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POWER-LAW GENERALIZATIONS FOR STUDYING DYNAMICAL SYSTEMS

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We present a more general description of the technique of Tsallis and co-workers, to study the behaviour of dynamical systems. We enlarge to any dimension the power-law generalization of the classical Lyapunov exponents, that was introduced by Tsallis and co-workers in one dimension. We apply the new generalization to the two-dimensional Hénon map and consider some cases.

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1. Introduction

To study the behaviour of dynamical systems, there is an important tool: the Lyapunov exponents. For a particular case of one-dimensional systems, the meaning of these numbers is the following: let $f : I \to I$ be an interval map. A discrete-time dynamical system can be presented by setting

$$x(n+1) = f(x(n)), \tag{1}$$

and the separation after time N is evaluated by

$$x_1(N) - x_2(N) = f'(c)[x_1(0) - x_2(0)] \,. \tag{2}$$

The growing rate of the separation obeys the exponential law

$$D_{x(0)}(f^n(\delta x(0))) \sim \mathrm{e}^{\lambda n}, \quad \lambda \in \mathrm{R}\,, \tag{3}$$

The number λ is the *Lyapunov exponent* for the dynamical system (1). An indicator for the occurrence of chaotic behaviour is the "sensitive dependence on initial conditions", i.e. there is an $\epsilon > 0$, such that for every neighborhood U of a point x, there is a $y \in U$ with $|f^N(x) - f^N(y)| > \epsilon$, for some integer N. Therefore, if the Lyapunov exponent is positive, chaotic behaviour occurs. In fact, we consider a change $x(0) \to x(0) + \delta x(0)$, for a variation of the time t, and we have $\delta x(t) \sim \delta x(0)\mathrm{e}^{\lambda t}$, so $\lambda > 0$ indicates a sensitive dependence on initial conditions.

In an arbitrary dimension d, the *Lyapunov exponents* for a differentiable map $f \ : \ \mathbf{R}^d \to \mathbf{R}^d$ are defined as follows: let L be a map defined on $\mathbf{R}^d$ which takes values in the space of linear transformations of $\mathbf{R}^d$ in $\mathbf{R}^d$ (or equivalently in the space $d \times d$ matrices). The map L can be taken, for instance, in such a way that for any $x \in \mathbf{R}^d$, $L_x \equiv L(x) : \mathbf{R}^d \to \mathbf{R}^d$ be the linear approximation (the differential map) of f in x. We denote

$$L_x^n \equiv L_{f^{n-1}(x)} \ldots L_{f(x)}\,,$$

where the dots are understood as products of matrices. For x, $v \in \mathbf{R}^d$, let

$$\lambda(x) = \lambda(x, v, f) = \lim_{n\to\infty} \frac{1}{n} \log ||L_x^n v|| \tag{4}$$

if the limit exists, where for $x = (x_1, x_2, \ldots, x_d)$ we take $||x|| = \sqrt{\sum_{i=1}^d x_i^2}$.

The number $\lambda(x) = \lambda(x, v, f)$ is called the Lyapunov exponent with respect to $L = L(f)$ in (x, v).

For a real number λ and $x \in \mathbf{R}^d$, we denote

$$E_\lambda(x) = \{v \in \mathbf{R}^d \ : \ \lambda(x, v, f) \leq \lambda\}\,.$$

Notice that if $\lambda_1 \geq \lambda_2$, then $E_{\lambda_1}(x) \supset E_{\lambda_2}(x)$. Furthermore, if $x \in \mathbf{R}^d$, there exists an integer $m(x) \leq d$, such that there is a collection of numbers λ_1, λ_2, $\ldots \lambda_{m(x)}$ and linear subspaces $\mathbf{R}^d = E_{\lambda_1}(x)$, $E_{\lambda_2}(x)$, $\ldots E_{\lambda_{m(x)}}(x) = \{0\}$, with

$$\lambda^1(x) > \lambda^2(x) > \ \ldots \ \lambda^{m(x)}(x)\,,$$

$$E_{\lambda_1}(x) \supset E_{\lambda_2}(x) \supset \ldots \supset E_{\lambda_{m(x)}}(x)\,.$$

