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# APPLICATION OF THE FRACTIONAL PROBLEM OF THE CALCULUS OF VARIATIONS AND THE FRACTIONAL PATH INTEGRAL APPROACH TO STOCHASTIC MODELING

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The fractional path integral approach is applied to stochastic models, in particular the financial derivatives and options pricing formulated within the framework of the fractional action-like variational approach recently introduced by the author. Many interesting features and consequences are revealed in some details.

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## 1. Introduction

Fractional calculus (FC) is considered today as a successful tool for describing dynamical systems with different levels of complexity and which are far-from equilibrium displaying scale-invariant properties, dissipation and long-range correlations that cannot be illustrated using traditional analytic functions and ordinary differential operators. Despite the fact that FC has been studied for over 300 years now, it has been regarded mainly as a mathematical curiosity until about 1992, where dynamical equations involving fractional derivatives and integrals were pretty much restricted to the realm of mathematics. Physicists and mathematicians have begun to explore the realm of applications of fractional calculus with ever new developments rapidly taking place in several fields including stochastic processes, finance and economics [1 − 12].

The fractional derivative/integral implies nonlocal effects either in space or time. On the other hand, they are nonlocal operators that do not correspond to a fractional derivative. However, a growing body of empirical evidence supports the importance of fractional integral in chaotic dynamics and economics where important economic data series might be fractionally integrated.

In fact, over the course of the last two decades, social scientists have made significant steps in modeling time-series data. One of the central areas that have made important developments is the study of long memory, in particular, the study of fractional integration. Fractional integration gives researchers more precise mathematical tools with which to describe in a better way their time series. By appropriately treating data as fractionally integrated when modeling time series data, considerable insight may be gained into the nature of political change. Fractional integration and, more generally, long memory can produce empirical data that appear to be stationary, but that nonetheless have high-order autocorrelations that are too large to be taken into account in applications [13].

Evidence of long memory has been found in finance and economics, in particular traditional business cycle indicators such as aggregate economic activity [14] and prices indices [15]. There is also strong evidence of long memory in asset price and exchange rate volatility [16]. Aggregate employment is shown to be fractionally integrated if few firms have long lifetimes given the turnover in new firms. Asset price volatility is shown to be fractionally integrated if a small number of asset positions last longer than would be predicted by the lifetimes of typical positions. An outstanding survey of the literature on fractional integration and long memory was provided by Baillie [17]. It is worth mentioning that the importance of random walks in finance (financial markets) has been known since the seminal work of Bachelier [18], considered as the first tentative model known in finance to describe stock market dynamics and give a price for a European call option which was completed nearly a hundred years ago and was further carried out by Mandelbrot [19], who introduced the concept of Levy flights and stable distributions [20, 21] in finance, and by the MIT school of Samuelson [22, 23]. In 1976, Cox and Ross used the Bachelier ordinary random walk approach to offer a discrete time analog to the well-known Black-Scholes option price. In addition, they obtained a special limiting process when the number of time steps is large, hence finding that the binomial model leads naturally to a Poisson jump process. Other offerings of the financial random walk formalism extend the binomial model by taking into account the crash as a third possible event, and observe the implications of it to the European option price [24].

In this work we essentially consider the continuous view-point based on the Riemann-Liouville fractional integral. While various fields of application of fractional derivatives and integrals are already well done, some others have just started in particular the study of fractional problems of the calculus of variations (COV) which is a subject of current strong research and investigations [25 – 29]. Recently, we proposed a novel approach known as the fractional action-like variational approach (FALVA) to model nonconservative dynamical systems where fractional time integral introduces only one fractional parameter $\alpha \in [0, 1]$, while in other models an arbitrary number of fractional parameters (orders of derivatives) appear [30]. The resulting equations of motion that result from the fractional action functional, namely the Euler-Lagrange equations, contain time-dependent dissipative terms where many applications with encouraging results were discussed in Refs. [31 – 50].

In this paper we consider some applications of the fractional path integral formalism of quantum mechanics to financial modeling based on FALVA. To fractionally generalize the Feynman path integral approach, the integration has been expanded from the standard integral to the fractional action integral for the free particle [51].

The Feynman path integral, which is in fact the integration over Brownian-like quantum mechanical paths, is defined as a limit of the sequence of finite sums. Hence, the path integral is non-standard but rather an infinite multidimensional integral consisting of a convolution of standard integrals. If an exact analytical solution is not obtainable straightforwardly, then powerful approximation techniques such as the WKB method can be applied for approximate solution to a path integral. Moreover, path integrals can be numerically evaluated by making use of the Monte Carlo simulation, or by a deterministic discretization scheme. Many authors reported in the last years very fast numerical methods for computing option price using Feynman path integrals. Path integrals are much more general than standard stochastic integrals with respect to semi-martingales. It is noteworthy that the path integral representation of averages can be obtained directly as the Feynman-Kac solution to the partial differential equation describing the time evolution of the quantum stochastic dynamical system, e.g. Schrödinger equation in quantum mechanics or Fokker-Planck-Kolmogorov diffusion equation in the theory of stochastic processes. More recently, the construction of the path integral representation which allows treating both fractional subdiffusion and fractional superdiffusion on an equal footing was introduced by Calvo and Sanchez [52]. In this paper, the fractional path integral is addressed in a different fashion starting from a fractional action integral as mentioned above. This particular choice avoids the use of fractional derivatives and fractional initial conditions.

In economics and finance, to develop a robust and consistent model of markets, we require dynamical modeling of actions augmented hopefully by algorithms in order to fit parameters in these models to real data.

It is interesting to note that starting with nonlinear, multivariate, nonlinear stochastic differential equation descriptions of the price evolution of cash and futures indices, one may build an algebraic cost function in terms of a Lagrangian. In fact, the basic principle is the absence of arbitrage [53]. In finance, it plays a role analogous to the least-action principle and the energy-conservation law in physics. Consequently, one can introduce naturally Lagrangian functions and action functionals for financial models, e.g. the Hamiltonian formulation of the evolution of an option price in the presence of stochastic volatility [54, 55]. Most financial models are based on some stochastic processes, and it is well-known that the path integral method is an integral formulation of the dynamics of a stochastic process as it allows computing the transition probability associated to a given financial stochastic process where the action functional for the underlying risk neutral price process defines a risk neutral measure on the set of all paths, i.e. the path integral formulation of the Black-Scholes (BS) partial differential equation and the model of barrier options, resorting to a Lagrangian path integral formulation [56].

