Printed ISSN 1330–0008
Online ISSN 1333–9125
CD ISSN 1333–8390
CODEN FIZAE4

THE EXACT SOLUTIONS TO LIENÁRD EQUATION WITH HIGH-ORDER NONLINEAR TERM AND APPLICATIONS

LIU CHENG-SHI

Department of Mathematics, Daqing Petroleum Institute, Daqing 163318, China
E-mail address: chengshiliu-68@126.com

Received 17 August 2008; Accepted 3 June 2009
Online 13 July 2009

Using the complete discrimination system for polynomials, we give the exact solutions to Lienárd equation with strong high-order nonlinear terms which include the elementary functions solutions, elliptic functions solutions and the solutions expressed with implicit functions. As application, we can easily obtain many exact solutions to some nonlinear equations of mathematical physics such as Kundu equation, generalized PC equation, Ablowitz equation, Rangwala-Rao equation, generalized BBM equation, generalized Pochhammer-Chree equation without dispersion term and generalized Ginzburg-Landau equation. To our knowledge, the known results on the exact travelling solutions to these equations in literature can be derived from our results as special cases.

PACS numbers: 02.30.Jr, 05.45.Yv UDC 532.591, 533.951

Keywords: Lienárd equation; complete discrimination system for polynomial, traveling wave solution, exact solution, nonlinear evolution equation

1. *Introduction*

The constructions of exact solutions to nonlinear differential equations have been investigated extensively using a large number of methods $[1-43]$, such as inverse scattering method, Backlund transformation, bilinear method, symmetric method, homogenous balance method, nonlinear variable separate method, tanh-method, elliptic function expansion method, project Riccati equation method, trial equation method, sub-equation method, function transformation method and so on. As a result, many equations have been solved and many new solutions have been claimed. However, under the traveling wave transformation, if the reduced ordinary differential equation has the following integral form

$$\pm(\xi_1 - \xi_0) = \int \frac{\mathrm{d}W}{\sqrt{F(W)}}\,, \tag{1}$$

then what we need is only to solve the integral (1). According to the different parameters, we give different solutions to the integral. This is the so called direct integral method. Although direct integral method is a routine method, to decide the parameter's scope is not easy. Therefore, the key steps are to decide the parameter's scopes and to solve the corresponding integral. A mathematical tool named the complete discrimination system for polynomial [27] is applied to this problem so that the parameter's scopes are solved easily. It is just because combined with complete discrimination system for polynomial the direct integral method becomes a powerful and efficient method. We utilize these complete discrimination systems to find the exact solutions to some nonlinear differential equations, and obtain abundant results [28 − 39].

In the present paper, we consider the following equations [40 − 43]:
Pochhammer-Chree equation

$$u_{tt} - u_{ttxx} - (a_1 u + a_3 u^3 + a_5 u^5)_{xx} = 0\,, \tag{2}$$

Kundu equation

$$iu_t + u_{xx} + \beta|u|^2 u + r|u|^4 u + \mathrm{i}\alpha(|u|^2 u)_x + \mathrm{i}s(|u|^2)_x u = 0\,, \tag{3}$$

Rangwala-Rao equation

$$u_{xt} - \beta_1 u_{xx} + u + \mathrm{i}T\beta_2|u|^2 u_x = 0\,, \quad (T = \pm 1)\,, \tag{4}$$

high-order nonlinear wave equation

$$u_{tt} - k u_{xx} + b_1 u + b_2 u^{p+1} + b_3 u^{2p+1} = 0\,, \tag{5}$$

generalized BBM equation

$$u_t + d\,u^p u_x + b u^{2p} u_x + c u_{xxt} = 0\,, \tag{6}$$

generalized Pochhammer-Chree equation

$$u_{tt} - u_{ttxx} + (b_1 u + b_2 u^{p+1} + b_3 u^{2p+1})_{xx} = 0\,, \tag{7}$$

and generalized Ginzburg-Landau equation

$$\mathrm{i}u_t + b_1 u_{xx} + b_2|u|^{2p} u + b_3|u|^{4p} u + \mathrm{i}b_4\big(|u|^{2p} u\big)_x + \mathrm{i}b_5\big(|u|^{2p}\big)_x u = 0, \quad p > 0\,. \tag{8}$$

If we consider the traveling wave solutions $u(x,t) = a(\xi)$ of Eq. (2) and the envelop traveling wave solution

$$u(x,t) = \mathrm{e}^{\mathrm{i}(\varphi(\xi) - wt)} a(\xi)\,, \quad \xi = x - vt\,, \tag{9}$$

of Eqs. (3) and (4), then $a(\xi)$ satisfies the following Lienárd equation

$$a''(\xi) + la + ma^3 + na^5 = 0\,. \tag{10}$$

Under the traveling wave transformation, Eqs. (5) − (7) are reduced to the ODE

$$a''(\xi) + la + ma^{p+1} + na^{2p+1} = 0\,. \tag{11}$$

and Eq. (8) is reduced to the ODE

$$a''(\xi) + la(\xi) + ma^{2p+1}(\xi) + na^{4p+1}(\xi) = 0\,. \tag{12}$$

Here Eq. (10) is a special case of Eq. (11) with $p = 2$. Eq. (10) has been studied extensively, so we don't discuss it, and we only discuss the Eqs. (11) and (12). It is obvious that the results for Eqs. (2) − (8) given in the references [40] and [43] are special cases of our results.

