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A DIRECT APPROACH FOR SOLVING THE BURGERS EQUATION

YUAN-XI XIE

Department of Physics and Electronic Information, Hunan Institute of Science and Technology, Yueyang Hunan 414000, P. R. China
E-mail address: xieyuanxi88@163.com

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By utilizing the transformation given in our article Fizika A **14** (2005) 233 and introducing an intermediate function $f(y)$, a direct approach is presented to solve the Burgers equation which does not require introducing the trial function. Based on it, abundant types of explicit exact solutions of the Burgers equation, including the solitary wave solutions, the singular traveling wave solutions, the triangle function periodic wave solutions, the rational solution etc., are successfully derived.

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1. Introduction

Explicit exact solutions of nonlinear partial differential equations (NPDEs) play a very important role in nonlinear science because they may give a deeper insight into the physical aspects of problems under investigation and are readily used in further applications. Thus searching for explicit exact solutions to NPDEs has become one of the most significant topics in nonlinear science. In recent years, a large variety of powerful methods have been developed to construct explicit exact solutions to NPDEs. Among these are the homogeneous balance method [1, 2], the hyperbolic tangent function expansion method [3 – 6], the trial function method [7 – 10], the sine-cosine method [11], the Jacobi elliptic function expansion method [12, 13], the superposition method [14], the auxiliary equation method [15 – 18], and so on. However, there are no general rules to solving NPDEs. As a consequence, it is still a very significant task to search for more powerful and efficient methods to solve NPDEs.

In the present article, we successfully find a series of explicit exact solutions of the Burgers equation without introducing the trial function, by utilizing the transformation given in Ref. [19] and using an intermediate function.

2. *Direct solving the Burgers equation*

The celebrated Burgers equation reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \alpha\frac{\partial^2 u}{\partial x^2} = 0\,. \tag{1}$$

In order to solve Eq. (1), we take the following transformation given in Ref. [19]

$$u = \frac{\partial v}{\partial x}, \qquad v = v(y), \qquad y = \mathrm{e}^{(kx-\omega t)}\,, \tag{2}$$

where k and ω are the wave number and angular frequency, respectively.

From Eqs. (2), it is not difficult to derive that

$$u = \frac{\partial v}{\partial x} = ky\frac{\mathrm{d}v}{\mathrm{d}y}\,, \tag{3}$$

$$\frac{\partial u}{\partial t} = -k\omega y\frac{\mathrm{d}v}{\mathrm{d}y} - k\omega y^2\frac{\mathrm{d}^2 v}{\mathrm{d}y^2}\,, \tag{4}$$

$$\frac{\partial u}{\partial x} = k^2 y\frac{\mathrm{d}v}{\mathrm{d}y} + k^2 y^2\frac{\mathrm{d}^2 v}{\mathrm{d}y^2}\,, \tag{5}$$

$$\frac{\partial^2 u}{\partial x^2} = k^3 y\frac{\mathrm{d}v}{\mathrm{d}y} + 3k^3 y^2\frac{\mathrm{d}^2 v}{\mathrm{d}y^2} + k^3 y^3\frac{\mathrm{d}^3 v}{\mathrm{d}y^3}\,. \tag{6}$$

Substituting Eqs. (3) – (6) into Eq. (1) results in

$$\alpha k^2 y^2\frac{\mathrm{d}^3 v}{\mathrm{d}y^3} + \left(3\alpha k^2 + \omega - k^2 y\frac{\mathrm{d}v}{\mathrm{d}y}\right) y\frac{\mathrm{d}^2 v}{\mathrm{d}y^2} - k^2 y\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2 + (\omega + \alpha k^2)\frac{\mathrm{d}v}{\mathrm{d}y} = 0\,. \tag{7}$$