Use of a path integral formulation has some advantages. First, it can provide fast and precise predictions for a large class of financial derivatives with path-dependent features, by means of a careful estimation of the transition probability and an appropriate choice of the integration limiting points needed to evaluate the averaged quantities of financial interest. Second, it opens the way to the use of quantum mechanical standard methods, i.e. the BS equation for the option price with constant volatility is reformulated in terms of a non-Hermitian Hamiltonian and hence standard methods used in quantum mechanics can be applied. In fact, the BS partial differential equation is a finance counterpart of the Schrödinger wave equation of quantum mechanics, and the risk neutral valuation formula is interpreted as the Feynman-Kac representation of the partial differential equation solution. Note that the BS model assumes that the log price is an arithmetic Brownian motion [20, 57].

Accordingly, if we went to go beyond the BS standard model in order to price options on assets whose log price is a Levy stable process, we need to assure our peace with pricing options in the presence of jumps. Every stochastic differential equation has correspondingly a path integral representation. In other words, the transitional probability density function can be explicitly expressed as a Feynman path integral. It is well-known that Brownian motion driven stochastic differential equations can not reproduce the fat Levy tail and the infinite Levy moments of observed distribution for price returns. We believe that it is possible (at least, theoretically) to represent the real distribution by a Feynman path integral. Thus, the path integral formalism provides a natural bridge between the risk neutral martingale pricing and the arbitrage free partial differential equation based pricing exhibiting non-Gaussian price fluctuations. An introductory overview of the path integral approach to financial modeling and options pricing is given in [56 and references therein]. Analytic and geometrical methods for heat kernel applications in finance are found in [58].

Finally, it should be noted that two practical methods for computation of path integrals are the familiar perturbation theory and the numerical simulation. Analytically, one looks at first for critical points of the standard action, which represent classical trajectories. After that, we expand the action in a functional Taylor series near these trajectories, leaving the quadratic terms in the exponent and expanding the rest in a power series. The Gaussian path integrals are the only that appear for which very similar techniques are available as for finite dimensional Gaussian integrals.

The paper is organized as follows: in Section 2, we give a brief overview of the FALVA and the corresponding fractional path integral. A general framework of the fractional path integral options pricing and the path integral representation (Feynman-Kac formula) for a fractional path dependent option by considering a single asset Black-Scholes model as an example is discussed in Section 3. Conclusions and perspectives are discussed in Section 4. We introduce the main notations, conventions and assumptions that underlie the remainder of the present work:

1. In the notation $t \to f(t)$, $t$ is a dummy variable.

2. Exactly the same function can be written, for example $(\dot{q}, q, \tau) \to f(\dot{q}, q, \tau)$; $\dot{q}, q, \tau$ are here dummy variables.

3. For $(\dot{q}, q, \tau) \to f(\dot{q}, q, \tau)$, the partial derivative of $f$ with respect to the first argument is denoted by $\partial L/\partial \dot{q}$.

4. No fractional derivatives of any type and any order will be introduced.

5. It is assumed that at least one stationary point for the fractional functional exists.

6. We expect that the reader is familiar with the standard Feynman path integral and Feynman-Kac formula.

7. We expect also that the reader is familiar with the BS model, martingales, integrability and with the notion of Brownian motion.

8. The non-local property of the fractional integral can be useful when dynamically hedging options. To the best of our knowledge, this work represents the first attempt to apply the concept of FALVA to financial modeling and option pricing.

## 2. *Fractional action-like variational approach, fractional path integral and motivations*

### 2.1. *FALVA*

FALVA is based on the concept of Riemann-Liouville fractional integral functionals with multi-time and multi-scales. In fact, this multi-time characteristic is important and plays a crucial role in many physical applications and is the main ingredient of the theory being developed by Udriste. In 2005, the author of the present paper introduced the one-dimensional FALVA problem as follows [31].

**Problem 2.1**: Find the stationary points of the integral functional

$$S[q(\bullet)] = \frac{1}{\Gamma(\alpha)} \int_a^t L(\dot{q}(\tau), q(\tau), \tau)(t-\tau)^{\alpha-1} \mathrm{d}\tau \equiv \int_a^t L(\dot{q}(\tau), q(\tau), \tau) \mathrm{d}g_t(\tau) \,, \quad (1)$$

under the initial condition $q(a) = q_a$, where $\dot{q} = \mathrm{d}q/\mathrm{d}\tau$, $\Gamma$ is the Euler gamma function, $0 < \alpha \leq 1$, $\tau$ is the intrinsic time, $t$ is the observer time, $t \neq \tau$, and the smooth Lagrangian function $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a $C^2$-function with respect to all its arguments. Here $L(\dot{q}(\tau), q(\tau), \tau)$ is the Lagrangian weighted with $(t-\tau)^{\alpha-1}/\Gamma(\alpha)$ and $\Gamma(\alpha+1)g_t(\tau) = t^\alpha - (t-\tau)^\alpha$ with the scaling property $g_{kt}(k\tau) = k^\alpha g_t(\tau),\ k > 0$.

**Remark 2.1**: In reality, the fractional smooth action integral (1) can be rewritten as the strictly singular Riemann-Liouville type fractional derivative Lagrangian

$$S_{\beta\in(0,1)}[q] = D_t^{-1+\beta} L(\dot{q}(t), q(t), t) = \int_0^t L(\dot{q}(t), q(t), t) \frac{\mathrm{d}\tau}{(t-\tau)^\beta} \overset{\beta \to 0}{\to} \int_0^t L(\dot{q}(t), q(t), t) \mathrm{d}\tau \,,$$

and thereby retrieve the standard action integral or functional integral. In our formalism, we have $\beta = 1 - \alpha,\ \alpha \in (0, 1)$. Such type of functionals is known in mathematical economy, describing, for instance, the so called "discounting" economical dynamics.

**Theorem 2.1**: If $q(\bullet)$ are solutions to the previous problem, i.e. $q(\bullet)$ are critical points of the function (1), then $q(\bullet)$ satisfy the following Euler-Lagrange equations

$$\frac{\partial L(\dot{q}(\tau), q(\tau), \tau)}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L(\dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}}\right) = \frac{1-\alpha}{t-\tau}\frac{\partial L(\dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}} . \qquad (2)$$

The previous arguments may be generalized to Lagrangian involving higher derivatives $L(q^{(m)}(\tau), \ldots, \dot{q}(\tau), q(\tau), \tau)$ as follows [59].