This paper is organized as follows. In Sec. 2, we give the exact solutions of Eq. (11). In Sec. 3, we give the exact solutions to Eq. (12). In Sec. 4, we discuss some applications. The last section is a summary.

2. Exact solution of the (2p+1)-order Lienárd equation

Integrating Eq. (11) once yields

$$(a')^2 = F(a) = c - la^2 - \frac{2m}{p+2}\,a^{p+2} - \frac{n}{p+1}\,a^{2p+2}\,, \tag{13}$$

where c is an integral constant. We consider the following cases.

2.1. Case $c = 0$

Take the transformation

$$w = a^p(\xi)\,. \tag{14}$$

Then Eq. (13) becomes

$$\int \frac{\mathrm{d}w}{w\sqrt{-l - \dfrac{2m}{p+2}w - \dfrac{n}{p+1}w^2}} = \pm p(\xi - \xi_0)\,. \tag{15}$$

Denote $\Delta = 4\big(m^2/(p+2)^2 - nl/(p+1)\big)$, which is the discriminant of the polynomial in (15). There are the following three cases.

Case 2.1.1. $\Delta = 0$. Then the solution of Eq. (15) is given by

$$\pm p\frac{m(p+1)}{n(p+2)}\sqrt{\left|\frac{n}{p+1}\right|}(\xi-\xi_0) = \ln\left|\frac{w-\dfrac{m(p+1)}{n(p+2)}}{w}\right|. \tag{16}$$

Case 2.1.2. $\Delta > 0$. Then the solution of Eq. (15) is given by

$$\pm\sqrt{-\frac{n}{p+1}}\,(\xi-\xi_0) = \frac{1}{\sqrt{\beta\gamma}}\ln\frac{\{\sqrt{-\gamma(w-\beta)}-\sqrt{-\beta(w-\gamma)}\,\}^2}{|w|}\,, \tag{17}$$

$$\pm\sqrt{-\frac{n}{p+1}}\,(\xi-\xi_0) = \frac{1}{\sqrt{\beta\gamma}}\ln\frac{\{\sqrt{\gamma(w-\beta)}-\sqrt{\beta(w-\gamma)}\,\}^2}{|w|}\,, \tag{18}$$

$$\pm\sqrt{-\frac{n}{p+1}}\,(\xi-\xi_0) = \frac{1}{\sqrt{-\beta\gamma}}\arcsin\frac{-\gamma(w-\beta)-\beta(w-\gamma)}{|w||\beta-\gamma|}\,, \tag{19}$$

where $n/(p+1) < 0$. And

$$\pm\sqrt{\frac{n}{p+1}}\,(\xi-\xi_0) = \frac{1}{\sqrt{-\beta\gamma}}\ln\frac{\{\sqrt{-\gamma(-w+\beta)}-\sqrt{\beta(w-\gamma)}\,\}^2}{|w|}\,, \tag{20}$$

$$\pm\sqrt{\frac{n}{p+1}}\,(\xi-\xi_0) = \frac{1}{\sqrt{-\beta\gamma}}\ln\frac{\{\sqrt{\gamma(-w+\beta)}-\sqrt{-\beta(w-\gamma)}\,\}^2}{|w|}\,, \tag{21}$$

$$\pm\sqrt{\frac{n}{p+1}}\,(\xi-\xi_0) = \frac{1}{\sqrt{\beta\gamma}}\arcsin\frac{-\gamma(-w+\beta)+\beta(w-\gamma)}{|w||\beta-\gamma|}\,, \tag{22}$$

where $n/(p+1) > 0$. In addition, in expressions (17−22), we have

$$\beta = \frac{\dfrac{2m}{p+2}+2\sqrt{\dfrac{m^2}{(p+2)^2}-\dfrac{nl}{p+1}}}{-\dfrac{2n}{p+1}}\,, \quad \gamma = \frac{\dfrac{2m}{p+2}-2\sqrt{\dfrac{m^2}{(p+2)^2}-\dfrac{nl}{p+1}}}{-\dfrac{2n}{p+1}}\,. \tag{23}$$

Case 2.1.3. $\Delta < 0$. Then the solution of Eq. (15) is given by

$$\pm\sqrt{-l}(\xi-\xi_0) = \frac{1}{\sqrt{l^2+s^2}}\ln\left|\frac{\dfrac{m}{(p+2)\sqrt{-l}}w+\sqrt{-l}-\sqrt{-\dfrac{n}{p+1}w^2-\dfrac{2m}{p+2}w-l}}{w}\right|, \tag{24}$$

where $l < 0$.

Remark 1. Substituting the formula (14) into the solutions of Eq. (15) will give the solutions of Eq. (13), and hence gives the solutions of Eq. (11). We omit these concrete formulas.

