$$z(R) = \pm b \,\ln \left[\frac{R}{b} + \left(\left(\frac{R}{b}\right)^2 - 1 \right)^{1/2} \right] . \tag{31}$$

As noted in Refs. [7] and [8], the embedding space has no physical meaning. The structure of Eq. (31) is an upper universe connected by a throat of radius b to a lower universe. The impression of a tube (throat) suggested by Eq. (31) is misleading. There is no tube in spacetime, because the regions with radial coordinate $R < b$ are not part of the spacetime. The throat has a spherical topology and becomes important only for geodesics that spiral in the direction of decreasing R (e.g., like water flowing down a drain). Additional discussions about the physics of wormholes and their shapes are found in Refs. [6 – 8].

The wormhole geometry cannot be produced from smooth distortions of flat spacetime. The creation of a wormhole geometry has not only a different geometry from the flat spacetime, but also a different topology [3].

In addition to the previous discussion, the MT wormhole metric has the following properties:

* The metric is independent of time.

* The metric is spherically symmetric because a surface of constant r and t has the geometry of a sphere.

* At very large r ($r \gg b$), the MT spacetime approaches flat spacetime.

The MT wormhole geometry also reduces to flat spacetime in the $b \to 0$ limit. This consistency check is indeed observed for the MT wormhole connection coefficients and curvature tensors presented below.

Christoffel symbols:

$$\begin{aligned} &\Gamma^r{}_{\theta\theta} = -r\,, && \Gamma^r{}_{\phi\phi} = -r\sin^2\theta\,, \\ &\Gamma^\theta{}_{\theta r} = r/(b^2+r^2)\,, && \Gamma^\theta{}_{\phi\phi} = -\cos\theta\sin\theta\,, \\ &\Gamma^\phi{}_{\phi r} = r/(b^2+r^2)\,, && \Gamma^\phi{}_{\phi\theta} = \cot\theta\,. \end{aligned} \tag{32}$$

Riemann curvature tensor:

$$\begin{aligned} &R^r{}_{\theta\theta r} = b^2/(b^2+r^2)\,, && R^r{}_{\phi\phi r} = b^2\sin^2\theta/(b^2+r^2)\,, \\ &R^\theta{}_{r\theta r} = -b^2/(b^2+r^2)^2\,, && R^\theta{}_{\phi\phi\theta} = -b^2\sin^2\theta/(b^2+r^2)\,, \\ &R^\phi{}_{r\phi r} = -b^2/(b^2+r^2)^2\,, && R^\phi{}_{\theta\phi\theta} = b^2/(b^2+r^2)\,. \end{aligned} \tag{33}$$

Ricci tensor. Only the R_{rr} element is nonzero within the MT wormhole geometry,

$$R_{rr} = -2b^2/(b^2+r^2)^2\,, \tag{34}$$

Scalar curvature. The scalar curvature is nonzero within the MT wormhole geometry,

$$R = -2b^2/(b^2+r^2)^2\,, \tag{35}$$

Einstein tensor. The diagonal elements of the Einstein tensor are nonzero within the MT wormhole geometry:

$$G_{rr} = -\frac{b^2}{(b^2+r^2)^2}\,, \quad G_{\theta\theta} = \frac{b^2}{b^2+r^2}\,, \quad G_{\phi\phi} = \frac{b^2\sin^2\theta}{b^2+r^2}\,, \quad G_{tt} = -\frac{b^2}{(b^2+r^2)^2}\,. \tag{36}$$

3.4. *Static spherical geometry*

The coordinates used to define the static spherical geometry [10] are $\{r, \theta, \phi, t\}$. The rr and tt metric tensor elements of the static spherical geometry are functions of r, namely exponential functions of $\lambda(r)$ and $\phi(r)$. The metric tensor is given by

$$g_{\mu\nu} = \begin{bmatrix} \mathrm{e}^{2\lambda(r)} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & -\mathrm{e}^{2\phi(r)} \end{bmatrix} \tag{37}$$

and the inverse metric tensor is

$$g_{\mu\nu} = \begin{bmatrix} \mathrm{e}^{-2\lambda(r)} & 0 & 0 & 0 \\ 0 & 1/r^2 & 0 & 0 \\ 0 & 0 & \mathrm{cosec}^2\theta/r^2 & 0 \\ 0 & 0 & 0 & -\mathrm{e}^{-2\phi(r)} \end{bmatrix}. \tag{38}$$

In the subsequent discussion, the derivative with respect to r is indicated by a prime. That is $\lambda' = \mathrm{d}\lambda/\mathrm{d}r$. Similarly, $\phi' = \mathrm{d}\phi/\mathrm{d}r$

The static spherical geometry reduces to the flat spacetime geometry in the limit $\lambda(r) \to 0$ and $\phi(r) \to 0$. This consistency check is verified by examining the tensors and connection coefficients noted below.