**Problem 2.2**: Find the stationary points of the integral functional

$$S^m[q(\bullet)] = \frac{1}{\Gamma(\alpha)}\int_a^t L(q^{(m)}(\tau), \ldots, \dot{q}(\tau), q(\tau), \tau)(t-\tau)^{\alpha-1}\mathrm{d}\tau , \qquad (3)$$

$m \geq 1$, under the initial condition $q^{(i)}(a) = q_a,\ i = 0, 1, 2, \ldots, m$, where $q^{(i)} = \mathrm{d}^i q/\mathrm{d}\tau^i$, $\Gamma$ is the Euler gamma function, $0 < \alpha \leq 1$, $\tau$ is the intrinsic time, $t$ is the observer time, $t \neq \tau$, and the smooth Lagrangian function $L : [a, b] \times \mathbb{R}^{n\times(m+1)} \to \mathbb{R}$ is a $C^{2m}$-function with respect to all its arguments.

**Theorem 2.2**: If $q(\bullet)$ are solutions to the problem 2.2, i.e. $q(\bullet)$ are critical points of the function (4), then $q(\bullet)$ satisfy the following higher-order Euler-Lagrange equations in (0+1) dimensions

$$\sum_{i=0}^{m}(-1)^i\frac{\mathrm{d}^i}{\mathrm{d}\tau^i}\partial_{i+2}L(q^{(m)}(\tau), \ldots, \dot{q}(\tau), q(\tau), \tau)$$

$$= \frac{1-\alpha}{t-\tau}\sum_{i=1}^{m} i(-1)^{i-1}\frac{\mathrm{d}^{i-1}}{\mathrm{d}\tau^{i-1}}\partial_{i+2}L(q^{(m)}(\tau), \ldots, \dot{q}(\tau), q(\tau), \tau) \qquad (4)$$

$$+ \sum_{k=2}^{m}\sum_{i=2}^{k}(-1)^{i-1}\frac{\Gamma(i-\alpha+1)}{(t-\tau)^i\Gamma(1-\alpha)}\begin{pmatrix} k \\ k-i \end{pmatrix}\frac{\mathrm{d}^{k-i}}{\mathrm{d}\tau^{k-i}}\partial_{k+2}L(q^{(m)}(\tau), \ldots, \dot{q}(\tau), q(\tau), \tau) .$$

Here $\partial_i L$ denotes the partial derivatives of $L(\bullet, \bullet, \ldots, \bullet)$ with respects to its $i$th argument. In the particular case when $m = 1$, Problem 2.2 reduces to Problem 2.1.

Problem 2.1 was explored recently to the multidimensional case as follows [48].

**Problem 2.3**: Find the stationary points of the four-dimensional integral functional

$$S^{\alpha}[q](\xi;\xi\in\Omega)=\frac{1}{\prod_{i=1}^{4}\Gamma(\alpha_i)}\iiiint_{\Omega(\xi)}L(\dot{q}(x),q(x),x)\prod_{i=1}^{4}(\xi_i-x_i)^{\alpha_i-1}\mathrm{d}x\,, \tag{5}$$

where the admissible paths are smooth functions $q:\Omega\subset\mathbb{R}^4\to M$, satisfying the Dirichlet boundary conditions on $\partial\Omega$. Here $x=(x_1,x_2,x_3,x_4)$ is the intrinsic time vector, $\xi=(\xi_1,\xi_2,\xi_3,\xi_4)\in\Omega$ is the observer time vector, $x\in\Omega(\xi)\subseteq\Omega$ with $x_i\neq\xi_i$ $(i=1,2,3,4)$, $\mathrm{d}x=\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4$ and $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, $0<\alpha_i<1$ and $(q_{x_1},q_{x_2},q_{x_3},q_{x_4},q,x_1,x_2,x_3,x_4)\to L(q_{x_1},q_{x_2},q_{x_3},q_{x_4},q,x_1,x_2,x_3,x_4)$ is a sufficiently smooth Lagrangian function.

In four-dimensions, for simplicity, we use the following notation: the 3D intrinsic space notations $(x_1,x_2,x_3)=(x,y,z)$ and the intrinsic time notation $(x_4)=(\tau)$, where associated, respectively, each $x_i\to\alpha_i\xi_i$.

**Theorem 2.3**: If $q(\bullet)$ are solutions to the previous problem, i.e. $q(\bullet)$ are critical points of the function (5), then $q(\bullet)$ satisfy the following Euler-Lagrange equations

$$\sum_{i=1}^{4}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial q_{x_i}}\right)+\frac{1-\alpha_i}{\xi_i-x_i}\left(\frac{\partial L}{\partial q_{x_i}}\right)\right]-\frac{\partial L}{\partial q}=0\,. \tag{6}$$

**Remark 2.2**: *The previous problems, in particular Problems 2.1 and 2.3, were explored for Lagrangian holding fractional derivatives [48], but in this paper, we will restrict ourselves to fractional integration and we leave the presence of fractional derivatives for a future work.*

### 2.2. *Fractional path integral*

In this work, the fractional path integral (FPI) is expected to describe the motion from the initial position $x_i(t_i)$ to the position $x_f(t_f)$ with a fractional quantum amplitude given by

$$K_\alpha(x_f,t_f;x_i,t_i)\propto\sum_{\{\gamma\}}\exp\left(\frac{\mathrm{i}}{\hbar}S_\alpha[\gamma]\right)\equiv\sum_{\{\gamma\}}\exp\left(\frac{\mathrm{i}}{\hbar}\frac{1}{\Gamma(\alpha)}\int_a^t L(t-\tau)^{\alpha-1}\mathrm{d}\tau\right), \tag{7}$$

where $\{\gamma\}$ is the set of all trajectories satisfying $x(t_i)=x_i$ and $x(t_f)=x_f$.

The standard result is expected to be resurrected in the $\alpha=1$ limit, and classical physics is expected to be recovered for $\hbar=0$ ($\hbar$ is the Planck's constant). Normally, to get an estimate of the mean square displacement of a free particle moving from an initial point $x(t_i) = x_i \rightarrow x(t_f = t) = x_f$, we follow the Feynman standard technique and write the fractional quantum mechanical kernel in the form

$$\left\langle x_f \left| \mathrm{e}^{-(T-t)H} \right| x_i \right\rangle \equiv K_\alpha(x_f, t_f; x_i, t_i) = \int_{x_i}^{x_f} \mathbb{D}[x(\tau)] \exp\left( \frac{\mathrm{i}}{\hbar} \frac{1}{\Gamma(\alpha)} \int_a^t L(t-\tau)^{\alpha-1} \mathrm{d}\tau \right)$$

$$\equiv \int_{x_i}^{x_f} \mathbb{D}[\bar{x}_\alpha(\tau)] \exp\left( \frac{\mathrm{i}}{\hbar} \int_a^t L \, \mathrm{d}g_t(\tau) \right), \tag{8}$$