For the sake of brevity, we assume that there exists the following relation between k and ω

$$\omega = -\alpha k^2\,. \tag{8}$$

In view of Eq. (8), Eq. (7) can be simplified as

$$\alpha y\frac{\mathrm{d}^3 v}{\mathrm{d}y^3} + 2\alpha\frac{\mathrm{d}^2 v}{\mathrm{d}y^2} - y\frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}^2 v}{\mathrm{d}y^2} - \left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2 = 0\,. \tag{9}$$

Let

$$z = \frac{\mathrm{d}v}{\mathrm{d}y}\,. \tag{10}$$

Then Eq. (9) becomes

$$\alpha y \frac{\mathrm{d}^2 z}{\mathrm{d}y^2} + 2\alpha \frac{\mathrm{d}z}{\mathrm{d}y} - yz\frac{\mathrm{d}z}{\mathrm{d}y} - z^2 = 0\,, \tag{11}$$

which is a second-order nonlinear ordinary differential equation (ODE). It is is very hard to solve it by usual ways. For this reason, we introduce an intermediate function $f(y)$ of the following form

$$f(y) = 2\alpha \frac{\mathrm{d}z}{\mathrm{d}y} - z^2\,. \tag{12}$$

Then

$$\frac{\mathrm{d}f(y)}{\mathrm{d}y} = 2\left(\alpha \frac{\mathrm{d}^2 z}{\mathrm{d}y^2} - z\frac{\mathrm{d}z}{\mathrm{d}y}\right). \tag{13}$$

Inserting Eq. (12) and Eq. (13) into Eq. (11) engenders

$$\frac{\mathrm{d}f(y)}{\mathrm{d}y} + \frac{2}{y} f(y) = 0\,. \tag{14}$$

Solving Eq. (14), we obtain

$$f(y) = \frac{c_1}{y^2}\,, \tag{15}$$

where c_1 is an integration constant.

Plugging Eq. (15) into Eq. (12) gives rise to

$$\frac{\mathrm{d}z}{\mathrm{d}y} - \frac{1}{2\alpha} z^2 - \frac{c_1}{2\alpha y^2} = 0\,, \tag{16}$$

which is the Riccati equation.

For Eq. (16), we consider two cases: (1) $c_1 = 0$ and (2) $c_1 \neq 0$. We give detailed discussion below.

Case (1): $c_1 = 0$

Under this condition, Eq. (16) is simple

$$\frac{\mathrm{d}z}{\mathrm{d}y} - \frac{1}{2\alpha} z^2\,. \tag{17}$$

Integrating Eq. (17) with respect to y yields

$$z = -\frac{2\alpha}{y + c_2}\,, \tag{18}$$

where c_2 is an integration constant.

Using Eq. (18) and Eq. (10), and integrating with respect to y results in

$$v = -2\alpha \ln(y + c_2) + c_3\,, \tag{19}$$

where c_3 is an integration constant.

Substituting Eq. (19) into Eq. (3) and using Eq. (2), we obtain the general travelling wave solution of the Burgers equation (1) as follows

$$u = -\frac{2\alpha k\, \mathrm{e}^{(kx-\omega t)}}{c_2 + \mathrm{e}^{(kx-\omega t)}}\,. \tag{20}$$

Making use of the following identity

$$\frac{\mathrm{e}^x}{\mathrm{e}^x + 1} = \frac{1}{2}\left(1 + \tanh\frac{x}{2}\right)\,, \tag{21}$$

and setting $c_2 = 1$ in Eq. (20), we get the so-called kink-type solitary wave solution of the Burgers equation (1) as follows

$$u = -\alpha k\, \tanh\frac{1}{2}(kx - \omega t) - \alpha k\,. \tag{22}$$

Similarly, choosing $c_2 = -1$ in Eq. (20) and making use of the following identity

$$\frac{\mathrm{e}^x}{\mathrm{e}^x - 1} = \frac{1}{2}\left(1 + \coth\frac{x}{2}\right)\,, \tag{23}$$

we obtain the following singular travelling wave solution of the Burgers equation (1)

$$u = -\alpha k\, \coth\frac{1}{2}(kx - \omega t) - \alpha k\,. \tag{24}$$

Making the following substitution

$$k \to \mathrm{i}k, \qquad \omega \to \mathrm{i}\,\omega \tag{25}$$

where i$=\sqrt{-1}$ is the imaginary unit, and making use of the following two identities

$$\tanh(\mathrm{i}x) = \mathrm{i}\tan x, \qquad \coth(\mathrm{i}x) = -\mathrm{i}\cot x\,, \tag{26}$$

