Nekoliko dokaza jedne trigonometrijske nejednakosti

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U ovom prilogu ćemo dati nekoliko raznih dokaza ove trigonometrijske nejednakosti:

$$\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\geqslant\sqrt{3}, \tag{1}$$

gdje su α, β, γ unutarnji kutovi trokuta ABC.

Dokaz 1. Imamo $\alpha+\beta+\gamma=180^\circ$, tj. $\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=90^\circ$, a odavde:

$$\begin{aligned}
&\frac{\alpha}{2}+\frac{\beta}{2}=90^\circ-\frac{\gamma}{2}\\
\Longleftrightarrow\ &\operatorname{tg}\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)=\operatorname{tg}\left(90^\circ-\frac{\gamma}{2}\right)\\
\Longleftrightarrow\ &\frac{\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}}{1-\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}}=\frac{1}{\operatorname{tg}\frac{\gamma}{2}}\\
\Longleftrightarrow\ &\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}+\operatorname{tg}\frac{\gamma}{2}\operatorname{tg}\frac{\alpha}{2}=1.
\end{aligned}\tag{2}$$

Koristeći nejednakost

$$x^2+y^2+z^2\geqslant xy+yz+zx,\quad x,y,z\in\mathbb{R}$$

dobivamo za $x=\operatorname{tg}\frac{\alpha}{2}$, $y=\operatorname{tg}\frac{\beta}{2}$, $z=\operatorname{tg}\frac{\gamma}{2}$

$$\operatorname{tg}^2\frac{\alpha}{2}+\operatorname{tg}^2\frac{\beta}{2}+\operatorname{tg}^2\frac{\gamma}{2}\geqslant\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}+\operatorname{tg}\frac{\gamma}{2}\operatorname{tg}\frac{\alpha}{2}. \tag{3}$$

Imamo sljedeću jednakost

$$\left(\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\right)^2=\operatorname{tg}^2\frac{\alpha}{2}+\operatorname{tg}^2\frac{\beta}{2}+\operatorname{tg}^2\frac{\gamma}{2}+2\left(\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}+\operatorname{tg}\frac{\gamma}{2}\operatorname{tg}\frac{\alpha}{2}\right),$$

a zbog (3) i (2)

$$\left(\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\right)^2\geqslant 3\left(\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}+\operatorname{tg}\frac{\gamma}{2}\operatorname{tg}\frac{\alpha}{2}\right),$$

tj.

$$\left(\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\right)^2\geqslant 3\implies\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\geqslant\sqrt{3}.$$

Jednakost u (1) vrijedi ako i samo ako je $\operatorname{tg}\frac{\alpha}{2}=\operatorname{tg}\frac{\beta}{2}=\operatorname{tg}\frac{\gamma}{2}\iff\alpha=\beta=\gamma=60^\circ$, tj. ako je trokut jednakostraničan.

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Dokaz 2. Najprije ćemo dokazati jednakost

$$\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}=\frac{4R+r}{s} \tag{4}$$

gdje su R i r radijusi opisane i upisane kružnice trokuta ABC, a s je njegov poluopseg. Neka su r_a, r_b, r_c radijusi pripisanih kružnica tog trokuta. Tada imamo:

$$\begin{aligned}
r_a+r_b+r_c-r&=\frac{P}{s-a}+\frac{P}{s-b}+\frac{P}{s-c}-\frac{P}{s}\\
&=P\left[\left(\frac{1}{s-a}+\frac{1}{s-b}\right)+\left(\frac{1}{s-c}-\frac{1}{s}\right)\right]\\
&=P\left[\frac{s-b+s-a}{(s-a)(s-b)}+\frac{s-s+c}{s(s-c)}\right]\\
&=P\left[\frac{2s-(a+b)}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right]=cP\cdot\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\\
&=cP\cdot\frac{2s^2-s(a+b+c)+ab}{s(s-a)(s-b)(s-c)}=\frac{abcP}{P^2}=\frac{abc}{P}=\frac{abc}{\frac{abc}{4R}}=4R,
\end{aligned}$$

tj.

$$r_a+r_b+r_c=4R+r. \tag{5}$$

(Ovdje smo koristili Heronovu formulu za površinu trokuta, $P=\sqrt{s(s-a)(s-b)(s-b)}$, kao i formulu $abc=4PR$.)