with the boundary conditions $x(t_i) = x_i$ and $x(t_f) = x_f$, and where

$$\int_{x_i}^{x_f} \mathbb{D}[x(\tau)] ... = \lim_{N \to \infty} \int_{-\infty}^{\infty} \prod_{j=1}^{N-1} (2\pi \mathrm{i} \epsilon \hbar)^{1/2} \mathrm{d}x_{j...}, \quad \epsilon = \frac{t_f - t_i}{N}, \tag{9}$$

$$\int_{x_i}^{x_f} \mathbb{D}[\bar{x}_\alpha(\tau)] ... = \lim_{N \to \infty} \int_{-\infty}^{\infty} \prod_{j=1}^{N-1} (2\pi \mathrm{i} \bar{\epsilon}_\alpha \hbar)^{1/2} \mathrm{d}x_{j...}, \ \bar{\epsilon}_\alpha = \frac{\left[t_f^\alpha - (t_f - \tau)^\alpha\right] - \left[t_i^\alpha - (t_i - \tau)^\alpha\right]}{N\Gamma(\alpha+1)} \tag{10}$$

denotes the sum over all paths between $(x_i, t_i) \rightarrow (x_f, t_f)$. $H$ in Eq. (8) denotes the Hamiltonian part. Normally, the wave function $\psi(x_f, t_f)$ at $(x_f, t_f)$ is given in terms of $\psi(x_i, t_i)$ at $(x_i, t_i)$ by the equation

$$\psi(x_f, t_f) = \int_{x_i}^{x_f} \mathrm{d}x_i K_\alpha(x_f, t_f; x_i, t_i) \psi(x_i, t_i). \tag{11}$$

This fractional equation describes, therefore, the evolution of the stochastic process in terms of the wave equation. The FPI was applied recently to quantum field theory [60] and subdiffusive processes [61] and many appealing consequences were revealed. It should be mentioned that in economy, one can interpret the option price as a ket $|f\rangle$ in the basis of $|x\rangle$. The pricing price is represented, therefore, by $p(x, x', T-t) = \langle x | \exp(-(T-t)H) | x' \rangle$. Here $T - t$ is referred to *time to maturity* [62, 63].

### *2.3. Motivations*

As we mentioned in the Introduction of the present paper, movement of asset price is normally modeled by stochastic differential equations (SDE) which have a path integral representation. More precisely, the transitional probability density function (pdf) can be explicitly expressed as a path integral. Moreover, the price of an option is explicitly represented as a multiple integral. Further, the discounted price of a dividendless security with a given future payoff is a martingale under risk-neutral probability measure in an ideal market. If $V_t$ is a dividendless option with a future payoff of $V_T$, $T > t \geq 0$, $M_t$ is a continuous time Markov process such that:

a) $M_t$ at time $t$ depends only on the available information at time $t$,

b) the expectation $\mathbb{E}_t[M_T] = \mathbb{E}_t[M_T|M_t] = M_t$, $T > t \geq 0$,

c) $\mathrm{d}M_t \triangleq M_{t+dt} - M_t$, $t \geq 0$, $\mathrm{d}t > 0$,

d) taking $T = t + \mathrm{d}t$, it follows that $\mathbb{E}[\mathrm{d}M_t] = 0$, $t \geq 0$,

The Feynman-Kac formula gives us a unique arbitrage-free price of an option in an ideal market as follows,

$$V_t = \mathbb{E}_t^Q \left[ \exp\left( -\mathrm{i} \int\limits_0^t r_u \mathrm{d}u \right) V_T \right], \quad T > t \geq 0\,. \tag{12}$$

Options price corresponding partial differential equations (PDE), i.e. the BS-PDE, will consequently be obtained from d) as follows: $\mathbb{E}_t^Q[\mathrm{d}M_t] = 0$, $t \geq 0$. The payoff function of the option is simply the terminal boundary condition to the resulting PDE.

Normally, a path-independent option, e.g. the European call option, is defined by its payoff at expiration at time $T$ by $O_T = \max(S_T, 0) = F(S_T)$, $T > 0$, where $F$ is a given function of the terminal asset (stock) price $S_T$ which pays no dividend and obeying a standard geometric Brownian motion or Wiener stochastic process $\mathrm{d}S_t/S_t = m\mathrm{d}t + \sigma\mathrm{d}W_t$ with the initial condition $S_0 = s_0$. Here $m$ and $\sigma$ denote the instantaneous expected rate of return on asset $S$ and the volatility. The filtration is generated by the stock price, similarly to the filtration generated by the Brownian motion $W_t$. These dynamics mean the asset price $S$ is lognormally distributed, i.e. $S_T = S_t \exp[\sigma(W_T - W_t) + (m - \sigma^2/2)(T - t)]$. The BS option pricing formula is based on a dynamic hedging argument, i.e. the price is the price because the risk can be hedged away by trading the asset itself. There are no transaction costs or taxes and trading in the cost is continuous. The option prices can be calculated via martingale methods and were the discounted expectation under the equivalent martingale measure making the discounted price process a martingale [62, 63]. Informally, the martingale representation theorem states that in equilibrium prices represented as the present discounted value of future payoffs from the asset must satisfy a martingale under a given measure. As $M_t = \mathrm{e}^{-rt}O(t, S_t)$ is a martingale under risk-neutral dynamics the standard absence of arbitrage argument, i.e. $\mathbb{E}_t^Q[\mathrm{d}M_t] = 0$, $t \geq 0$ leads us to constructing a replicating portfolio consisting of the underlying asset and the risk-free bond and to the BS-PDE for the present value of the option at time $t$ preceding expiration

$$\frac{\partial O_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 O_t}{\partial S_t^2} + rS_t \frac{\partial O_t}{\partial S_t} - rO_t = 0\,. \tag{13}$$