2.2. Case $c \neq 0$

In principle, we can give the corresponding solutions of Eq. (13) according to the complete discrimination system for polynomial $F(a)$. But this is a rather complicated thing for the polynomials of the sixth order or still high order. We only take $p = 4$ as an example to illustrate our method. Take the transformation

$$w = a^2(\xi)\,. \tag{25}$$

Then Eq. (13) becomes

$$\int \frac{\mathrm{d}w}{\sqrt{w\left(c - lw - \dfrac{2m}{p+2}w^3 - \dfrac{n}{p+1}w^5\right)}} = \pm 2(\xi - \xi_0)\,. \tag{26}$$

Denote

$$f(w) = w^5 + qw^3 + rw + s\,, \tag{27}$$

where

$$q = \frac{2m(p+1)}{n(p+2)}\,, \quad r = \frac{l(p+1)}{n}\,, \quad s = \frac{c(p+1)}{n}\,. \tag{28}$$

From Refs. [27] and [29], the complete discrimination for the polynomial $f(w)$ is given by

$$\begin{gathered} D_2 = -q\,, \\ D_3 = 40rq - 12q^3, \\ D_4 = 12q^4r - 88r^2q^2 + 125qs^2 + 160r^3, \\ D_5 = 2000qs^2r^2 - 900rs^2q^3 + 16q^4r^3 + 108q^5s^2 - 128r^4q^2 + 256r^5 + 3125s^4, \\ E_2 = 160r^2q^3 - 48rq^5 + 625s^2q^2, \\ F_2 = -8rq\,. \end{gathered} \tag{29}$$

Rewrite (26) as

$$\int \frac{\mathrm{d}w}{\sqrt{Awf(w)}} = \pm 2(\xi - \xi_0)\,, \tag{30}$$

where $A = -n/(p+1)$. There are the following eight cases to be discussed.

Case 2.2.1. $D_5 = 0$, $D_4 = 0$, $D_3 > 0$, $E_2 \neq 0$. We have

$$f(w) = (w-\alpha)^2(w-\beta)^2(w-\gamma)\,, \tag{31}$$

where α, β and γ are real, and $\alpha \neq \beta \neq \gamma$. Then

$$\pm 2(\xi - \xi_0) = \int \frac{\mathrm{d}w}{(w-\alpha)(w-\beta)\sqrt{Aw(w-\gamma)}}$$

$$= \frac{1}{\alpha-\beta}\int\left(\frac{1}{w-\alpha} - \frac{1}{w-\beta}\right)\frac{\mathrm{d}w}{\sqrt{Aw(w-\gamma)}}\,. \tag{32}$$

The corresponding solutions are given by

$$\pm 2(\xi - \xi_0) = \frac{1}{\alpha-\beta}\frac{1}{\sqrt{A\alpha(\alpha-\gamma)}}\ln\frac{\left(\sqrt{\epsilon_1 Aw(\alpha-\gamma)} - \sqrt{\epsilon_1 A\alpha(w-\gamma)}\,\right)^2}{|w-\alpha|}$$

$$-\frac{1}{\alpha-\beta}\frac{1}{\sqrt{-A\beta(\beta-\gamma)}}\arcsin\frac{Aw(\beta-\gamma)+A\beta(w-\gamma)}{|A\gamma(w-\beta)|}\,, \tag{33}$$

$$\pm 2(\xi - \xi_0) = \frac{1}{\alpha-\beta}\frac{1}{\sqrt{A\alpha(\alpha-\gamma)}}\ln\frac{\left(\sqrt{\epsilon_1 Aw(\alpha-\gamma)} - \sqrt{\epsilon_1 A\alpha(w-\gamma)}\,\right)^2}{|w-\alpha|}$$

$$-\frac{1}{\alpha-\beta}\frac{1}{\sqrt{A\beta(\beta-\gamma)}}\ln\frac{\left(\sqrt{\epsilon_2 Aw(\beta-\gamma)} - \sqrt{\epsilon_2 A\beta(w-\gamma)}\,\right)^2}{|w-\beta|}\,, \tag{34}$$

$$\pm 2(\xi - \xi_0) = \frac{1}{\alpha-\beta}\frac{1}{\sqrt{-A\alpha(\alpha-\gamma)}}\arcsin\frac{Aw(\alpha-\gamma)+A\alpha(w-\gamma)}{|A\gamma(w-\alpha)|}$$

$$-\frac{1}{\alpha-\beta}\frac{1}{\sqrt{A\beta(\beta-\gamma)}}\ln\frac{\left(\sqrt{\epsilon_2 Aw(\beta-\gamma)} - \sqrt{\epsilon_2 A\beta(w-\gamma)}\,\right)^2}{|w-\beta|}\,, \tag{35}$$

$$\pm 2(\xi - \xi_0) = \frac{1}{\alpha-\beta}\frac{1}{\sqrt{-A\alpha(\alpha-\gamma)}}\arcsin\frac{Aw(\alpha-\gamma)+A\alpha(w-\gamma)}{|A\gamma(w-\alpha)|}$$

$$-\frac{1}{\alpha-\beta}\frac{1}{\sqrt{-A\beta(\beta-\gamma)}}\arcsin\frac{Aw(\beta-\gamma)+A\beta(w-\gamma)}{|A\gamma(w-\beta)|}\,, \tag{36}$$

where $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$.