Eqs. (22) and (24) can be simplified as

$$u = \alpha k \tan\frac{1}{2}(kx - \omega t) - \mathrm{i}\,\alpha k\,, \tag{27}$$

$$u = -\alpha k \cot \frac{1}{2}(kx - \omega t) - \mathrm{i}\, \alpha k \,. \tag{28}$$

Case (2): $c_1 \neq 0$

Under this condition, to solve Eq. (16), we take the following transformation

$$z = -2\alpha \frac{1}{w(y)} \frac{\mathrm{d}w(y)}{\mathrm{d}y} \,. \tag{29}$$

Then Eq. (16) becomes

$$y^2 \frac{\mathrm{d}^2 w}{\mathrm{d}y^2} + \frac{c_1}{4\alpha^2} w = 0 \,, \tag{30}$$

which is the Euler equation.

According to the regular method for solving the Euler equation, we may set

$$y = \mathrm{e}^{\eta} \,. \tag{31}$$

Then Eq. (30) becomes the following second-order linear constant-coefficient ODE

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\eta^2} - \frac{\mathrm{d}w}{\mathrm{d}\eta} + \frac{c_1}{4\alpha^2} w = 0 \,. \tag{32}$$

For Eq. (32), three cases need to be considered: (1) $c_1 = \alpha^2$, (2) $c_1 < \alpha^2$ and (3) $c_1 > \alpha^2$. We give detailed discussions below.

Case 2.1: $c_1 = \alpha^2$

In this case, solving Eq. (32) and considering Eq. (31) yields

$$w = (c_4 + c_5 \ln y)\sqrt{y} \,, \tag{33}$$

where c_4 and c_5 are integration constants.

Inserting Eq. (33) and Eq. (29) into Eq. (10) and integrating it with respect to y leads to

$$v = -\alpha \left[\ln y + 2 \ln(c_4 + c_5 \ln y) \right] + c_6 \,, \tag{34}$$

where c_6 is an integration constant.

Substituting Eq. (34) into Eq. (3) and using simultaneously Eq. (2), we acquire

$$u = -\frac{2c_5 \alpha k}{c_4 + c_5(kx - \omega t)} - \alpha k \,, \tag{35}$$

which is a rational solution of the Burgers equation (1).

Case 2.2: $c_1 < \alpha^2$

In this case, solving Eq. (32) and utilizing Eq. (31) yields

$$w = c_7 y^{\left(\alpha+\sqrt{\alpha^2-c_1}\right)/(2\alpha)} + c_8 y^{\left(\alpha-\sqrt{\alpha^2-c_1}\right)/(2\alpha)}, \tag{36}$$

where c_7 and c_8 are integration constants.

Putting Eq. (36) and Eq. (29) into Eq. (10) and integrating it with regard to y results in

$$v = \left(\sqrt{\alpha^2 - c_1} - \alpha\right) \ln y - 2\alpha \ln\left(c_8 + c_7\, y^{\sqrt{\alpha^2-c_1}/\alpha}\right) + c_9\,, \tag{37}$$

where c_9 is an integration constant.

