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*LETTER TO THE EDITOR*

# SUBDIFFUSION OVER FRACTIONAL QUANTUM PATHS WITHOUT FRACTIONAL DERIVATIVE

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It is shown that subdiffusion processes in quantum dynamical systems can be realized without implementing any kind of time-fractional derivative if a fractional action integral is used in the theory.

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The standard Brownian diffusion process governs the scaling of a random walk, where a tracer in a two dimensional spacetime is governed by the variance of the displacement of a particle distribution $\sigma^2(t) \propto t^\gamma$, $\gamma = 1$. In other words, if we denote by $P(x,t)$ the relative probability of particles at location $x \in \mathbb{R}$ at time $t$, the classical diffusion equation $\partial P/\partial t = \frac{1}{2}\partial^2 p/\partial x^2$ can be solved by means of the Fourier transform which inverts to a normal probability distribution with the mean zero and standard deviation $t^{1/2}$. In the last decade, it has been realized that the understanding of complex dynamical systems has required the implementation of the subdiffusion (slower spreading rate, i.e. $\gamma < 1$) and superdiffusion (faster spreading rate, i.e. $\gamma > 1$) anomalous random processes. Anomalous superdiffusion can be modeled using infinite variance particle jumps that lead to space-fractional derivatives in the governing diffusion equation. Anomalous subdiffusion can be modeled using IID infinite mean waiting times between particle jumps, leading to a fractional time derivative in the governing diffusion equation [1]. The aim of this short communication is to prove that the anomalous subdiffusion can be realized in quantum physical phenomena without implementing any kind of time-fractional derivative.

Over the last decades, the usefulness of the fractional calculus in applications that are found in different fields of science with various aspects and in different ways, as well as its merits in pure mathematics, has become more and more evident and interesting [2]. A number of definitions have emerged over the years including Riemann-Liouville, Grunwald-Letnikov, Riesz, Weyl, Caputo and Erdelyi-Kober fractional derivatives. The fact that there is obviously more than one way to define non-integer order derivatives, not all being equivalent, is one of the most challenging and rewarding aspects of this special mathematical field. In this work, we essentially consider the continuous view-point based on the Riemann-Liouville fractional integral. Perhaps the easiest way to access to the idea of the Riemann-Liouville fractional integral is given by Caputo's well known representation of an $n$-fold integral as a convolution integral

$$I^n y(t) = \int_0^t \int_0^{t_{n-1}} \dots \int_0^{t_1} y(t_0)\mathrm{d}t_0 \dots \mathrm{d}t_{n-2}\mathrm{d}t_{n-1} = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} y(\tau)\mathrm{d}\tau, \quad (1)$$

$$n \in \mathbb{N}, \quad t \in \mathbb{R}_+,$$

where $I^n$ is the $n$-fold integral operator. After replacing the discrete factorial $(n-1)!$ with the Euler's continuous gamma function $\Gamma(n)$, one obtains the Riemann-Liouville definition of a non-integer order integral as follows

$$I^n y(t) = \frac{1}{\Gamma(n)}\int_0^t (t-\tau)^{\alpha-1} y(\tau)\mathrm{d}\tau, \quad 0 < \alpha \leq 1, \quad t \in \mathbb{R}_+ . \quad (2)$$

This approach turns out to be useful in treating generalized diffusion processes in the theory of probability and stochastic processes. While various fields of application of fractional derivatives and integrals are already well done, some others have just been started, in particular the study of fractional problems of the calculus of variations (COV) and respective Euler-Lagrange type equations that are a subject of current intensive research and investigations [2]. Recently, we proposed a novel approach known as fractional action-like variational approach (FALVA) to model nonconservative dynamical systems where fractional time integral introduces only one fractional parameter $\alpha \in [0, 1]$, while in other models an arbitrary number of fractional parameters (orders of derivatives) appear [3]. The standard functional action is replaced by a fractional functional integral action as follows

$$S[f(t)] \to S_\alpha[f(t)] = \frac{1}{\Gamma(\alpha)}\int_{t_0=0}^t L(\dot{x}(\tau), x(\tau), \tau)\,(t-\tau)^{\alpha-1}\mathrm{d}\tau$$

$$= \int_{t_0=0}^t L(\dot{x}, x, \tau)\mathrm{d}T_t(\tau), \quad [t_0, t] \in \mathbb{R}_+ , \quad (3)$$

thus revealing interesting features. Here $L(\dot{x}, x, \tau)$ is the Lagrangian weighted with $(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, and $\Gamma(\alpha+1) = t^{\alpha} - (t-\tau)^{\alpha-1}$. We consider in fact two time variables: the intrinsic time $\tau$ and the observer time $t$. The Feynman path integral, which is in fact the integration over Brownian-like quantum mechanical paths, ought to be generalized to describe classical and quantum dynamical systems at both the classical and quantum levels. To generalize the Feynman path integral approach, the integration has been expanded from the standard integral to the fractional action integral for a free particle.