Christoffel symbols:

$$\begin{aligned} &\Gamma^r{}_{rr} = \lambda'(r)\,, && \Gamma^r{}_{\theta\theta} = -r\mathrm{e}^{-2\lambda(r)}\,, && \Gamma^r{}_{\phi\phi} = -\mathrm{e}^{-2\lambda(r)} r \sin^2\theta\,, \\ &\Gamma^r{}_{tt} = \mathrm{e}^{-2\lambda(r)+2\phi(r)}\phi'(r)\,, && \Gamma^\theta{}_{\theta r} = 1/r\,, && \Gamma^\theta{}_{\phi\phi} = -\cos\theta\sin\theta\,, \\ &\Gamma^\phi{}_{\phi r} = 1/r && \Gamma^\phi{}_{\phi\theta} = \cot\theta\,, && \Gamma^t{}_{tr} = \phi'(r)\,. \end{aligned} \tag{39}$$

Riemann curvature tensor:

$$\begin{aligned} &R^r{}_{\theta\theta r} = -\mathrm{e}^{-2\lambda(r)} r\lambda'(r), && R^r{}_{\phi\phi r} = -\mathrm{e}^{-2\lambda(r)} r \sin^2\theta\lambda'(r), \\ &R^r{}_{ttr} = \mathrm{e}^{-2\lambda(r)+2\phi(r)}(\lambda'(r)\phi'(r) - \phi'^2(r) - \phi''(r)), && R^\theta{}_{r\theta r} = \lambda'(r)/r, \\ &R^\theta{}_{\phi\phi\theta} = (-1 + \mathrm{e}^{-2\lambda(r)})\sin^2\theta, && R^\theta{}_{tt\theta} = -\mathrm{e}^{-2\lambda(r)+2\phi(r)}\phi'(r)/r, \\ &R^\phi{}_{r\phi r} = \lambda'(r)/r, && R^\phi{}_{\theta\phi\theta} = 1 - \mathrm{e}^{-2\lambda(r)}, \\ &R^\phi{}_{tt\phi} = -\mathrm{e}^{-2\lambda(r)+2\phi(r)}\phi'(r)/r, && R^t{}_{\theta t\theta} = -\mathrm{e}^{-2\lambda(r)} r\phi'(r), \\ &R^t{}_{rtr} = \lambda'(r)\phi'(r) - \phi'^2(r) - \phi''(r), && R^t{}_{\phi t\phi} = -\mathrm{e}^{-2\lambda(r)} r \sin^2\theta\phi'(r). \end{aligned} \tag{40}$$

Ricci tensor. Only the diagonal Ricci tensor elements are nonzero within the static spherical geometry:

$$\begin{aligned}
R_{rr} &= \Big(\lambda'(r)\big(2 + r\phi'(r)\big) - r\big(\phi'^2(r) + \phi''(r)\big)\Big)\Big/ r\,, \\
R_{\theta\theta} &= \mathrm{e}^{-2\lambda(r)}\big(-1 + \mathrm{e}^{2\lambda(r)} + r\lambda'(r) - r\phi'(r)\big)\,, \\
R_{\phi\phi} &= \mathrm{e}^{-2\lambda(r)} \sin^2\theta\big(-1 + \mathrm{e}^{2\lambda(r)} + r\lambda'(r) - r\phi'(r)\big)\,, \\
R_{tt} &= \mathrm{e}^{-2\lambda(r)+2\phi(r)}\Big(\big(2 - r\lambda'(r)\big)\phi'(r) + r\phi'^2(r) + r\phi''(r)\Big)\Big/ r\,.
\end{aligned} \tag{41}$$

Scalar curvature. The scalar curvature is nonzero within the static spherical geometry,

$$R = \frac{1}{r^2}\Big(2\mathrm{e}^{-2\lambda(r)}\big(-1 + \mathrm{e}^{2\lambda(r)} - 2r\phi'(r) - r^2\phi'^2(r) + r\lambda'(r)(2 + r\phi'(r)) - r^2\phi''(r)\big)\Big). \tag{42}$$

