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SOLITON PERTURBATION THEORY FOR THE KAWAHARA EQUATION

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The Kawahara equation is studied along with its perturbation terms. The adiabatic dynamics of the soliton amplitude and the velocity of the soliton is obtained by the aid of soliton perturbation theory.

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1. Introduction

The theory of nonlinear evolution equations is an ongoing topic of research for decades [1 − 20]. This paper is going to study one of the classical nonlinear evolution equations that is known as the Kawahara equation (KE). The dimensionless form of the KE is given by

$$q_t + qq_x + q_{xxx} - q_{xxxxx} = 0. \tag{1}$$

This dispersive equation was proposed by Kawahara in 1972 as an important dispersive equation that arises in the context of shallow water waves [3].

The KE given by (1) is not integrable by the classical method of inverse scattering transform as this equation will fail the Painleve test of integrability [1]. However, in the last few years, very powerful methods of integrability of nonlinear evolution equations of this type were developed. They include the Wadati trace method, pseudo-spectral method, tanh-sech method, sine-cosine method and the Riccati equation expansion method [7]. It is to be noted that one of the major disadvantage of these modern methods of integrability is that one can obtain only the one-soliton solution of such a nonlinear evolution equation and not a multi-soliton solution. Also, these methods are unable to compute a closed form solution for the soliton radiation. Using the sine-cosine method, the one-soliton solution of (1) is given by [14−16]

$$q(x,t) = \frac{A}{\cosh^4 B(x-\bar{x})}\,, \tag{2}$$

where

$$A = \frac{105}{169} \tag{3}$$

and

$$B = \frac{2}{\sqrt{13}}\,. \tag{4}$$

Here A represents the amplitude of the soliton, B is the inverse width of the soliton and $\bar{x}$ represents the center position of the soliton. Therefore, the velocity of the soliton is given by

$$v = \frac{\mathrm{d}\bar{x}}{\mathrm{d}t}\,. \tag{5}$$

In this paper, the interest will be focused on the one-soliton solution although it is possible to obtain the multi-soliton solution by means of the Hirota's bilinear method.

2. *Mathematical properties*

Equation (1) has at least two integrals of motion [1] that are known as linear momentum (M) and energy (E). These are, respectively, given by

$$M = \int_{-\infty}^{\infty} q\,\mathrm{d}x = \frac{4A}{3B} \tag{6}$$

and

$$E = \int_{-\infty}^{\infty} q^2\,\mathrm{d}x = \frac{32A^2}{35B}\,. \tag{7}$$

These conserved quantities are calculated by using the one-soliton solution given by (2). The center of the soliton $\bar{x}$ is given by the definition

$$\bar{x} = \frac{\int_{-\infty}^{\infty} xq(x,t)\mathrm{d}x}{\int_{-\infty}^{\infty} q(x,t)\mathrm{d}x} = \frac{\int_{-\infty}^{\infty} xq(x,t)\mathrm{d}x}{M}\,, \tag{8}$$

where M is defined in (6). Thus, the velocity of the soliton is given by

$$v = \frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \frac{\int_{-\infty}^{\infty} x q_t \mathrm{d}x}{\int_{-\infty}^{\infty} q \mathrm{d}x} = \frac{\int_{-\infty}^{\infty} x q_t \mathrm{d}x}{M} \tag{9}$$

On using (1) and (9), the velocity of the soliton reduces to

$$v = \frac{36}{169}\,. \tag{10}$$

3. Perturbation terms

The perturbed KE that will be studied in this paper is given by

$$q_t + q q_x + q_{xxx} - q_{xxxxx} = \epsilon R\,, \tag{11}$$

where ϵ is the perturbation parameter, $0 < \epsilon \ll 1$ [1, 2, 12], while R gives the perturbation terms. In the presence of perturbation terms, the momentum and the energy of the soliton do not stay conserved. Instead, they undergo adiabatic changes that lead to the adiabatic deformation of the soliton amplitude, width and a slow change in the velocity [1, 2, 12]. Using (7), the law of adiabatic deformation of the soliton energy is given by [1, 2]