Case 2.2.2. $D_5 = 0$, $D_4 = 0$, $D_3 = 0$, $D_2 \neq 0$, $F_2 \neq 0$. We have

$$f(w) = (w-\alpha)^3 (w-\beta)^2 \,, \tag{37}$$

where α and β are real, and $\alpha \neq \beta$. Then

$$\pm 2(\xi-\xi_0) = \int \frac{\mathrm{d}w}{(w-\alpha)(w-\beta)\sqrt{Aw(w-\alpha)}} = \frac{1}{\alpha-\beta}\int \left(\frac{1}{w-\alpha} - \frac{1}{w-\beta}\right) \frac{\mathrm{d}w}{\sqrt{Aw(w-\alpha)}} \,. \tag{38}$$

The corresponding solutions are given by

$$\pm 2(\xi-\xi_0) = \frac{1}{\alpha-\beta}\frac{-2}{A\alpha}\sqrt{\frac{Aw}{w-\alpha}}$$

$$-\frac{1}{\alpha-\beta}\frac{1}{\sqrt{-A\beta(\beta-\alpha)}} \arcsin \frac{Aw(\beta-\alpha)+A\beta(w-\alpha)}{|A\alpha(\nu-\beta)|} \,, \tag{39}$$

$$\pm 2(\xi-\xi_0) = \frac{1}{\alpha-\beta}\frac{-2}{A\alpha}\sqrt{\frac{Aw}{w-\alpha}}$$

$$-\frac{1}{\alpha-\beta}\frac{1}{\sqrt{A\beta(\beta-\alpha)}} \ln \frac{\left(\sqrt{\epsilon_2 Aw(\beta-\alpha)} - \sqrt{\epsilon_2 A\beta(w-\alpha)}\right)^2}{|w-\beta|} \,, \tag{40}$$

where $\epsilon_2 = \pm 1$.

Case 2.2.3. $D_5 = 0$, $D_4 = 0$, $D_3 = 0$, $D_2 \neq 0$, $F_2 = 0$. Then we have

$$f(w) = (w-\alpha)^4 (w-\beta) \,, \tag{41}$$

where α and β are real, and $\alpha \neq \beta$. Then

$$\pm 2(\xi-\xi_0) = \int \frac{\mathrm{d}w}{(w-\alpha)^2\sqrt{Aw(w-\beta)}}$$

$$= \frac{-1}{A\alpha(\alpha-\beta)}\left[\frac{\sqrt{Aw(w-\beta)}}{w-\alpha} + A(\alpha-\frac{1}{2}\beta)\int \frac{\mathrm{d}w}{(w-\alpha)\sqrt{Aw(w-\beta)}}\right] . \tag{42}$$

The corresponding solutions are given by

$$\pm 2(\xi-\xi_0) = \frac{-1}{A\alpha(\alpha-\beta)}\left[\frac{\sqrt{Aw(w-\beta)}}{w-\alpha}\right.$$

$$+A\left(\alpha-\frac{1}{2}\beta\right)\frac{1}{\sqrt{A\alpha(\alpha-\beta)}}\ln\frac{\left(\sqrt{\epsilon_2 Aw(\alpha-\beta)}-\sqrt{\epsilon_2 A\alpha(w-\beta)}\right)^2}{|w-\alpha|}\Bigg], \tag{43}$$

$$\pm 2(\xi-\xi_0)=\frac{-1}{A\alpha(\alpha-\beta)}\left[\frac{\sqrt{Aw(w-\beta)}}{w-\alpha}\right.$$

$$\left.+A\left(\alpha-\frac{1}{2}\beta\right)\frac{1}{\sqrt{-A\alpha(\alpha-\beta)}}\arcsin\frac{Aw(\alpha-\beta)+A\alpha(w-\beta)}{|A\beta(w-\alpha)|}\right], \tag{44}$$

where $\epsilon_2=\pm 1$.

Case 2.2.4. $D_5=0$, $D_4=0$, $D_3<0$, $E_2\neq 0$. We have

$$f(w)=((w-l)^2+s^2)^2(w-\alpha)\,, \tag{45}$$

where $s>0$. Then

$$\pm 2(\xi-\xi_0)=\frac{\beta_1}{\alpha\rho^2}\left(\arctan\frac{s}{w-l}+\arctan\frac{\gamma_2(w-l)+\delta_2}{\gamma_1(w-l)+\delta_1-\sqrt{Aw(w-\alpha)}}\right)$$

$$+\frac{\beta_2}{2\alpha\rho^2}\ln\left|\frac{(w-l)^2+s^2}{\left(\gamma_1(w-l)+\delta_1-\sqrt{Aw(w-\alpha)}\right)^2+\left(\gamma_2(w-l)+\delta_2\right)^2}\right|. \tag{46}$$

where

$$\beta_1=\pm\sqrt{\frac{1}{2}\rho^2+\frac{1}{2}(Al(l-\alpha)-As^2)},\quad \beta_2=\pm\sqrt{\frac{1}{2}\rho^2-\frac{1}{2}(Al(l-\alpha)-As^2)},$$

$$\rho^2=\sqrt{4\left(\frac{A}{2}(2l-\alpha)s\right)^2+(Al(l-\alpha)-As^2)^2}\,,$$

$$\gamma_1=\frac{1}{\rho^2}\left(As\beta_2+A\left(l-\frac{1}{2}s\right)\beta_1\right),\quad \gamma_2=\frac{1}{\rho^2}\left(As\beta_2-A\left(l-\frac{1}{2}s\right)\beta_1\right),$$

$$\delta_1=\frac{1}{\rho^2}\left(Al(l-\alpha)\beta_1+A\left(l-\frac{1}{2}s\right)s\beta_2\right),\ \delta_2=\frac{1}{\rho^2}\left(-Al(l-\alpha)\beta_2+A\left(l-\frac{1}{2}s\right)s\beta_1\right), \tag{47}$$

and we choose β_1 and β_2 such that $\beta_1\beta_2=As(l-\alpha/2)$.