If $v \in E_{\lambda_i}(x) - E_{\lambda_{i+1}}(x)$, then $\lambda(x, v) = \lambda_i(x)$. The *Lyapunov spectrum* is now defined as

$$Sp_x(L, f) = \{\lambda_i(x) \ : \ i = 1, 2 \ldots, m(x)\} .$$

If it is considered a $f-$invariant ergodic measure μ, the functions $x \to m(x)$, $x \to E_{\lambda_i}(x)$ and $x \to \lambda^i(x)$ are constant except for μ–null sets. For more details about this subject and for the demonstrations of the facts mentioned above see, e.g., Refs. [1] or [2].

Tsallis et al. [3] proposed a power-law generalization of the Lyapunov exponents associated to one-dimensional discrete dynamical systems in order to study the behaviour at the edge of chaos, or, more generally, at critical values of control parameters of this kind of systems. They have been presented in the form

$$x(n+1) = f_\alpha(x(n)) , \tag{5}$$

with f_α a family of interval maps depending on a bifurcation parameter α. Of course, there is a dependence of the exponent on the parameter and for the critical values α_c, i.e. those for which $\lambda(\alpha_c) = 0$, the description of the system in these points could not be well understood. Thus in Ref. [3], a one-parameter family $\{\lambda_q\}$ has been introduced, such that the Lyapunov λ is contained as one of its members, specifically when $q \to 1$. From now on, we will refer to $\lambda_{q=1}(x)$ as $\lambda(x)$, where $q = 1$ actually means $q \to 1$. The aim of the above mentioned paper is to get a parameter q such that $\lambda_q(\alpha_c) \neq 0$ when $\lambda(\alpha_c) = 0$. The results of the above quoted paper of Tsallis et al. have raised many developments in several directions, (see Ref. [4]). In the preset article, we propose to formulate a presentation of the techniques from Ref. [3] for dynamical systems with any number of dimensions.

It should be remarked that the dependence is not continuous in general, and this fact causes problems in the physical context. Taking into account that Lyapunov exponents measure the stability of the system, it would be good that in an experiment they cannot be modified substantially by perturbations of the internal parameters of the system. In a purely experimental situation, the quantities measured for predictions could be "smoothed" by instrumental procedures.

Another feature to observe is the following: in the case of having, at least in a neighborhood of α_c, a continuous dependence on the bifurcation parameter, we would have eventually a transition $\lambda(\alpha_c - \epsilon) \to \lambda(\alpha_c + \epsilon)$ from chaotic to stable (or stable to chaotic) state, and α_c is a threshold of chaos. In general, this situation is not expected, and so we could not ensure by a numerical computation that transition occurs (even for a small ϵ), i.e., that α_c is a genuine threshold of chaos. We shall use the term "possible threshold of chaos".

In this article we treat dynamical systems in discrete time, hence we denote them by

$$x(n+1) = f_\Lambda(x(n)) , \tag{6}$$

where Λ is a bifurcation parameter set $(\alpha_1, \alpha_2, \ldots \alpha_d)$, i.e., the behaviour of the system can change by varying some α_i. The technique proposed in Ref. [3] for the one-dimensional systems is applied in our context when the Lyapunov spectrum is such that for $q = 1$, one of the exponents vanishes and the others are negative. So the behaviour for the vanishing one is not well described.

In the study of chaotic behaviour, we frequently find invariant sets with a complex mathematical structure. The multifractal analysis essentially deals with decompositions of these kind of sets. One of the main problems is to evaluate some dimensions, e.g. the Hausdorff dimension, correlation dimension. In this article, we estimate some of these dimensions. Herein we treat discrete dynamical systems, and for the continuos case the considered dynamics are of the flows $\{\varphi_t\}_{t\in\mathbf{R}}$. Lyapunov exponents $\lambda\left(\{\varphi_t\}\right)$ for the flow $\{\varphi_t\}$ are calculated as $\lambda\left(\varphi_{t=1}\right)$.