This has the form of the backwards Chapman-Kolmogorov equation with the risk-free rate $r$. The quantum mechanical version of this equation is obtained by the change of variable $x = \ln S_t$, and we obtain the beautiful result

$$\frac{\partial O_t}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 O_t}{\partial S_t x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial O_t}{\partial S_t} - rO_t = 0\,, \tag{14}$$

augmented by $O(\mathrm{e}^{x_T}, T) = F(\mathrm{e}^{x_T})$. The PDE (14) is exactly the Chapman-Kolmogorov equation, corresponding to the Fokker-Planck PDE (the $2^{\mathrm{nd}}$ Kolmogorov equation). This well-known formula was found to be very useful for pricing many other options, such as, geometric Asian option. A unique solution to the previous problem is given by the Feynman-Kac formula where the average can be represented as a path integral over the set of all paths defined as a limit of the sequence of finite-dimensional multiple integrals as represented in Eq. (9). We give here the final result and we refer the reader to reference [56] for the details of the calculation

$$O_t(S,t) = \mathrm{e}^{-r(T-t)}\mathbb{E}_t^Q\left[F(\mathrm{e}^{x_T}), 0\right]$$

$$= \mathrm{e}^{-r(T-t)} \int\limits_{-\infty}^{+\infty} \left( \int\limits_{x(t)=x}^{x(T)=x_T} F(\mathrm{e}^{x_T}) \exp\left(-\int\limits_t^T L_{BS}\,\mathrm{d}\tau\right) \mathbb{D}[x(\tau)]\right) \mathrm{d}x_T \tag{15}$$

defined on the path $x(\tau)$, $t \le \tau \le T$. Here

$$A - L_{BS} = \frac{1}{2\sigma^2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau} - \left(r - \frac{1}{2}\sigma^2\right)\right)^2$$

is the BS Lagrangian function and $B - \mathbb{E}_t^Q$ denotes averaging over the risk-neutral measure conditional on the initial price $S$ at time $t$ under a certain risk-neutral measure $Q$.

Our main aim in the next section is to explore the general framework of the fractional path integral options pricing by considering again a single-asset BS model as an example. Then, we develop the fractional path integral formalism for a multi-asset economy with asset and time-dependent volatilities and correlations based on the fractional extension of Eq. (15) as follows,

$$\bar{O}_t(S,t) = \mathrm{e}^{-r(T-t)}\bar{\mathbb{E}}_t^Q\left[F(\mathrm{e}^{X_T}), 0\right]$$

$$= \mathrm{e}^{-r(T-t)} \int\limits_{-\infty}^{+\infty} \left( \int\limits_{x(t)=x}^{x(T)=x_T} F(\mathrm{e}^{X_T}) \exp\left(-\frac{1}{\Gamma(\alpha)}\int\limits_a^T L_{BS}(T-\tau)^{\alpha-1}\,\mathrm{d}\tau\right) \mathbb{D}[x(\tau)]\right) \mathrm{d}x_T \tag{16}$$

$$\equiv \mathrm{e}^{-r(T-t)} \int\limits_{-\infty}^{+\infty} \left( \int\limits_{x(t)=x}^{x(T)=x_T} F(\mathrm{e}^{X_T}) \exp\left( -\int\limits_{a}^{T} L_{BS}\, \mathrm{d}\pi(\tau) \right) \mathbb{D}[x_\alpha(\bar{\tau})] \right) \mathrm{d}x_T, \quad a \leq \tau \leq T\,, \tag{17}$$

where $\bar{O}_t(S,t)$ and $\bar{\mathbb{E}}_t^Q\left[F(\mathrm{e}^{X_T}),0\right]$ denote the fractional counterpart of $O_t(S,t)$ and $\mathbb{E}_t^Q\left[F(\mathrm{e}^{X_T}),0\right]$, respectively.

## *3. Risk neutral valuation and the standard Wiener-Feynman path integrals*

To explore the fractional counterpart, we will perform the calculation for mathematical convenience with respect to the new scaling time $\pi(\tau)$ with the scaling property $\pi(\lambda\tau) = \lambda^\alpha \pi(\tau)$. We first write the Black-Scholes Lagrangian with respect to the new scaling time as follows,

$$L_{BS} = \frac{1}{2\sigma^2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau} - \mu\right)^2 \to \bar{L}_{BS} = \frac{1}{2\sigma^2}\left(\frac{\mathrm{d}x}{\mathrm{d}g_t(\tau)} - \mu\right)^2 = \frac{1}{2\sigma^2}\left(\frac{\mathrm{d}x}{\mathrm{d}g_t(\tau)}\right)^2 + \frac{\mu^2}{2\sigma^2} - \frac{\mu}{\sigma^2}\left(\frac{\mathrm{d}x}{\mathrm{d}g_t(\tau)}\right) \tag{18}$$

where $\mu = r - \sigma^2/2$, and hence the FALVA gives easily

$$S_{BS}\big[x(g_t(t'))\big] = \frac{1}{\Gamma(\alpha)} = \int\limits_a^T L_{BS}(T-\tau)^{\alpha-1}\, \mathrm{d}\tau = \int\limits_a^T L_{BS}\, \mathrm{d}g_t(\tau)$$

$$= \int\limits_a^T \left[ \frac{1}{2\sigma^2}\left(\frac{\mathrm{d}x}{\mathrm{d}g_t(\tau)}\right)^2 + \frac{\mu^2}{2\sigma^2} - \frac{\mu}{\sigma^2}\left(\frac{\mathrm{d}x}{\mathrm{d}g_t(\tau)}\right) \right] \mathrm{d}g_t(\tau) \tag{19}$$

$$= \frac{\mu^2 g_t(\tau)}{2\sigma^2} - \frac{\mu}{\sigma^2}(x(T) - x) + \frac{1}{2\sigma^2} \int\limits_a^T \left(\frac{\mathrm{d}x}{\mathrm{d}g_t(\tau)}\right)^2 \mathrm{d}g_t(\tau). \tag{20}$$

Then, discretizing paths and substituting

$$\int\limits_a^T \cdots \mathrm{d}g_t(\tau) \to \sum_{i=0}^{N-1} \cdots \Delta a(g_t(\tau))\,,$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}g_t(\tau)} \to \frac{x_{i+1} - x_i}{\Delta a(g_t(\tau)}\,,$$

we find

$$S_{BS}\big[x(g_t(t'))\big]=\frac{\mu^2 g_t(\tau)}{2\sigma^2}-\frac{\mu}{\sigma^2}\big(x(g_t(T))-x(g_t(t))\big)+\frac{1}{2\sigma^2\Delta a(g_t(\tau))}\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2\,. \tag{21}$$

In our formalism, the time to expiration $T$ is discretized into $N$ equal time steps $\Delta a(g_t(\tau)) = (g_t(T) - g_t(a))/N$ bounded by $N + 1$ equally spaced time points, $g_{ti}(\tau) = g_t(\tau) + i\Delta g_t(\tau)$, $i = 0,\ldots,N$, where $\Gamma(\alpha + 1)g_t(\tau) = t^\alpha - (t - \tau)^\alpha$.