Case 2.2.5. $D_5=0$, $D_4>0$. We have

$$f(w)=(w-\alpha)^2(w-\alpha_1)(w-\alpha_2)(w-\alpha_2)\,,\quad \alpha_1>\alpha_2>\alpha_3. \tag{48}$$

Then

$$\pm 2(\xi - \xi_0) = \int \frac{\mathrm{d}w}{(w-\alpha)\sqrt{Aw(w-\alpha_1)(w-\alpha_2)(w-\alpha_3)}} \,. \tag{49}$$

The corresponding solutions are given by

$$\pm 2(\xi - \xi_0) = \frac{2\delta}{(a-\alpha c)\gamma} \left\{ cF(\varphi, k) + \frac{\delta}{b-\alpha d} \Pi(\varphi, \frac{a-\alpha c}{b-\alpha d}, k) \right\}, \tag{50}$$

where we reorder α_1, α_2, α_3, 0, so that $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$. Then, when $A > 0$, $w \geq \alpha_1$ or $w \leq \alpha_4$ (other case can be obtained similarly, we omit them), the parameters in (50) are given by

$$a = \alpha_2(\alpha_1 - \alpha_4)\,, \qquad b = -\alpha_1(\alpha_2 - \alpha_4)\,, \qquad c = \alpha_1 - \alpha_4\,,$$

$$d = \alpha_2 - \alpha_4\,, \qquad \delta = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_4)\,,$$

$$\frac{\delta}{\gamma} = \frac{1}{\sqrt{|A|(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}\,, \qquad k^2 = \frac{(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}\,, \tag{51}$$

$$F(\varphi, k) = \int\limits_0^{\varphi} \frac{\mathrm{d}\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}\,, \tag{52}$$

$$\Pi(\varphi, n, k) = \int\limits_0^{\varphi} \frac{\mathrm{d}\varphi}{(1 + n \sin^2 \varphi)\sqrt{1 - k^2 \sin^2 \varphi}}\,. \tag{53}$$

Case 2.2.6. $D_5 = 0$, $D_4 = 0$, $D_3 > 0$, $E_2 = 0$. We have

$$f(w) = (w-\alpha)^3 (w-\beta)(w-\rho)\,, \tag{54}$$

where we reorder α, β, ρ, 0, so that $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$. Then, when $A > 0$, $w \geq \alpha_1$ or $w \leq \alpha_4$ (other case can be obtained similarly, we omit them), we have

$$\pm 2(\xi - \xi_0) = \frac{-2c}{k^2 \gamma} \left\{ (c + k^2 d) F(\varphi, k) - cE(\varphi, k) \right\}, \quad \alpha_1 = \alpha\,, \tag{55}$$

$$\pm 2(\xi - \xi_0) = \frac{2d}{\gamma} \left\{ (c+d)F(\varphi, k) - dE(\varphi, k) - d\mathrm{cotan}(\varphi \sqrt{1 - k^2 \sin^2 \varphi}) \right\}, \quad \alpha_2 = \alpha\,, \tag{56}$$

where these parameters are the same as in (51) and (52). Note,

$$E(\varphi, k) = \int_0^{\varphi_i} \sqrt{1 - k^2 \sin^2 \psi}\, \mathrm{d}\psi \,. \tag{57}$$

Case 2.2.7. $D_5 = 0$, $D_4 = 0$, $D_3 < 0$, $E_2 = 0$. We have

$$f(w) = (w - \alpha)^3 \big((w - l)^2 + s^2\big) \,, \tag{58}$$

where we reorder α, 0 so that $\alpha_1 > \alpha_2$. Then

$$\pm 2(\xi - \xi_0) = \frac{cd}{\gamma} F(\varphi, k) + \frac{c^2}{k\gamma} \arcsin(k \sin\varphi), \quad \alpha_1 = \alpha \,, \tag{59}$$

$$\pm 2(\xi - \xi_0) = \frac{-d^2}{\gamma\sqrt{1 - k^2}} \ln \frac{\sqrt{1 - k^2 \sin^2\varphi} + \sqrt{1 - k^2} \sin\varphi}{\cos\varphi} - \frac{cd}{\gamma} F(\varphi, k) \,, \quad \alpha_2 = \alpha \,, \tag{60}$$

where

$$a = \frac{1}{2}(\alpha_1 + \alpha_2)c - \frac{1}{2}(\alpha_1 - \alpha_2)d \,, \qquad b = \frac{1}{2}(\alpha_1 + \alpha_2)d - \frac{1}{2}(\alpha_1 - \alpha_2)c \,,$$

$$c = \alpha_1 - l - \frac{s}{m_1} \,, \qquad d = \alpha_1 - l - s m_1 \,, \qquad E = \frac{s^2 + (\alpha_1 - l)(\alpha_2 - l)}{s(\alpha_1 - \alpha_2)} \,,$$

$$m_1 = E \pm \sqrt{E^2 + 1} \,, \qquad m^2 = \frac{1}{1 + m_1^2} \,, \tag{61}$$

and we choose m_1 such that $Am_1 < 0$. Other parameters are the same as in the former case.