The scheme and goals of this article are: in Sec. 2 we present a power-law generalization in the spirit of Tsallis et al. for abstract dynamical systems and illustrate this procedure for the two-dimensional Hénon map.

2. *Power-law generalizations*

In Ref. [3], the authors proposed the following power-law generalization of the Lyapunov exponents

$$\left|\frac{\Delta x(t)}{\Delta x(0)}\right| \sim \left[(1-q)\lambda^q\right]^{1/(1-q)} \times t^{1/(1-q)} .$$

Let $f \;:\; \mathbf{R}^d \to \mathbf{R}^d$ be a differentiable map and as in the introduction, $L_x \equiv L(x) \;:\; \mathbf{R}^d \to \mathbf{R}^d$ is the linear approximation of f in x. Inspired in the above formula for the one-dimensional case, we set

$$\lambda_q(x) = \lambda_q(x,v,f) = \lim_{n\to\infty}\left[\frac{(1/n)\left(1-||L_x^n v||^{(1-q)}\right)}{q-1}\right], \quad \text{for} \quad q \neq 1 . \tag{7}$$

Now we have a power law: $||L_x^n v|| \sim \left[(1-q)\lambda^q\right]^{1/(1-q)} \times n^{1/(1-q)}$, therefore $||L_x^n v||$ and $n^{1/(1-q)}$ change asymptotically in the same way.

To compare the notations with those in the paper of Tsallis et al.: $||L_x^n v||$ corresponds to $|\Delta x(t)/\Delta x(0)|$ and n to the "time" t.

We insist that the objective is try to describe the behaviour of dynamical systems for which in the Lyapunov spectrum occurs when $\lambda^1(\Lambda_c)$ vanish at a critical value Λ_c and the other $\lambda^1(\Lambda_c)$ are negative. The proposed power-law generalization enables us to find an optimal $q_{\rm op}$ such that $\lambda_{q_{\rm op}}(\Lambda_c) \neq 0$, and so the behaviour of the system can be described.

Remark: λ mean the numbers obtained by taking the limit $q \to 1$ in Eq. (7), i.e. the classical Lyapunov exponents.

The parameter q may be fitted in the case when the boundary of the possibly fractal set $(n, ||L_{x,\Lambda_c}||)$ has a shape that can be fitted with a straight line. It is hoped that this fractality "decreases" as the bifurcation set is moved away from Λ_c. Here L_{x,Λ_c} means the linear approximation of f restricted to the set Λ_c.

We illustrate these ideas in the two-dimensional discrete Hénon map,

$$\begin{cases} x(n+1) = y(n) + 1 - \alpha(x(n))^2, \\ y(n+1) = \beta x(n)\,. \end{cases} \tag{8}$$

where we set $\Lambda = (\alpha, \beta)$. Some features about this map are well known: for the special values $\alpha = 1.4$, $\beta = 0.3$, there is positive Lyapunov exponent $\lambda^1(\alpha, \beta)$, and the other one is $\lambda^2 = \log|J| - \lambda^1$, where J is the Jacobian of the map, $|J| = \beta$. Numerically it is known that $\lambda^1(1.4, 0.3) \simeq 0.42$, and so $\lambda^2(1.4, 0.3) \simeq 1.62$ [5].

We fix now $\beta = 0.3$ and sweep α in order to find a critical value $\Lambda_c = (\alpha_c, 0.3)$, such that $\lambda^1(\alpha_c, 0.3 = 0)$.

In Fig. 1 we see a plot of λ^1 versus α for β fixed at 0.3 and $0.0 \leq \alpha \leq 1.4$. The first positive λ^1 value occurs at $\alpha = 1.058047982596\ldots$. To the right of this α, although many $\lambda^1 > 0$ appear, one also finds a scattering of subintervals on which $\lambda^1 < 0$, and each such λ^1 corresponds to a periodic attractor. The fact that a period 7 attractor occurs for $\alpha = 1.3$ had been already noticed [6].