In this work, we define the fractional path integral (FPI) by the following four guidelines [4]:

1) The FPI is expected to describe the motion from the initial position $x_i(t_i = 0)$ to the position $x_f(t_f)$ with a fractional quantum amplitude given by

$$K_{\alpha}(x_f, t;\, 0,\, 0) \propto \sum_{\{\gamma\}} \exp\left(\frac{\mathrm{i}}{\hbar} S_{\alpha}[\gamma]\right), \tag{4}$$

where $\{\gamma\}$ is the set of all trajectories satisfying $x(t_i = 0) = x_i$ and $x(t_f) = x_f$.

2) The standard result is expected to be resurrected in the $\alpha = 1$ limit and classical physics is expected to be recovered for $\hbar = 1$ ($\hbar$ is the Planck's constant).

3) The particle jumps satisfy the spatial fractional diffusion equation [5]

$$\frac{\partial P(\vec{x}, t)}{\partial t} = -\frac{\hbar^{\beta-1}}{(2m)^{\beta/2}} (-\nabla^2)^{\beta/2} P(\vec{x}, t), \quad 1 < \beta < 2\,, \tag{5}$$

where $(-\nabla^2)^{\beta/2}$ is the fractional Laplacian, often called the Riesz fractional derivative in term of the Riesz potential $I_d^s$ of order $s$ and dimension $d$ that reads [6]

$$I_d^s\, \varphi(x) = \frac{\Gamma((d-s)/2)}{\pi^{s/2} 2^s \Gamma(s/2)} \int\limits_{\Omega} \frac{\varphi(\xi)}{||x-\xi||^{d-s}}\, \mathrm{d}\Omega(\xi), \quad 0 < s < 2\,, \tag{6}$$

where $\Omega$ is the integral domain.

4) The fractional action of the one-dimensional free particle in Minkowski's space follows from Eq. (5) and is given by [5]

$$S_{\alpha,\beta} = \frac{f(\alpha,\beta)}{\Gamma} \int\limits_0^t \left[(\dot{x})^2\right]^{\beta/(2(\beta-1))} (t-\tau)^{\alpha-1}\, \mathrm{d}\tau\,, \tag{7}$$

where $f(\alpha,\beta)$ has dimensions $[\mathrm{E}][\mathrm{T}]^{\alpha+1}/[\mathrm{L}]^{\beta/(\beta-1)}$ and is given by

$$f(\alpha,\beta) = (\beta-1)\left[\frac{(2m)^{\beta/2}}{\beta^{\beta} R_{\alpha,\beta}}\right]^{1/(\beta-1)}. \tag{8}$$

In Eq. (8), $R_{\alpha,\beta}$ is a constant with dimensions $[R_{\alpha,\beta}]=[\mathrm{E}]^{1-\beta/2}[\mathrm{T}]^{\alpha(1-\beta)+1}$.

To get an estimate of the mean-square displacement of a free particle moving from an initial point $x(t_i=0)=x_i=0\rightarrow x(t_f=t)=x_f$, we follow the Feynman standard technique and write the fractional quantum mechanical kernel in the form [7]

$$K_\alpha(x_f,t;0,\,0)=\int\limits_0^{x_f}\mathbb{D}[x(\tau)]\exp\left(\frac{\mathrm{i}}{\hbar}S_\alpha[x(\tau)]\right)$$

$$=\int\limits_0^{x_f}\mathbb{D}[x(\tau)]\exp\left(\frac{\mathrm{i}}{\hbar}\frac{f(\beta)}{\Gamma(\alpha)}\int\limits_0^t\left(\left[(\dot{x})^2\right]^{\beta/(2(\beta-1))}-V(x(\tau),\tau)\right)(t-\tau)^{\alpha-1}\mathrm{d}\tau\right). \tag{9}$$