Case 2.2.8. All remaining cases are such that $f(w)$ has not multiple roots, then the corresponding solutions are given by hyperelliptic functions. We omit them.

Remark 2. Substituting the formula (25) into the solutions of Eq. (15) given in the cases 2.2.1– 2.2.8 yields the solutions of Eq. (13), and hence gives the solutions of Eq. (11). We omit these formulae.

3. Exact solutions of (4p+1)-order Lienárd equation

Integrating Eq. (12) yields

$$a_\xi^2 = -la^2 - \frac{m}{p+1} a^{2(p+1)} - \frac{n}{2p+1} a^{4p+2} + 2c \,. \tag{62}$$

Taking the transformation

$$a(\xi) = \pm(W(\xi))^{-1/(2p)}\,, \tag{63}$$

and substituting it into Eq. (62) gives

$$W_\xi^2 = -4p^2 l W^2 - \frac{4p^2 m}{p+1} W - \frac{4p^2 n}{2p+1} + 8p^2 c W^{2+1/p}\,. \tag{64}$$

There are the following two cases to be discussed.

3.1. Case $c = 0$

Then Eq. (64) becomes

$$\pm(\xi - \xi_0) = \frac{1}{2|p|} \int \frac{\mathrm{d}W}{\sqrt{-lW^2 - (mW/(p+1)) - (n/(2p+1))}}\,. \tag{65}$$

The corresponding solutions of Eq. (65) are given by

$$\pm(\xi - \xi_0) = \frac{1}{2|p|\sqrt{-l}} \ln\left| -2W - \frac{m}{p+1} + 2\sqrt{-l}\sqrt{-lW^2 - \frac{mW}{p+1} - \frac{n}{2p+1}}\right|, \quad l < 0\,, \tag{66}$$

$$\pm(\xi - \xi_0) = \frac{1}{2|p|\sqrt{l}} \arcsin \frac{2lW + m/(p+1)}{\sqrt{(m/(p+1))^2 - 4nl/(2p+1)}}, \quad l > 0\,. \tag{67}$$

Therefore, we have

$$W = \frac{\left(\mathrm{e}^{2p\sqrt{-l}(\xi-\xi_0)} + m/(p+1)\right)^2 - 4nl/(2p+1)}{-4l\mathrm{e}^{2p\sqrt{-l}(\xi-\xi_0)}}, \quad l < 0\,, \tag{68}$$

$$W = \frac{\left(\mathrm{e}^{-2p\sqrt{-l}(\xi-\xi_0)} + m/(p+1)\right)^2 - 4nl/(2p+1)}{-4l\mathrm{e}^{-2p\sqrt{-l}(\xi-\xi_0)}}, \quad l < 0\,, \tag{69}$$

$$W = \frac{\left(-\mathrm{e}^{2p\sqrt{-l}(\xi-\xi_0)} + m/(p+1)\right)^2 - 4nl/(2p+1)}{4l\mathrm{e}^{2p\sqrt{-l}(\xi-\xi_0)}}, \quad l < 0\,, \tag{70}$$

$$W = \frac{\left(-\mathrm{e}^{-2p\sqrt{-l}(\xi-\xi_0)} + m/(p+1)\right)^2 - 4nl/(2p+1)}{4l\mathrm{e}^{-2p\sqrt{-l}(\xi-\xi_0)}}, \quad l < 0\,, \tag{71}$$

$$W = \frac{1}{2l}\left[\pm\sqrt{\left(\frac{m}{p+1}\right)^2 - \frac{4nl}{2p+1}}\sin\left(\sqrt{l}\,2p(\xi-\xi_0)\right) - \frac{m}{p+1})\right], \quad l > 0\,. \quad (72)$$

3.2. Case $c \neq 0$

If $p = 1$ or $p = 1/2$, then this is reduced to the former cases. If $p = 2$ or $p = 1/3$, we have discussed it in another paper [31]. We discuss other cases in the following. First, we point out that if p is a positive integer, then we take Eq. (62), and if p is a fraction $1/q$, then we take Eq. (64). When p is taken as p_1/q, where p_1 and q are positive integers with $(p_1, q) = 1$, we take the transformation

$$W = u^q. \quad (73)$$

Substituting (73) into Eq. (64) yields

$$q^2 u^{2(q+1)} u_\xi^2 = -4p_1^2 l u^{2q} - \frac{4p_1^2 m}{p_1+1} - \frac{4p_1^2 n}{2p_1+1} + 8p_1^2 c u^{q+2p_1}. \quad (74)$$

Rewrite Eq. (74) in the integral form

$$\pm\frac{2p_1}{q}(\xi-\xi_0) = \int \frac{u^{q+1}\mathrm{d}u}{\sqrt{-lu^{2q} - (m/(p_1+1))u^q - n/(2p_1+1) + 2cu^{q+2p_1}}}\,. \quad (75)$$