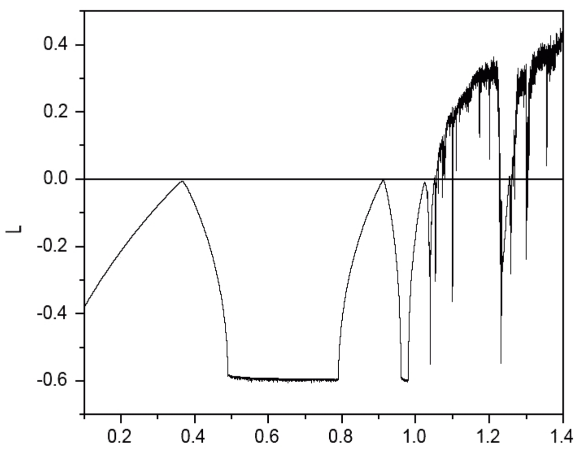

Fig. 1. Lyapunov exponents for the Hénon map, with $\beta = 0.3$, vs. $\alpha \in [0, 1.4]$.

We shall focus our attention at the values $\alpha_c = 1.058047982596\ldots$ and $\alpha_c = 1.26182278611\ldots$ which yield values of $\lambda^1 = 0.95 \times 10^{-10}$ and $\lambda^1 = 0.43 \times 10^{-10}$, respectively.

Recall that $||L^n_x|| = \prod_{i=0}^{n-1} ||L_{f^{i-1}(x)}||$, thus we consider the estimation

$$L(n) = \log||L^n_{(x(n),y(n))\Lambda_c} v|| = \sum_{i=0}^{n-1} \log||L_{f^{i-1}(x(i),y(i))\Lambda_c} v||\,, \tag{9}$$

with the initial coordinate values $x(0) = y(0) = 0$ and the initial vector $v_0 = (1, 0)$. However, the same result is obtained with different unitary vectors, e.g. $(0, 1)$, $\big(1/\sqrt{2}, 1/\sqrt{2}\big)$, etc.

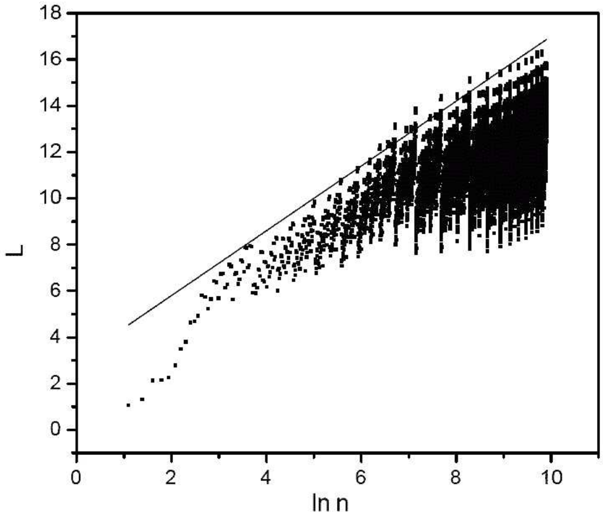

Fig. 2. log-log *of* $\left\| L^n_{x\Lambda_c} \right\|$ *vs. the number of iterations* n*, for* $x(0) = y(0) = 0$, $\alpha = \alpha_c = 1.26182278611$.

A plot of $L(n)$ vs. $\log n$ (Fig. 2) with $n = 8000$ and $\alpha_c = 1.26182278611\ldots$, reveals the fractal nature of this set, as expected, and besides that the points for large values of n are placed on a line with the slope m; it suggests that the optimal choice for the parameter q, taking into account the asymptotic behaviour expressed below Eq. (7), is

$$\frac{1}{1 - q_{\rm op}} = m \simeq 1.4\,. \tag{10}$$

which leads to $q_{\rm op} \simeq 0.285714$, and therefore $\lambda_1^{q_{\rm op}} = 0.035\ldots$, which indicates that there is a chaotic behaviour, and Λ_c is a possible threshold of chaos.