where

$$\int\limits_0^{x_f}\mathbb{D}[x(\tau)]\ldots=\lim_{n\to\infty}(2\pi\mathrm{i}\varepsilon\hbar)^{1/2}\int\limits_{-\infty}^{\infty}\prod_{j=1}^{N-1}(2\pi\mathrm{i}\varepsilon\hbar)^{1/2}\mathrm{d}x_j\ldots,\quad \varepsilon=\frac{t}{N}, \tag{10}$$

denotes the sum over all paths between $(0,\,0)\rightarrow(x_f,t_f)$, and $V(x(\tau),\tau)$ is the potential energy of the mass $m$. The wave function $\psi(x_f,t)$ at $(x_f,t)$ is given in terms of $\psi(0,\,0)$ at $(0,\,0)$ by the equation

$$\psi(x_f,t)=\int\limits_0^{x_f}\mathrm{d}x_iK_\alpha(x_f,t;0,\,0)\psi(0,\,0). \tag{11}$$

This fractional equation describes the evolution of the quantum mechanical system in terms of the wave equation.

We will proceed by calculating the fractional quantum-mechanical amplitude for a free particle ($V(x)=0$) using Eqs. (1) to (8):

$$K_\alpha^0(x_f,t;0,\,0)=\int\limits_0^{x_f}\mathbb{D}[x(\tau)]\exp\left(\frac{\mathrm{i}}{\hbar}\frac{f(\beta)}{\Gamma(\alpha)}\int\limits_0^t(t-\tau)^{\alpha-1}\left[(\dot{x})^2\right]^{\beta/(2(\beta-1))}\mathrm{d}\tau\right) \tag{12}$$

$$=\lim_{n\to\infty}(2\pi\mathrm{i}\varepsilon\hbar)^{1/2}\int\limits_{-\infty}^{\infty}\prod_{j=1}^{N-1}(2\pi\mathrm{i}\varepsilon\hbar)^{1/2}\mathrm{d}x_j\times\prod_{j=1}^{N}\exp\left(\frac{\mathrm{i}f(\beta)}{\hbar}\int\limits_0^t\left[(\dot{x})^2\right]^{\beta/(2(\beta-1))}\mathrm{d}T_t\right) \tag{13}$$

$$=\lim_{n\to\infty}(2\pi\mathrm{i}\varepsilon\hbar)^{1/2}\int\limits_{-\infty}^{\infty}\prod_{j=1}^{N-1}(2\pi\mathrm{i}\varepsilon\hbar)^{1/2}\mathrm{d}x_j\times\prod_{j=1}^{N}f(\beta)\exp\left(\frac{\mathrm{i}}{\hbar}\frac{X_{j,\beta}}{\bar{\varepsilon}_\alpha}\right), \tag{14}$$

where

$$\bar{\varepsilon}_\alpha = \frac{t^\alpha - (t-\tau)^\alpha}{N\Gamma(\alpha+1)}, \tag{15}$$

and

$$X_{j,\beta} = x_j^{\beta/(2(\beta-1))}. \tag{16}$$

After $N-1$ steps, after performing the integration and establishing a recursion procedure, we find

$$K_\alpha^0(x_f,t_f;x_i,t_i) = \big(2\pi i\hbar(t_f-t_i)\big)^{-1/2} \exp\left(\frac{if(\beta)\Gamma(\alpha+1)}{\hbar}\,\frac{x_f^{\beta/(\beta-1)}}{t^\alpha-(t-\tau)^\alpha}\right), \tag{17}$$

and, consequently, the mean displacement after simple scaling is given by

$$\Delta x \propto (\Delta t)^{\alpha(\beta-1)/\beta}, \quad 0<\alpha\leq 1, \quad 0<\beta\leq 2\,, \tag{18}$$

which corresponds to a subdiffusion process as $0<\alpha(\beta-1)/\beta\leq 1/2$. Thus we argue that subdiffusion processes in quantum dynamical systems can be realized without implementing any kind of time-fractional derivative if a fractional action integral is used in the theory. Example for the subdiffusion processes are charge transport in amorphous semiconductors. The present result is interesting and several other applications and consequences need to be studied in a future work, in particular the fractional classical Brownian motion with dissipation, a quantum Brownian dissipative motion. A proper and truly unifying foundation of fractional functional integral should be found and a lot of mathematical research still remains to be done.

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## PODDIFUZIJA PREKO DIJELNIH KVANTNIH PUTEVA BEZ DIJELNIH IZVODA

Pokazuje se da se poddifuzijski procesi u kvantnim dinamičkim sustavima mogu ostvariti bez uključivanja bilo koje vrste vremensko-dijelnog izvoda ako se u teoriji rabi dijelni učinski integral.