Einstein tensor. The diagonal elements of the Einstein tensor are nonzero within the static spherical geometry:

$$\begin{aligned}
G_{rr} &= \frac{1}{r^2}\big(1 - \mathrm{e}^{2\lambda(r)} + 2r\phi'(r)\big)\,, \\
G_{\theta\theta} &= \mathrm{e}^{-2\lambda(r)} r\big(\phi'(r) + r\phi'^2(r) - \lambda'(r)(1 + r\phi'(r)) + r\phi''(r)\big)\,, \\
G_{\phi\phi} &= \mathrm{e}^{-2\lambda(r)} r \sin^2\theta\Big(\phi'(r) + r\phi'^2(r) - \lambda'(r)(1 + r\phi'(r)) + r\phi''(r)\Big)\,, \\
G_{tt} &= \frac{1}{r^2}\mathrm{e}^{-2\lambda(r)+2\phi(r)}\big(-1 + \mathrm{e}^{2\lambda(r)} + 2r\lambda'(r)\big)\,.
\end{aligned} \tag{43}$$

3.5. Friedmann-Robertson-Walker (FRW) geometry

The FRW geometry describes the time evolution of a homogeneous, isotropic space that expands in time as $a(t)$ increases and contracts as $a(t)$ decreases [3]. The function $a(t)$ contains all information about the temporal evolution of the universe.

In addition to the scaling factor $a(t)$, a constant k is included in the FRW metric. The constant k determines the classification of the universe (i.e., $k = +1$ indicates a closed universe, $k = 0$ indicates a flat universe, and $k = -1$ indicates an open universe). Although the conventional terminology flat, closed, and open are used to distinguish the three possible homogeneous and isotropic geometries of space, it is more physical to distinguish these features in terms of their spatial curvature [3].

Homogeneity requires that the spatial curvature be the same at each point in these geometries.

The flat case has zero spatial curvature everywhere. The closed and open cases have constant positive and constant negative curvature, respectively.

Following the previous discussion of the MT wormhole geometry, embedding can be constructed for the possible homogeneous and isotropic geometries for the

FRW metric. If the $t = constant$, $\theta = \pi/2$ two-surface is considered, the flat and closed embedding diagrams correspond to a plane and sphere, respectively. These are constant zero-curvature and positive curvature surfaces, respectively. A $t = constant$, $\theta = \pi/2$ slice of the open FRW geometry can't be embedded as an axisymmetric surface in flat three-dimensional space. That surface has a constant negative curvature. The reader is referred to Hartle [3] for additional discussion of FRW embedding diagrams.

The coordinates used to define the FRW geometry are $\{r, \theta, \phi, t\}$, and the metric tensor is given by

$$g_{\mu\nu} = \begin{bmatrix} \dfrac{a^2(t)}{1-kr^2} & 0 & 0 & 0 \\ 0 & r^2a^2(t) & 0 & 0 \\ 0 & 0 & r^2a^2(t)\sin^2\theta & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{44}$$

and the inverse metric tensor is

$$g_{\mu\nu} = \begin{bmatrix} \dfrac{1-kr^2}{a^2(t)} & 0 & 0 & 0 \\ 0 & \dfrac{1}{r^2a^2(t)} & 0 & 0 \\ 0 & 0 & \dfrac{\mathrm{cosec}^2\theta}{r^2a^2(t)} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{45}$$

The FRW geometry reduces to the flat spacetime geometry in the limit $a \to 1$ and $k \to 0$. This consistency check is verified by examining the tensors and connection coefficients noted below.

Christoffel symbols:

$$\begin{aligned}
&\Gamma^r{}_{rr} = \frac{kr}{1-kr^2}\,, && \Gamma^r{}_{\theta\theta} = (-1+kr^2)r, && \\
&\Gamma^r{}_{\phi\phi} = (-1+kr^2)r\sin^2\theta, && \Gamma^r{}_{tr} = \frac{\dot{a}(t)}{a(t)}\,, && \\
&\Gamma^\theta{}_{\theta r} = \frac{1}{r}\,, && \Gamma^\theta{}_{\phi\phi} = -\cos\theta\sin\theta\,, && \Gamma^\theta{}_{t\theta} = \frac{\dot{a}(t)}{a(t)}\,, \\
&\Gamma^\phi{}_{\phi r} = \frac{1}{r}\,, && \Gamma^\phi{}_{\phi\theta} = \cot\theta\,, && \Gamma^\phi{}_{t\phi} = \frac{\dot{a}(t)}{a(t)}\,, \\
&\Gamma^t{}_{rr} = \frac{a(t)\dot{a}(t)}{1-kr^2}\,, && \Gamma^t{}_{\theta\theta} = r^2a(t)\dot{a}(t)\,, && \Gamma^t{}_{\phi\phi} = r^2a(t)\dot{a}(t)\sin^2\theta\,.
\end{aligned} \tag{46}$$