Imamo dalje:

$$r_a=\frac{P}{s-a}=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a}=s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Kako je

$$\operatorname{tg}\frac{\alpha}{2}=\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}},$$

te $\cos\alpha=\dfrac{b^2+c^2-a^2}{2bc}$, dobivamo

$$\operatorname{tg}\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

odnosno

$$\operatorname{tg}\frac{\alpha}{2}=\frac{r_a}{s},\quad \text{te analogno}\quad \operatorname{tg}\frac{\beta}{2}=\frac{r_b}{s}\quad \text{i}\quad \operatorname{tg}\frac{\gamma}{2}=\frac{r_c}{s}.$$

Zbrajajući ove tri jednakosti dobivamo, zbog (5):

$$\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}=\frac{4R+r}{s},$$

a ovo je (4).

Dokazat ćemo sada i nejednakost

$$4R + r \geqslant s\sqrt{3}. \tag{6}$$

Polazimo od nejednakosti

$$\left(r_a + r_b + r_c\right)^2 \geqslant 3\left(r_a r_b + r_b r_c + r_c r_a\right). \tag{7}$$

Imamo:

$$\begin{aligned} r_a r_b + r_b r_c + r_c r_a &= \frac{P^2}{(s-a)(s-b)} + \frac{P^2}{(s-b)(s-c)} + \frac{P^2}{(s-c)(s-a)} \\ &= P^2 \cdot \frac{s-c+s-a+s-b}{(s-a)(s-b)(s-c)} = P^2 \cdot \frac{3s-(a+b+c)}{\frac{P^2}{s}} = s^2, \end{aligned}$$

tj.

$$r_a r_b + r_b r_c + r_c r_a = s^2. \tag{8}$$

Sada iz (5), (7) i (8) slijedi $(4R+r)^2 \geqslant 3s^2$, tj. $4R + r \geqslant s\sqrt{3}$, a ovo je nejednakost (6).

Konačno iz (4) i (6) slijedi (1).

Dokaz 3. Koristeći poznate formule

$$\operatorname{tg}\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \operatorname{tg}\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \quad \text{i} \quad \operatorname{tg}\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

dobivamo

$$\begin{aligned} &\operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2} \\ &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} + \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \\ &= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-a)^2}} + \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-b)^2}} + \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-c)^2}} \\ &= \frac{P}{s(s-a)} + \frac{P}{s(s-b)} + \frac{P}{s(s-c)} = \frac{rs}{s}\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right) \\ &= r \cdot \frac{(s-b)(s-c) + (s-a)(s-c) + (s-a)(s-b)}{(s-a)(s-b)(s-c)} \\ &= r \cdot \frac{3s^2 - 2s(a+b+c) + ab + bc + ca}{\frac{P^2}{s}} \\ &= r \cdot \frac{3s^2 - 4s^2 + s^2 + r^2 + 4Rr}{r^2 s} = \frac{r^2 + 4Rr}{rs} \overset{(6)}{\geqslant} \frac{rs\sqrt{3}}{rs} = \sqrt{3}, \end{aligned}$$

tj.

$$\operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2} \geqslant \sqrt{3}.$$

Napomena 1. Ovdje smo koristili jednakost $ab + bc + ca = s^2 + r^2 + 4Rr$.