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 2\epsilon \int_{-\infty}^{\infty} x R \mathrm{d}x\,, \tag{12}$$

while the adiabatic law of change of the velocity of the soliton is given by [12]

$$v = \frac{36}{169} + \frac{\epsilon}{M} \int_{-\infty}^{\infty} x R \mathrm{d}x\,. \tag{13}$$

4. Examples

In this paper, the perturbation terms that are going to be considered are

$$\begin{aligned} R = {} & \alpha q + \beta q_{xx} + \gamma q_x q_{xx} + \delta q^m q_x + \lambda q_{xxx} + \nu q q_x q_{xx} + \sigma q_x^3 + \xi q_x q_{xxxx} \\ & + \eta q_{xx} q_{xxx} + \rho q_{xxxx} + \psi q_{xxxxx} + \kappa q q_{xxxxx}\,. \end{aligned} \tag{14}$$

So, the perturbed Kawahara equation is

$$q_t + qq_x + q_{xxx} - q_{xxxxx} \tag{15}$$
$$= \epsilon\Big[\alpha q + \beta q_{xx} + \gamma q_x q_{xx} + \delta q^m q_x + \lambda q_{xxx} + \nu q q_x q_{xx} + \sigma q_x^3 + \xi q_x q_{xxxx} + \eta q_{xx} q_{xxx} + \rho q_{xxxx} + \psi q_{xxxxxx} + \kappa q q_{xxxxx}\Big].$$

The perturbation terms due to αq appears due to the shoaling and βq_{xx} is a dissipative term [5]. The term $\delta q^m q_x$ is due to higher nonlinear dispersion while ψq_{xxxxxx} represents the higher spatial dispersion. In (14), m is a positive integer and $1 \leq m \leq 4$. The term ρq_{xxxx} will provide the higher stabilizing term and must, therefore, be taken into account. The remaining coefficients appear in the context of the Whitham hierarchy [13].

5. *Applications*

In the presence of the above perturbation terms, the adiabatic variation of the energy of the soliton is given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{64\epsilon A^2}{35B}\left(\frac{68608}{429}\kappa - \alpha + \frac{16}{9}\beta B^2 - \frac{1024}{99}\rho B^4\right). \tag{16}$$

The expression for the velocity after the inclusion of the perturbation terms given in (14) is

$$v = \frac{36}{169} + \frac{4\epsilon}{\sqrt{\pi}} \tag{17}$$
$$\times\left[\frac{\delta A^m}{3(m+1)}\frac{\Gamma(2m+2)}{\Gamma(2m+\frac{5}{2})} + \frac{64AB^2}{315}\Big\{11(\gamma-2\lambda) - 64B^2(3\xi-\eta)\Big\} + \frac{1024A^2B^2}{189189}(\nu-38\sigma)\right].$$

It is to be noted that in order to compute the adiabatic variation of the energy of the soliton as well as the change of the velocity of the soliton, the one-soliton solution given by (2) is used. This is the trend that is used in soliton perturbation theory [1]. However, it is possible to obtain the one-soliton solution of the perturbed Kawahara equation given by (15) by the aid of multiple-scale perturbation theory or by the homotopy perturbation theory, just as in the case of KdV equation, modified KdV equation and higher order KdV equation that was done in 1981 [12] and the nonlinear Schrödinger's equation that was done in 2003 [2]. In this context some other relevant references are [3, 5, 10].

6. *Conclusions*

In this paper, soliton perturbation theory is used to study the Kawahara equation. This theory gives the ability to compute the adiabatic variation of the soliton energy and hence the adiabatic variation of the soliton amplitude. This finally leads to the computation of the long-term behaviour of the soliton energy. Also, it is shown that the velocity undergoes a slow change due to the perturbation terms.

In the future, the integration of the perturbed Kawahara equation will be carried out by the aid of multiple-scale perturbation analysis. Thus the quasi-stationary soliton, in the presence of such perturbation terms, will be obtained. These results will be reported in a future publication.

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SOLITONSKA TEORIJA SMETNJE ZA KAWAHARA-OVU JEDNADŽBU

Proučavamo Kawahara-ovu jednadžbu s članovima smetnje. Izveli smo adijabatsku dinamiku solitonske amplitude i brzinu solitona primjenom solitonske teorije smetnje.