Riemann curvature tensor:

$$
\begin{aligned}
&R^{r}{}_{\theta\theta r} = -r^2(\dot{a}^2(t)+k), && R^{r}{}_{\phi\phi r} = -r^2(\dot{a}^2(t)+k)\sin^2\theta, && R^{r}{}_{ttr} = \frac{\ddot{a}(t)}{a(t)}\,,\\
&R^{\theta}{}_{r\theta r} = \frac{\dot{a}^2(t)+k}{1-kr^2}\,, && R^{\theta}{}_{\phi\phi\theta} = -r^2(k+\dot{a}^2(t))\sin^2\theta, && R^{\theta}{}_{tt\theta} = \frac{\ddot{a}(t)}{a(t)}\,,\\
&R^{\phi}{}_{r\phi r} = \frac{\dot{a}^2(t)+k}{1-kr^2}\,, && R^{\phi}{}_{\theta\phi\theta} = r^2(k+\dot{a}^2(t))\,, && R^{\phi}{}_{tt\phi} = \frac{\ddot{a}(t)}{a(t)}\,,\\
&R^{t}{}_{rtr} = \frac{a(t)\ddot{a}(t)}{1-kr^2}\,, && R^{t}{}_{\theta t\theta} = r^2 a(t)\ddot{a}(t)\,, && R^{t}{}_{\phi t\phi} = r^2 a(t)\ddot{a}(t)\sin^2\theta.
\end{aligned}
\tag{47}
$$

Ricci tensor. Only the diagonal Ricci tensor elements are nonzero within the FRW geometry:

$$
\begin{aligned}
R_{rr} &= \frac{2k+2\dot{a}^2(t)+a(t)\ddot{a}(t)}{1-kr^2}\,, & R_{\theta\theta} &= r^2(2k+2\dot{a}^2(t)+a(t)\ddot{a}(t)),\\
R_{\phi\phi} &= r^2(2k+2\dot{a}^2(t)+a(t)\ddot{a}(t))\sin^2\theta, & R_{tt} &= -\frac{3\ddot{a}(t)}{a(t)}\,.
\end{aligned}
\tag{48}
$$

Scalar curvature. The scalar curvature is nonzero within the FRW geometry:

$$
R = \frac{6\big(k+\dot{a}^2(t)+a(t)\ddot{a}(t)\big)}{a(t)^2}\,. \tag{49}
$$

Einstein tensor. The diagonal elements of the Einstein tensor are nonzero within the FRW geometry

$$
\begin{aligned}
G_{rr} &= \frac{k+\dot{a}^2(t)+a(t)\ddot{a}(t)}{-1+kr^2}\,, & G_{\theta\theta} &= -r^2(k+\dot{a}^2(t)+a(t)\ddot{a}(t))\,,\\
G_{\phi\phi} &= -r^2\sin^2\theta(k+\dot{a}^2(t)+a(t)\ddot{a}(t))\,, & G_{tt} &= \frac{3(k+\dot{a}^2(t))}{a^2(t)}\,.
\end{aligned}
\tag{50}
$$

4. *Conclusion*

Using spherical coordinates, the affine connection coefficients, the Riemann curvature tensor, Ricci tensor, scalar curvature, and Einstein tensor are determined for flat spacetime, the Schwarzschild geometry, the Morris-Thorne wormhole geometry, the Friedmann-Robertson-Walker geometry, and a static spherical geometry. This approach provides students with a logical and consistent treatment of the basic quantities and spacetime geometries associated with general relativity, gravitation, and differential geometry. In addition, the approach provides a physical description of these quantities and their interrelationships.

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IZRAZI ZA ZAKRIVLJENOST U OPĆOJ TEORIJI RELATIVNOSTI

Daje se jasan opis niza prostorno-vremenskih geometrija preko veznih koeficijenata i tenzora zakrivljenosti koji se rabe u teoriji. Navode se afini vezni koeficijenti, Riemannov tenzor zakrivljenosti, Riccijev tenzor, skalarna zakrivljenost i Einsteinov tenzor, i oni se raspravljaju za ravan prostor, Schwarzschildovu geometriju, Morris-Thorneovu geometriju crvotočina, te Friedmann-Robertson-Walkerovu i statičku sfernu geometriju.