Hence Eq. (21) in the standard time is written as

$$S_{BS}\big[x(g_t(t'))\big]=\frac{\mu^2\big[t^\alpha-(t-\tau)^\alpha\big]}{2\Gamma(\alpha+1)\sigma^2}-\frac{\mu}{\sigma^2}\big(x(g_t(T))-x(g_t(t))\big)+\frac{1}{2\sigma^2\Delta a(g_t)}\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2, \tag{22}$$

where

$$\Delta a(g_t)=\frac{g_t(T)-g_t(a)}{N}=\frac{1}{N\Gamma(\alpha+1)}\big[(t-a)^\alpha-(t-T)^\alpha\big],\quad i=0,\ldots,N\,. \tag{23}$$

The fractional path integral is now written as follows,

$$\int\limits_{x(t)=x}^{x(T)=x_T} F(\mathrm{e}^{X_T})\exp\left(-\frac{1}{\Gamma(\alpha)}\int\limits_a^T L_{BS}(T-\tau)^{\alpha-1}\,\mathrm{d}\tau\right)\mathbb{D}[x(\tau)] \tag{24}$$

$$=\lim_{N\to\infty}\int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty}F(\mathrm{e}^{X_T})\exp\left(-\int\limits_a^T L_{BS}(x_i)\,\mathrm{d}g_t(\tau(x_i))\right)\frac{\mathrm{d}x_1(g_t)\cdots\mathrm{d}x_{N-1}(g_t)}{\left(\sqrt{2\pi\sigma^2\Delta a(g_t)}\right)^{N-1}}$$

and consequently the payoff is written as

$$\bar{O}_t(S,t)=\mathrm{e}^{-r(T-t)}\bar{\mathbb{E}}_t^Q\left[F(\mathrm{e}^{X_T}),0\right] \tag{25}$$

$$=\mathrm{e}^{-r(T-t)}\int\limits_{-\infty}^{+\infty}F(\mathrm{e}^{X_T})\exp\left(\frac{\mu}{\sigma^2}\big(x(g_t(T))-x(g_t(t))\big)-\frac{\mu^2 g_t(\tau)}{2\sigma^2}\right)K_\alpha(x_T,T;x,t)\,\mathrm{d}x_T\,,$$

where

$$K_\alpha(x_T,T;x,t)\equiv\lim_{N\to\infty}\int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty}\exp\left(-\frac{1}{2\sigma^2\Delta a(g_t(\tau))}\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2\right)\frac{\mathrm{d}x_1(g_t)\cdots\mathrm{d}x_{N-1}(g_t)}{\left(\sqrt{2\pi\sigma^2\Delta a(g_t)}\right)^{N-1}}$$

Making use of the Gaussian integration formula [56], we find after simple mathematical manipulation

$$K_\alpha(x_T, T; x, t) = \frac{1}{\sqrt{2\pi\sigma^2 N \Delta a(g_t)}} \exp\left(-\frac{(x_T - x)^2}{2N\Delta a(g_t)\sigma^2}\right) \tag{26}$$

$$= \sqrt{\frac{\Gamma(\alpha+1)}{2\pi\sigma^2\big[(t-a)^\alpha - (t-T)^\alpha\big]}} \exp\left(-\frac{\Gamma(\alpha+1)}{\big[(t-a)^\alpha - (t-T)^\alpha\big]} \frac{(x_T - x)^2}{2\sigma^2}\right) \tag{27}$$

$$\xrightarrow{\alpha=1} \sqrt{\frac{1}{2\pi\sigma^2(T-a)}} \exp\left(-\frac{1}{T-a}\frac{(x_T - x)^2}{2\sigma^2}\right), \tag{28}$$

where Eq. (28) is the standard result. Equation (27) implies

$$|x_i - x_{i-1}| \propto \big[(t-a)^\alpha - (t-T)^\alpha\big]^{1/2}, \quad 0 < \alpha \le 1.$$

The functional measure defined by Eq. (24) is generated by a fractional stochastic process (fractional Levy motion). Further, Eq. (27) is Gaussian-like unless the volatility is time-dependent, i.e. $\sigma_\alpha^2 \equiv \sigma^2\big[(t-a)^\alpha - (t-T)^\alpha\big]/\Gamma(\alpha+1)$ with a dependence on the past. It is noteworthy that stochastic volatility models must possess efficient numerical methods for pricing European options and hence they are only accessible for stochastic volatility models with time-homogeneous parameters. Empirical facts from many markets suggest nevertheless that the parameters of the volatility smile are different for different option expiries. Therefore, we anticipate that the previous results may be useful to model stochastic volatility model with time-dependent skew. The conclusions in Ref. [63] are that, as far as one aims to conserve market completeness, a volatility model depending on the whole past trajectory of the asset should be investigated.

In fact, it is easy to prove that Eq. (27) is the solution of the modified diffusion equation,

$$\frac{\sigma^2}{2}\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha+1)} \frac{\alpha\sigma^2\big[(t-a)^\alpha - (t-T)^\alpha\big] - x^2\Gamma(\alpha+1)}{\sigma^2\big[(t-a)^\alpha - (t-T)^\alpha\big] - x^2\Gamma(\alpha+1)} \frac{\partial^2 K_\alpha}{\partial x^2} = -\frac{\partial K_\alpha}{\partial a}, \tag{29}$$

with a diffusion coefficient which depends on space and time. For a large distance, the dominant term is the multiplicative factor $(t-a)^{\alpha-1}\sigma^2/(2\Gamma(\alpha+1))$ which approximates to the asymptotic power law $(t-a)^{\alpha-1}$. For a small distance, the asymptotic power law behaves also like $(t-a)^{\alpha-1}$. For a large distance, Eq. (29) takes the special form,

$$\frac{\sigma^2}{2}\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha+1)} \frac{\partial^2 K_\alpha}{\partial x^2} = -\frac{\partial K_\alpha}{\partial a}. \tag{30}$$

By defining the new time variable $\hat{\tau} = t - a$, Eq. (30) is written as

$$\frac{\sigma^2}{2}\frac{\hat{\tau}^{\alpha-1}}{\Gamma(\alpha+1)} \frac{\partial^2 K_\alpha}{\partial x^2} = -\frac{\partial K_\alpha}{\partial \hat{\tau}}, \tag{31}$$

and hence the invariance of Eq. (31) under the transformation $(x, \hat{\tau}) \to (\xi^{\alpha/2}x, \xi\hat{\tau})$ is apparent. Obviously, when $\alpha = 1$, Eq. (31) is reduced to the standard diffusion equation and the fractional functional action is reduced to the standard form. Notice that the term $\hat{\tau}^{\alpha-1}$ is the leading term in the fractional action functional (7) and it is quite amazing that the same term appears in the diffusion coefficient. This remarkable fact may help to select the right choice of fractional integral in each application of the approach described in the paper. It is noteworthy that diffusion with time-dependent diffusivity has been observed in different aspects, including turbulence [64], composite materials [65, 66], biological systems [67] and anomalous transport [68] and many others [69]. Obviously, for $\alpha = 1$, we retrieve the standard diffusion equation associated to standard Brownian motion. It is noteworthy that the Green's function for diffusion with constant drift rate $\mu$ is obtained by multiplying the zero-drift Green's function by the drift-dependent factor, i.e.