In Fig. 3 is shown how the fractality "decreases" as the bifurcation parameter is moved away from the threshold of chaos, i.e. the value $\alpha_c = 1.26182278611\ldots$. Indeed this is a genuine threshold of chaos, at least locally. Investigations about the behaviour of Lyapunov exponents of the two-dimensional Hénon map, were done by Feit [6].

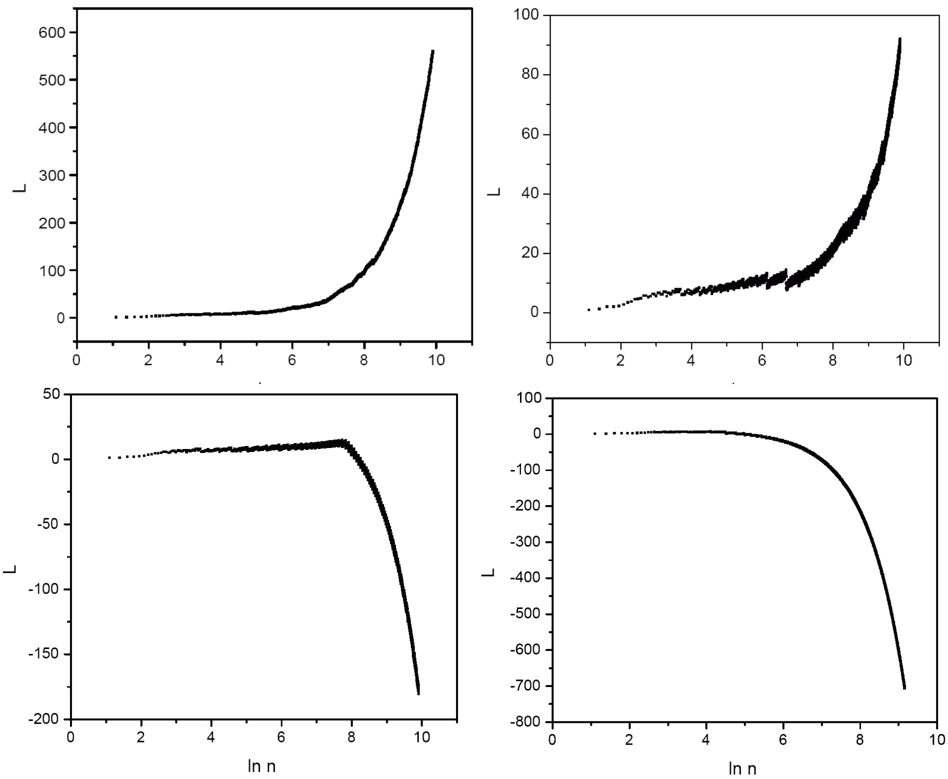

Fig. 3. log-log *of* $\left\| L^n_{x\Lambda_c} \right\|$ *vs the number of iterations* n*, for* $x(0) = y(0) = 0$*,* $\alpha_c = 1.26182278611$*; (a)* $\alpha = \alpha_c + 10^{-3}$*; (b)* $\alpha = \alpha_c + 10^{-5}$*;(c)* $\alpha = \alpha_c - 10^{-5}$*; (d)* $\alpha = \alpha_c - 10^{-3}$.

With a similar procedure applied to $\alpha_c = 1.058047982596\ldots$ (Fig. 4), we obtained

$$\frac{1}{1 - q_{\mathrm{op}}} = m \simeq -4.285\,.$$

which leads to $q_{\mathrm{op}} \simeq 1.23333$ and $\lambda_1^{q_{\mathrm{op}}}(\Lambda_c) = -0.000909\ldots$, which indicates non-chaotic behaviour. In Fig. 5 we observe the same behaviour as in Fig. 3 while one moves away from the threshold of chaos.

In the last two columns of the tables above, we calculate the Hausdorff dimension (d) and the correlation dimension (ν) [7] for the fractal set in Fig. 2.