For example, when $q = 3$ and $p_1 = 2$, then we have a seventh-order polynomial in the integrand. If the polynomial has three real roots with multiplicities two and a single real root, then the right-hand side of (75) is given by

$$\int \frac{u^4\mathrm{d}u}{(u-\alpha_1)(u-\alpha_2)(u-\alpha_3)\sqrt{u-\alpha_4}}\,, \quad (76)$$

and its solution is expressed by

$$\int \frac{u^4\mathrm{d}u}{(u-\alpha_1)(u-\alpha_2)(u-\alpha_3)\sqrt{u-\alpha_4}} = \int \frac{(u-A)\mathrm{d}u}{\sqrt{u-\alpha_4}} + \sum_{i=1}^{3}\int \frac{A_i\mathrm{d}u}{(u-\alpha_i)\sqrt{u-\alpha_4}}$$

$$= \frac{2}{3}(u-\alpha_4)^{3/2} + 2(\alpha_4 - A)\sqrt{u-\alpha_4} + \sum_{i=1}^{3} A_i I_i\,, \quad (77)$$

where

$$I_i = \frac{1}{\sqrt{\alpha_i-\alpha_4}}\ln\left|\frac{\sqrt{u-\alpha_4}-\sqrt{\alpha_i-\alpha_4}}{\sqrt{u-\alpha_4}+\sqrt{\alpha_i-\alpha_4}}\right|, \quad \alpha_i - \alpha_4 > 0\,, \quad (78)$$

$$I_i = \frac{-2}{\sqrt{-\alpha_i+\alpha_4}} \arctan\sqrt{-\frac{u-\alpha_4}{\alpha_i-\alpha_4}}, \quad \alpha_i - \alpha_4 < 0\,, \tag{79}$$

and

$$A = -(\alpha_1+\alpha_2+\alpha_3), \quad A_i = D_i/D, \quad i = 1,\,2,\,3, \tag{80}$$

$$D = -\begin{vmatrix} 1 & 1 & 1 \\ \alpha_2+\alpha_3 & \alpha_1+\alpha_3 & \alpha_1+\alpha_2 \\ \alpha_2\alpha_3 & \alpha_1\alpha_3 & \alpha_1\alpha_2 \end{vmatrix} = \alpha_1^2(\alpha_3-\alpha_2)+\alpha_2^2(\alpha_1-\alpha_3)+\alpha_3^2(\alpha_2-\alpha_1), \tag{81}$$

where D_i is obtained by replacing the i^{th} volume vector in D by $(a_1, a_2, a_3)^\tau$, where

$$a_2 = -\Big[\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3+(\alpha_1+\alpha_2+\alpha_3)A\Big],$$

$$a_1 = (\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)A+\alpha_1\alpha_2\alpha_3\,, \quad a_0 = -A\alpha_1\alpha_2\alpha_3\,. \tag{82}$$

Other cases can be given similarly, but we omit them for simplicity.

4. Applications

For brevity, we consider only Eq. (3) and Eq. (8) as examples to discuss applications. To write the concrete solutions is only a simple exercise, so we omit them.

4.1. Kundu equation

Under the following condition

$$\varphi'(\xi) = \frac{\nu}{2} - \frac{3\alpha+2s}{4}a^2(\xi)\,, \tag{83}$$

$a(\xi)$ satisfies [43]

$$a''(\xi)+\left(\frac{\nu^2}{4}-w\right)a(\xi)+\left(\beta-\frac{\alpha\nu}{2}\right)a^3(\xi)+\left[\gamma+\frac{1}{16}(3\alpha+2s)(\alpha-2s)\right]a^5(\xi)=0\,, \tag{84}$$

which is just the Lienárd Eq. (10), where $l = (\nu^2/4) - w$, $m = \beta - \alpha\nu/2$ and $n = \gamma + (1/16)(3\alpha+2s)(\alpha-2s)$.

Remark 3. The results in Refs. [40] and [41] and those in Eqs. (2−4) in Ref. [42] are special cases of our results.

4.2. *Generalized Ginzburg-Landau equation*

Under the following condition

$$\varphi'(\xi) = \frac{\nu}{2b_1} - \frac{b_4 + 2p(b_4 + b_5)}{2b_1(1+p)} a^{2p}(\xi)\,, \tag{85}$$

$a(\xi)$ satisfies [43]

$$a''(\xi) + la(\xi) + ma^{2p+1}(\xi) + na^{4p+1}(\xi) = 0\,, \tag{86}$$

where

$$l = \frac{\nu^2 + 4b_1 w}{4b_1^2}\,, \qquad m = \frac{2b_1 b_2 - b_4 \nu}{2b_1^2}\,,$$

$$n = \frac{4b_1 b_3 (1+p)^2 + (b_4 - 2pb_5)(b_4 + 2p(b_4 + b_5))}{4b_1^2(1+p)^2}\,. \tag{87}$$

Using the results of Eq. (86) in Sec. 3, we can give many exact solutions to Eq. (8). To our knowledge, the results for the case $c \neq 0$ are novel.

Remark 4. It is obvious that our method is simpler than the method in Ref. [42].

5. *Summary*

We use the complete discrimination system for polynomials to obtain the exact solutions to Lienárd equations with $(2p+1)$- and $(4p+1)$-order nonlinear term. The solutions include elementary function solutions, elliptic function solutions and the exact solutions expressed with implicit functions. According to these results, we can obtain abundant exact traveling wave solutions to many nonlinear evolution equations. Among these, many solutions are new. Our method is so simple and direct that it can become a general principle. This shows that the complete discrimination system for polynomials plays an important role in solving the nonlinear differential equations.

References

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolutions and Inverse Scattering*, Cambridge University Press, Cambridge (1991).