$$K^{\mu}_{\alpha}(x_T, T; x, t) = \exp\left(\frac{\mu}{\sigma^2}\big(x(g_t(T)) - x(g_t(t))\big) - \frac{\mu^2 g_t(\tau)}{2\sigma^2}\right) K_\alpha(x_T, T; x, t) \quad (32)$$

$$= \sqrt{\frac{\Gamma(\alpha+1)}{2\pi\sigma^2\big[(t-a)^\alpha - (t-T)^\alpha\big]}} \exp\left(-\frac{\Gamma(\alpha+1)}{\big[(t-a)^\alpha - (t-T)^\alpha\big]}\,\frac{(x_T - x - \mu(T-a))^2}{2\sigma^2}\right). \quad (33)$$

It is easy to check that $K^{\mu}_{\alpha}(x_T, T; x, t)$ is the solution of the following modified diffusion equation,

$$\frac{\sigma^2}{2}\frac{\partial^2 K^\mu_\alpha}{\partial x^2} + \frac{\mu(T-a)\Gamma(\alpha+1)}{\big[(t-a)^\alpha - (t-T)^\alpha\big]}\frac{\partial K^\mu_\alpha}{\partial x} = -\Gamma(\alpha+1)\frac{\partial K^\mu_\alpha}{\partial a} + \frac{\Gamma^2(\alpha+1)}{2\sigma^2}K^\mu_\alpha \quad (34)$$

$$\times\left\{\frac{(x_T-x)^2 - \mu^2(T-a)^2}{\big[(t-a)^\alpha - (t-T)^\alpha\big]^2} - \frac{(x_T-x) - \mu(T-a)}{(t-a)^\alpha - (t-T)^\alpha}\left[\frac{(x_T-x) - \mu(T-a)}{(t-a)^\alpha - (t-T)^\alpha}(t-a)^{\alpha-1} + 2\mu\right]\right\}.$$

Evidently, for $\alpha = 1$, we retrieve the standard diffusion equation with drift. Surprisingly, Eq. (34) is similar to the BS equation with space and time-dependent parameters which play an interesting role in finance [70–72]. The RHS of Eq. (34) may be interpreted as a potential $V(x, a)$; $a \le \tau \le T$. The option price $\bar{O}_t(S, t)$ satisfies then the fractional BS-like equation

$$\frac{\sigma^2}{2}\frac{\partial^2 \bar{O}_t}{\partial x^2} + \frac{\mu(T-a)\Gamma(\alpha+1)}{\big[(t-a)^\alpha - (t-T)^\alpha\big]}\frac{\partial \bar{O}_t}{\partial x} = -\Gamma(\alpha+1)\frac{\partial \bar{O}_t}{\partial a} + \frac{\Gamma^2(\alpha+1)}{2\sigma^2}\bar{O}_t \quad (35)$$

$$\times\left\{\frac{(x_T-x)^2 - \mu^2(T-a)^2}{\big[(t-a)^\alpha - (t-T)^\alpha\big]^2} - \frac{(x_T-x) - \mu(T-a)}{(t-a)^\alpha - (t-T)^\alpha}\left[\frac{(x_T-x) - \mu(T-a)}{(t-a)^\alpha - (t-T)^\alpha}(t-a)^{\alpha-1} + 2\mu\right]\right\}.$$

It can be interpreted as the fractional BS-like equation with time and space-dependent continuous dividend fractional yield with

$$V(x, a) = \frac{\Gamma^2(\alpha+1)}{2\sigma^2} \quad (36)$$

$$\times\left\{\frac{(x_T-x)^2-\mu^2(T-a)^2}{\big[(t-a)^\alpha-(t-T)^\alpha\big]^2}-\frac{(x_T-x)-\mu(T-a)}{(t-a)^\alpha-(t-T)^\alpha}\left[\frac{(x_T-x)-\mu(T-a)}{(t-a)^\alpha-(t-T)^\alpha}(t-a)^{\alpha-1}+2\mu\right]\right\}.$$

It is now easy to compute the propagator for a path-dependent option defined by its payoff at expiration $O(T) = F(S(t'))$, where $F(S(t'))$ is a given functional on price paths $\{S(t'),\ a \le t' \le \tau\}$ rather than a function dependent just on the terminal asset price. Assuming the risk-neutral price process $\mathrm{d}S_t/S_t = r\mathrm{d}t + \sigma\,\mathrm{d}W_t$ and assuming that $F = f(S_T)\exp(-I[S_\alpha(t')])$, where $f(S_T)$ depends only on $S_T$, and $I[S_\alpha(t')] = \int\limits_a^T V(x(t'),t')\mathrm{d}\pi(t')$ is the fractional action of some potential $V(x(t'),t')$, $x = \ln S$, then one finds after similar mathematical manipulations of the above calculations

$$\bar{O}(S,t)=\mathrm{e}^{-r(T-t)}\int\limits_{-\infty}^{+\infty}F(\mathrm{e}^{X_T})\exp\left(\frac{\mu}{\sigma^2}\big(x(g_t(T))-x(g_t(t))\big)-\frac{\mu^2 g_t(\tau)}{2\sigma^2}\right)K_\alpha^\mu(x_T,T;x,t) \tag{37}$$

where

$$K_\alpha^\mu(x_T,T;x,t)=\int\limits_{x(t)=x}^{x(T)=x_T}\exp\left(-\int\limits_a^T\left[\frac{1}{2\sigma^2}\left(\frac{\mathrm{d}x}{\mathrm{d}g_t(\tau)}\right)^2+V\right]\mathrm{d}g_t(\tau)\right)\mathbb{D}[x(\tau)]\,\mathrm{d}x_T \tag{38}$$

$$=\sqrt{\frac{\Gamma(\alpha+1)}{2\pi\sigma^2\big[(t-a)^\alpha-(t-T)^\alpha\big]}}\exp\!\left(-\frac{\Gamma(\alpha+1)}{\big[(t-a)^\alpha-(t-T)^\alpha\big]}\frac{(x_T-x-\mu(T-a))^2}{2\sigma^2}-V\frac{(T-a)^\alpha}{\alpha}\right).$$