The Hausdorff dimension d is a purely geometric measure, because it is independent of the frequency with which a trajectory visits various parts of the attractor. The correlation dimension ν is obtained from the correlations between random points on the attractor. As it can be observed, $\nu < d$ for all values of the α parameter.

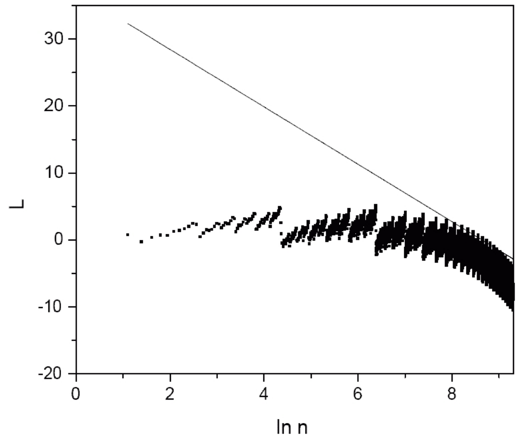

Fig. 4. log-log *of* $\left\|L^n_{x\,\Lambda_c}\right\|$ *vs the number of iterations* n*, for* $x(0) = y(0) = 0$, $\alpha = \alpha_c = 1.058047982596$.

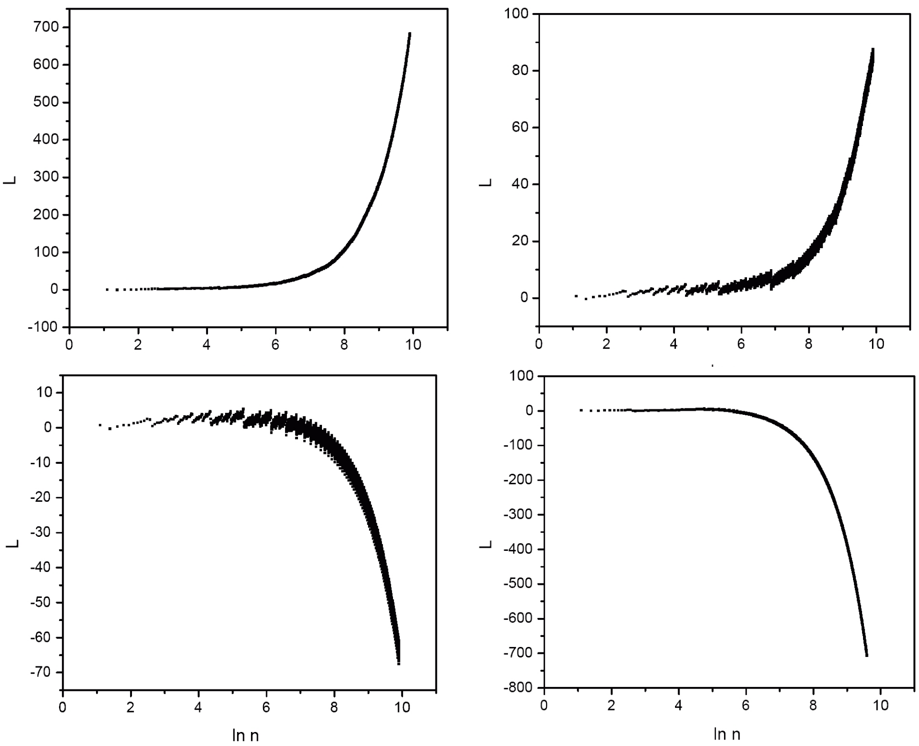

Fig. 5. log-log *of* $\left\|L^n_{x\,\Lambda_c}\right\|$ *vs the number of iterations* n*, for* $x(0) = y(0) = 0$, $\alpha_c = 1.058047982596$*; (a)* $\alpha = \alpha_c + 10^{-3}$*; (b)* $\alpha = \alpha_c + 10^{-5}$*;(c)* $\alpha = \alpha_c - 10^{-5}$*; (d)* $\alpha = \alpha_c - 10^{-3}$.