Plugging Eq. (37) into Eq. (3) and applying simultaneously Eq. (2), we get

$$u = -\frac{2k\sqrt{\alpha^2 - c_1}\ \mathrm{e}^{\left(\sqrt{\alpha^2-c_1}/\alpha\right)(kx-\omega t)}}{c_8/c_7 + \mathrm{e}^{\left(\sqrt{\alpha^2-c_1}/\alpha\right)(kx-\omega t)}} + k\left(\sqrt{\alpha^2 - c_1} - \alpha\right). \tag{38}$$

Similarly, making use of the foregoing identity (21) and setting $c_8/c_7 = 1$ in Eq. (38), we obtain the so-called kink-type solitary wave solution of the Burgers equation (1) as follows

$$u = -k\sqrt{\alpha^2 - c_1}\ \tanh\frac{\sqrt{\alpha^2 - c_1}}{2\alpha}(kx - \omega t) - \alpha k\,. \tag{39}$$

Similarly, making use of the identity (23) and setting $c_8/c_7 = -1$ in Eq. (38), we get the singular travelling wave solution of the Burgers equation (1) of the following form

$$u = -k\sqrt{\alpha^2 - c_1}\ \coth\frac{\sqrt{\alpha^2 - c_1}}{2\alpha}(kx - \omega t) - \alpha k\,. \tag{40}$$

Case 2.3: $c_1 > \alpha^2$

Utilizing a similar procedure as above, we find the following solutions of the Burgers equation (1)

$$u = \mp\frac{2k\mathrm{i}\sqrt{c_1 - \alpha^2}\ \mathrm{e}^{\pm\mathrm{i}\left(\sqrt{c_1-\alpha^2}/\alpha\right)(kx-\omega t)}}{c_{11}/c_{10} + \mathrm{e}^{\pm\mathrm{i}\left(\sqrt{c_1-\alpha^2}/\alpha\right)(kx-\omega t)}} + k\mathrm{i}\sqrt{c_1 - \alpha^2} - \alpha k\,. \tag{41}$$

Making use of the following identity

$$\frac{\mathrm{e}^{\pm\,\mathrm{i}x}}{\mathrm{e}^{\pm\,\mathrm{i}x} + 1} = \frac{1}{2}\left(1 \pm \mathrm{i}\tan\frac{x}{2}\right), \tag{42}$$

and setting $c_{11}/c_{10} = 1$ in Eq. (41), we acquire the triangle-function periodic wave solution of the Burgers equation (1) as follows

$$u = k\sqrt{c_1 - \alpha^2}\ \tan\frac{\sqrt{c_1 - \alpha^2}}{2\alpha}(kx - \omega t) - \alpha k\,. \tag{43}$$

Similarly, making use of the following identity

$$\frac{\mathrm{e}^{\pm\,\mathrm{i}x}}{\mathrm{e}^{\pm\,\mathrm{i}x} - 1} = \frac{1}{2}\left(1 \mp \mathrm{i}\cot\frac{x}{2}\right)\,, \tag{44}$$

and setting $c_{11}/c_{10} = -1$ in Eq. (41), we obtain the triangle-function periodic wave solution of the Burgers equation (1) as follows

$$u = -k\sqrt{c_1 - \alpha^2}\ \cot\frac{\sqrt{c_1 - \alpha^2}}{2\alpha}(kx - \omega t) - \alpha k\,. \tag{45}$$

Finally, it should be mentioned that we have verified that the above solutions do indeed satisfy the original equation under the constraint (8) by means of Mathematica.

3. Conclusions

Using the transformation given in Ref. [19] and introducing an intermediate function, we put forward a direct method for solving the Burgers equation without introducing the trial function. As a result, rich types of explicit exact solutions of the Burgers equation, comprising the solitary wave solutions, the singular traveling wave solutions, the triangle function periodic wave solutions and the rational solution have been successfully constructed by this approach.

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IZRAVAN NAČIN RJEŠAVANJA BURGERSOVE JEDNADŽBE

Primjenom pretvorbe objavljene u našem radu Fizika A **14** (2005) 233 i uvodeći međufunkciju $f(y)$, razvili smo izravan način rješavanja Burgersove jednadžbe bez uvođenja probne funkcije. Na toj smo osnovi izveli niz eksplicitnih egzaktnih rješenja Burgersove jednadžbe, uključujući solitonska valna rješenja, singularna rješenja za putujuće valove, trokutna periodička valna rješenja i racionalna rješenja.