[2] R. M. Miura, ed., *Bachlund Transformation*, Lecture Notes in Mathematics 515, Springer-Verlag, New York (1976).

[3] R. Hirota, *Direct Method in Soliton Theory*, in *Solitons*, eds. R. K. Bullough and P. J. Caudrey, Topics in Current Physics 17, Springer Verlag, New York (1980) p. 157.

[4] E. S. Cheb-Terrab, L. G. S. Duarte and L. A. C. P. da Mota, Comput. Phys. Commun. **101** (1997) 254.

[5] E. S. Cheb-Terrab, L. G. S. Duarte and L. A. C. P. da Mota Comput. Phys. Commun. **108** (1998) 90.
[6] M. Gao, Y. Kato and M. Ito, Comput. Phys. Commun. **148** (2002) 242.
[7] M. L. Wang, Y. B. Zhou and Z. B. Li, Phys. Lett. A **216** (1996) 67.
[8] S. Y. Lou, J. Math. Phys. **40** (1999) 6491.
[9] D. Baldwin, Ü. Göktas and W. Hereman, Comput. Phys. Commun. **162** (2004) 203.
[10] Zhi-Bin Li and Yin-Ping Liu, Comput. Phys. Commun. **163** (2004) 191.
[11] M. L. Wang, Phys. Lett. A **199** (1995) 169.
[12] E. G. Fan, Phys. Lett. A **277** (2000) 212.
[13] S. K. Liu, Z. T. Fu, S. D. Liu and Q. Zhao, Phys. Lett. A **290** (2001) 72.
[14] Z .Y. Yan, Chaos, Solitons and Fractals **15** (2003) 575.
[15] S. Y. Lou and J. Z. Lu, J. Phys. A **29** (1996) 4029.
[16] E. G. Fan, J. Phys. Soc. Japan **71** (2002) 2663.
[17] E. G. Fan, Chaos, Solitons and Fractals **16** (2003) 819.
[18] E. G. Fan, J. Phys. A **35** (2002) 6853.
[19] E. G. Fan, J. Phys. A **33** (2000) 6925.
[20] C. S. Liu, Acta. Phys. Sin. **54** (2005) 2505 (in Chinese).
[21] C. S. Liu, Acta. Phys. Sin. **54** (2005) 4506 (in Chinese).
[22] Z. T. Fu et al., Phys. Lett. A **299** (2002) 507.
[23] Z. T. Fu et al., Phys. Lett. A **290** (2001) 72.
[24] E. G. Fan, Phys. Lett. A **305** (2002) 383.
[25] Y. Z. Peng, Phys. Lett. A **314** (2003) 402.
[26] Sirendaoreji and Sun, J. Phys. Lett. A **298** (2002) 133.
[27] L. Yang, X. R. Hou and Z. B. Zeng, Science in China, Ser E **39** (1996) 628.
[28] C. S. Liu and X. H. Du, Acta Phys. Sin. **54** (2005) 1709.
[29] C. S. Liu, Chin. Phys. Lett. **21** (2004) 1369.
[30] C. S. Liu, Chin. Phys. **14** (2005) 1710.
[31] C. S. Liu, Commun. Theor. Phys. **43** (2005) 787.
[32] C. S. Liu, Commun. Theor. Phys. **44** (2005) 799.
[33] C. S. Liu, Commun. Theor. Phys. **45** (2006) 219.
[34] C. S. Liu, Chin. Phys. **16** (2007) 1832.
[35] X. H. Du and C. S. Liu, Commun. Theor. Phys. **46** (2006) 5.
[36] C. S. Liu, Commun. Theor. Phys. **45** (2006) 395.
[37] C. S. Liu, Commun. Theor. Phys. **45** (2006) 991.
[38] C. S. Liu, Commun. Theor. Phys. **48** (2007) 601.
[39] C. S. Liu, Commun. Theor. Phys. **49** (2008) 158.
[40] W. G. Zhang, Acta Math. Appl. **21** (1998) 249 (in Chinese).
[41] Naranmandula et al., Acta Sin. Phys. **53** (2004) 11 (in Chinese).
[42] W. G. Zhang, Acta. Math. Sci. A **23** (2003) 679 (in Chinese).
[43] W. G. Zhang et al., Commun. Theor. Phys. **41** (2004) 849.

TOČNA RJEŠENJA LIENÁRDOVE JEDNADŽBE S NELINEARNIM ČLANOM VIŠEG REDA I PRIMJENE

Primjenom potpunog diskriminacijskog sustava za polinome, izveli smo točna rješenja Lienárdove jednadžbe s jako nelinearnim članom višeg reda koja uključuju elementarne funkcije, eliptičke funkcije i rješenja izražena implicitnim funkcijama. Kao primjene, mogu se lako dobiti mnoga točna rješenja nekih nelinernih jednadžbi matematičke fizike, kao što su Kunduova jednadžba, poopćena PC jednadžba, Ablowitzova jednadžba, Rangwala-Raova jednadžba, poopćena BBM jednadžba, poopćena Pochhammer-Chreeva jednadžba bez disperzijskog člana te poopćena Ginzburg-Landauova jednadžba. Prema našim saznanjima, u literaturi poznata točna rješenja tih jednadžbi za putujuće valova mogu se izvesti iz naših rezultata kao posebni slučajevi.