It is easy to check that $K_\alpha^\mu(x_T,T;x,t)$ is the solution of the equation

$$\frac{\sigma^2}{2}\frac{\partial^2 K_\alpha^\mu}{\partial x^2}+\frac{\mu(T-a)\Gamma(\alpha+1)}{\big[(t-a)^\alpha-(t-T)^\alpha\big]}\frac{\partial K_\alpha^\mu}{\partial x}=-\Gamma(\alpha+1)\frac{\partial K_\alpha^\mu}{\partial a} \tag{39}$$

$$=K_\alpha^\mu\frac{(T-a)^{\alpha-1}}{\alpha}V+\frac{\Gamma^2(\alpha+1)}{2\sigma^2}K_\alpha^\mu$$

$$\times\left\{\frac{(x_T-x)^2-\mu^2(T-a)^2}{\big[(t-a)^\alpha-(t-T)^\alpha\big]^2}-\frac{(x_T-x)-\mu(T-a)}{(t-a)^\alpha-(t-T)^\alpha}\left[\frac{(x_T-x)-\mu(T-a)}{(t-a)^\alpha-(t-T)^\alpha}(t-a)^{\alpha-1}+2\mu\right]\right\}.$$

This equation holds also for the option price and is written as

$$\frac{\sigma_\alpha^2}{2}\frac{\partial^2 K_\alpha^\mu}{\partial x^2}+\bar{\mu}_\alpha\frac{\partial K_\alpha^\mu}{\partial x}=-\frac{\partial K_\alpha^\mu}{\partial a}+K_\alpha^\mu\big[\bar{r}_\alpha+V(x,a)\big], \tag{40}$$

where

$$\sigma_\alpha^2 = \frac{\sigma^2}{\Gamma(\alpha+1)}\,, \tag{41}$$

$$\bar{\mu}_\alpha = \frac{\mu(T-a)}{\big[(t-a)^\alpha - (t-T)^\alpha\big]}\,, \tag{42}$$

$$\bar{r}_\alpha = \frac{(T-a)^{\alpha-1}}{\Gamma(\alpha+2)} V\,. \tag{43}$$

We entitle Eq. (40) by the fractional BS equation. The present model exhibits similar behavior to option pricing with fractional volatility driven by fractional Brownian motion [8]. Those parameters need to be confronted to observations when applied to a specific model. Time-varying parameters are an important concern because in applied economics, one may want to integrate the market's view on the direction of the future behaviour of variables which the call price depends on, thus offering more elasticity to the model. Moreover, their term structures reflect expectation and dynamics of market factors.

Unlike the standard European options, the estimation of barrier options with time-dependent parameters is not an inconsequential extension, and has been the center of some recent work [73]. A general and appealing solution of Eq. (40) is given in Ref. [74]. An alternative derivation of the BS solution with time-dependent parameters was given through the use of a generalized change of variable technique. The derived solution shows that the price of a European call option on a non-dividend paying equity is decomposed as a product of three simple terms consisting of a BS price for the case of constant-coefficient in a non-dividend-paying setup, the ratio of two strike prices and a modified factor characterizing the parameterized time. In reality, the market process exhibits approximate self-similar properties, therefore mathematical simplicity suggests looking for descriptions in terms of fractional Brownian motion.

In reality, the standard BS model is based on unrealistic assumption about the geometric nature of Brownian motion with constant non-stochastic fine-tuning of the portfolio and no transaction fees. Moreover, the assumption of zero interest rate or known constant interest rate is unrealistic. The empirical limitations of BS model originated a vast amount of alternative modified models [75 − 77], with effort to elucidate the deviations from BS model by introducing supplementary degrees of freedom. Most of these alternative models have no compact closed-form solution and numerical solutions most often do not reproduce appropriately the data profiles. It is worth mentioning that geometric Brownian motion models well the lack of memory in liquid markets where the autocorrelation of price changes decays to negligible values in a short period of time, consistent with the absence of long-term statistical arbitrage. We expect interesting consequences in more realistic models with time-dependent skew [78, 79].

## 4. Conclusions and perspectives

To the best of our knowledge, this work represents the first attempt to build a fractional finance modeling and option pricing based on criteria of fractional action integral operators. Our contribution is meant to serve as an informal introduction and not as a rigorous and comprehensive treatment of the topic discussed through this paper. We have laid out the groundwork for non-local fractional finance using the methodology of fractional integral operator within the framework of the fractional action-like variational approach by introducing the basic settings. By performing the fractional action integral for the Black-Scholes Lagrangian, we have obtained a diffusion equation with time-dependent diffusivity which exhibits a similarity with the fractional Brownian motion and therefore may have interesting consequences on jump-diffusion processes in continuous time. The paths exhibit a non-Gaussian process, non-stationary random process whose increments are independent and distributed in a similar way to the Levy sable distribution which plays a crucial role because they are the attractors of distributions of sums of random variables with divergent second moments according to the generalized central limit theorem. Moreover, the fractional distributions obtained through this work display slowly decaying tails describing stochastic processes with large events. Further, their self-similarity properties make them practical in the description of fractal processes.

We find a departure from the normal distribution which in fact is observed when the irregular random behaviour of stock price changes is superposed on another regular periodic pattern. There is a definite support of periodic behaviour of price changes matching to intervals of a day, week, quarter and year, according to the rhythm of individual action. Obtained Black-Scholes equation is not enough complicated, and we expect solutions to be found using advanced numerical techniques and computational simulations. In this paper we have shown how to use the fractional action integral to derive fractional Brownian motion and fractional Black-Scholes price of a call option. An extension is the application of this type of fractional Black-Scholes formula to price different derivatives augmented by numerical simulations.

We expect developing more appropriate stochastic optimal portfolio models using the results obtained. Concurrent research efforts are needed to confirm or falsify, develop or disprove fractional BS equation with time-dependent fractional parameters and our preliminary findings.

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## PRIMJENA INTEGRO-DIFERENCIJALNE VARIJACIJSKE RAZLOMNE ZADAĆE I RAZLOMNOG PRISTUPA INTEGRALIMA PO PUTEVIMA STOHASTIČKOM MODELIRANJU

Proučavaju se stohastički modeli primjenom integrala po putevima, a posebno se razlažu novčane izvodnice i mogućnosti u određivanju cijena u okviru razlomnog djelotvornog varijacijskog pristupa nedavno uvedenog autorom. Mnoge se zanimljive odlike i posljedice otkrivaju djelomično.