Dokaz 4. Promatrat ćemo funkciju
$$y = \operatorname{tg}\frac{x}{2}, \quad x \in (0, \pi).$$
Imamo
$$y' = \frac{1}{2\cos^2\frac{x}{2}}, \quad \text{te} \quad y'' = \frac{\sin\frac{x}{2}}{2\cos^3\frac{x}{2}} > 0,$$
što znači da je promatrana funkcija konveksna te, na osnovu Jensenove nejednakosti, imamo
$$\operatorname{tg} x_1 + \operatorname{tg} x_2 + \operatorname{tg} x_3 \geqslant 3\operatorname{tg}\frac{x_1 + x_2 + x_3}{3},$$
a odavde stavljajući $x_1 = \frac{\alpha}{2}$, $x_2 = \frac{\beta}{2}$, $x_3 = \frac{\gamma}{2}$:
$$\operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2} \geqslant 3\operatorname{tg}\frac{\alpha + \beta + \gamma}{6},$$
tj.
$$\operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2} \geqslant 3\operatorname{tg} 30^\circ,$$
te
$$\operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2} \geqslant \sqrt{3}.$$

Dokaz 5. Koristit ćemo supstitucije
$$a = y + z, \quad b = z + x, \quad c = x + y, \quad x, y, z > 0,$$
gdje su a, b, c duljine stranica trokuta.

Iz formule $\operatorname{tg}\frac{\alpha}{2} = \sqrt{\frac{(s-b)(sc)}{s(s-a)}}$ kao i jednakosti koje dobijemo iz gornjih supstitucija:
$$s = \frac{a + b + c}{2} = x + y + z, \quad s - a = x, \quad s - b = y \quad \text{i} \quad s - c = z,$$
dobivamo
$$\operatorname{tg}\frac{\alpha}{2} = \sqrt{\frac{yz}{x(x + y + z)}},$$
kao i analogne formule
$$\operatorname{tg}\frac{\beta}{2} = \sqrt{\frac{xz}{y(x + y + z)}}, \quad \operatorname{tg}\frac{\gamma}{2} = \sqrt{\frac{xy}{z(x + y + z)}}.$$
Sada dobivamo:
$$\begin{aligned} &\operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2} \\ &= \sqrt{\frac{yz}{x(x + y + z)}} + \sqrt{\frac{xz}{y(x + y + z)}} + \sqrt{\frac{xy}{z(x + y + z)}}, \\ \Longrightarrow &\left(\operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2}\right)^2 \end{aligned}$$

$$\begin{aligned}&= \frac{yz}{x(x+y+z)} + \frac{xz}{y(x+y+z)} + \frac{xy}{z(x+y+z)}\\&\quad + 2\left(\frac{z}{x+y+z} + \frac{x}{x+y+z} + \frac{y}{x+y+z}\right)\\&= \frac{y^2z^2 + x^2z^2 + x^2y^2}{xyz(x+y+z)} + 2 \geqslant \frac{yz\cdot xz + xz\cdot xy + yz\cdot xy}{xyz(x+y+z)} + 2\\&= \frac{xyz(x+y+z)}{xyz(x+y+z)} + 2 = 1 + 2 = 3,\end{aligned}$$

tj.

$$\operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2} \geqslant \sqrt{3}.$$

Napomena 2. Dokazat ćemo nejednakost

$$\operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2} \leqslant \frac{9Rr}{2P}. \tag{9}$$

Dokaz 6. Iz (4) imamo

$$\operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2} = \frac{4R+r}{s}.$$

Dokazat ćemo nejednakost:

$$\begin{aligned}&\frac{4R+r}{s} \leqslant \frac{9Rr}{2P}\\ \iff &\frac{4R+r}{s} \leqslant \frac{9Rr}{2rs} \ \Big/ \cdot 2rs\\ \iff &2r(4R+r) \leqslant 9Rr\\ \iff &Rr \geqslant 2r^2 / : r\\ \implies &R \geqslant 2r,\end{aligned}$$

a ovo je poznata Eulerova nejednakost. Dakle, (9) vrijedi.

Jednakost u (9) vrijedi ako i samo ako je $R = 2r \iff a = b = c$, tj. ako i samo ako je trokut jednakostraničn.

Iz nejednakosti (1) i (9) slijedi dvostruka nejednakost

$$\sqrt{3} \leqslant \operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2} \leqslant \frac{9Rr}{2P},$$

tj.

$$\sqrt{3} \leqslant \operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2} \leqslant \frac{9R}{2s}.$$

Literatura

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