TABLE 1. Characteristic dimensions for the fractal set of Fig. 2 with α_c = 1.2618.....

	$\lim_{n\to\infty} \|\|L_x^n v\|\|$	Hénon 2D $x(0)=0$ $y(0)=0$	Hausdorff dimension	Correlation dimension
$\lambda^1 > 0$	$\mathrm{e}^{\lambda^1 n}$	$\alpha = \alpha_c + 10^{-5}$	0.58424	0.53737
$\lambda^1 < 0$	$\mathrm{e}^{\lambda^1 n}$	$\alpha = \alpha_c - 10^{-5}$	0.52221	0.45721
$\lambda^1 = 0$ $\lambda_q^1 > 0$	$\left[(1-q)\lambda_q^1 n\right]^{1/(1-q)}$	$\alpha_c = 1.2618...$ $q \cong 0.285714$	0.53861	0.50034

TABLE 2. Characteristic dimensions for the fractal set of Fig. 2 with $\alpha_c = 1.05904$.

	$\lim_{n\to\infty} \|\|L_x^n v\|\|$	Hénon 2D $x(0)=0$ $y(0)=0$	Hausdorff dimension	Correlation dimension
$\lambda^1 > 0$	$\mathrm{e}^{\lambda^1 n}$	$\alpha = \alpha_c + 10^{-5}$	0.59302	0.55878
$\lambda^1 < 0$	$\mathrm{e}^{\lambda^1 n}$	$\alpha = \alpha_c - 10^{-5}$	0.41428	0.40987
$\lambda^1 = 0$ $\lambda_q^1 > 0$	$\left[(1-q)\lambda_q^1 n\right]^{1/(1-q)}$	$\alpha_c = 1.05804...$ $q \cong 1.23333$	0.44412	0.44400

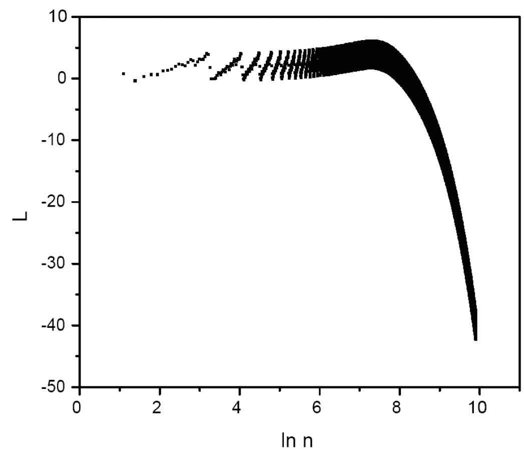

Fig. 6. log-log of $\left\| L^n_{x\,\Lambda_c} \right\|$ *vs the number of iterations* n*, for* $x(0) = y(0) = 0$, $\alpha = \alpha_c = 1.06328909597577$.

Another interesting value of the parameter α for which $\lambda^1(\alpha, 0.3) \simeq 0$ is $\alpha \simeq 1.063289095$. For this value we calculate $\lambda^1_{q_{\rm op}}(\Lambda_c) = 0.66 \times 10^{-9}$ (Fig. 6). It can be numerically shown that period-doubling can occur, leading to the onset of chaos [8].

3. Conclusion

The results for the model studied here show that the techniques introduced by Tsallis can be applied to analyze chaotic behaviour in dynamical systems with any number of dimensions.

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POOPĆENJE POTENCIJSKOG ZAKONA ZA PROUČAVANJE DINAMIČKIH SUSTAVA

Izlažemo poopćen opis metode Tsallisa i sur. za proučavanje svojstava dinamičkih sustava. Proširujemo poopćenje potencijskog zakona klasičnih Lyapunovih eksponenata na proizvoljan broj dimenzija po uzoru na poopćenje koje su uveli Tsallis i sur. za jednu dimenziju. Primjenjujemo to poopćenje na dvodimenzijsku Hénonovu mapu i razmatramo neke slučajeve.