Printed ISSN 1330–0008
Online ISSN 1333–9125
CD ISSN 1333–8390
CODEN FIZAE4

# HIGHER-ORDER FRACTIONAL FIELD EQUATIONS IN (0+1) DIMENSIONS AND PHYSICS BEYOND THE STANDARD MODEL

RAMI AHMAD EL-NABULSI

*Department of Nuclear and Energy Engineering, Cheju National University, Ara-dong 1, Jeju 690-756, South Korea E-mail address: nabulsiahmadrami@yahoo.fr*

Received 22 June 2008; Revised manuscript received 29 November 2009
Accepted 17 March 2010 Online 12 July 2010

I discuss the Lagrangian procedure for the treatment of higher-order fractional field equations in (0+1) dimensions based on the fractional action-like variational problems with higher-order derivatives. We explore the case in which the derivatives appear only in the invariant d'Alembertian operator. Some interesting consequences are revealed.

PACS numbers: 45.10.Hj, 45.20.Jj UDC 531.314, 532.511

Keywords: fractional action-like variational approach, higher-order derivatives, field theory in (0+1)-dimensions, massless particles, tachyons

## 1. Introduction

One of the greatest of Newton's and Leibnitz's legacies is their introduction of what later became known as the calculus because it was through calculus that the laws of physics came to be cast in terms of ordinary and partial differential equations. These differential equations are normally based on derivatives in space and time that are of integer order. Fractional dynamics is the study of physical dynamical systems that can be cast in terms of solutions to differential equations which are of fractional order to which the fractional calculus can be applied. The fractional derivatives and integrals describe more accurately the complex physical systems and at the same time, investigate more about simple dynamical systems. It is worth-mentioning that fractional calculus has led to many breakthroughs in theoretical and applied physics. Although fractional calculus has been studied for over 300 years now, it has been regarded mainly as a mathematical curiosity until about 1992, when dynamical equations involving fractional derivatives and integrals were pretty much restricted to the realm of mathematics. However, since that time, physicists have begun to explore the many applications of fractional calculus to many problems in physics, including hydrology, viscoelastivity, heat conduction, polymer physics, chaos and fractals, biophysics and thermodynamics, Brownian random walks with memory, modelling dispersion and turbulence, oscillating vortex chain, control theory, transfer equation in a medium with fractal geometry, stochasting modelling for ultraslow diffusion, kinetic theories, far-from equilibrium statistical models manifesting scale invariance, non-local correlations and extensive symmetry breaking, plasma physics, modelling mechanical and electrical properties of real materials, description of rheological properties of rocks, dynamics in complex media, wave propagation in complex and porous media, astrophysics, cosmology, quantum field theory, potential theory, and so on [1 − 11]. Further, fractional differential equations of different types can be used to describe a range of stochastic processes that do not conform to conventional statistical models. They are finding increasing value in modelling processes that are quasi-deterministic, i.e. neither fully deterministic or fully stochastic, and as such, can be used to analyze systems for which conventional statistical analysis is inadequate and where deterministic models become intractable. We refer the reader interested on fractional theory to the comprehensive book [2].

In reality, dealing with fractional derivatives is not more complex than with usual differential operators. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models in which such effects are, in fact, neglected. Today, there exist many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types, including Grunwald-Letnikov fractional derivative, Caputo fractional derivative, etc. The definition of the fractional order derivative and integral are not unique where several definitions exist, e.g. Grunwald-Letnikov, Caputo, Weyl, Feller, Erdelyi-Kober, Riesz fractional derivatives, fractional Liouville operators, and so on, but the Riemann-Liouville (RL) fractional derivatives and integrals are the most recurrently used and have been popularized when fractional integration is performed in the dynamical system under study [1 − 11].

A subject of current strong research concerns the study of fractional problems of the calculus of variations (COV) with its corresponding fractional Hamiltonian formalism and respective Euler-Lagrange type equations [12]. Different forms of Euler-Lagrange equations were obtained in literature depending on the action and type of fractional derivative used. The major problem with most of the fractional approaches treated in the literature is the presence of non-local fractional differential operators and the adjoint of a fractional differential operator used to describe the dynamics is not the negative of itself. Further, the derived Euler-Lagrange equations depend on left and right fractional derivatives, even when the dynamics depend only on one of them. Other complicated problems arise during the mathematical manipulations as the appearance of a very complicated Leibniz rule (the derivative of product of functions) and the non-presence of any fractional analogue of the chain rule. The formulation of the fractional problems of the COV still needs more elaboration as the problem is deeply related to the fractional quantization procedure and to the presence of non-local fractional differential operators.

In order to better model non-conservative dynamical systems, we proposed a novel one-dimensional (1D) approach entitled fractional action-like variational approach (FALVA) based on the concept of left Riemann-Liouville fractional integral functionals with one parameter, but not on fractional-order derivatives of the same order [13, 14]. The derived Euler-Lagrange equations are similar to the standard one, but with the presence of fractional generalized external force acting on the system. Many encouraging results were obtained and discussed [9 − 40]. The generalization of the classical Noether's theorem for the context of the fractional calculus of variations has been derived recently [32 − 34]. The variational calculus of fractional order was used by Jumarie in Refs. [41, 42] to derive the Hamilton-Jacobi equation and a fractional Lagrangian approach to the one-dimensional optimal control theory with fractional cost functional. The multi-dimensional fractional action-like problems of the calculus of variations were explored recently in Ref. [19]. More recently, the author discussed the application of the multi-dimensional fractional approach to fractional field theories where the fractional Euler-Lagrange equations have been derived for classical fields and the fractional Dirac operators and fractional Bohner-Weitzenböck of multiple orders in fractional spinor fields have been introduced [15].

The interest in fractional field theory is a relatively new one. The readers may be refereed to Refs. [15 − 18] and references therein for the importance of fractional dynamics in quantum field theory. The principle of local gauge invariance were applied to fractional fields where an analytic mass formula of a non-relativistic fractional charged particle moving in a constant magnetic fractional field was derived [43], opening consequently a new exciting branch in fractional field theories. Recently, Hermann [44, 45] applied the concept of fractional derivative to derive a fractional Schödinger type wave equation by a quantization of the classical non-relativistic Hamiltonian. This equation was considered by the author as an alternative tool for a suitable explanation of the charmonium spectrum, normally described by a phenomenological potential.

Our main aim in the present work is to explore the fractional Lagrangian formalism for the treatment of higher-order fractional field equations. No fractional derivatives of any type and any order will be introduced.

The higher-order field equations are acquiring increasing importance due to the consideration of higher-order gravity theories (supergravity, superstring, M-theory), with Lagrangians containing terms quadratic in the curvature tensors [46]. It was recently shown that many of the central ideas constructed in quantum field theory can be exemplified simply and straightforwardly by using toy models in (0+1) dimensions. Because quantum field theory in (0+1) dimensions is equivalent to quantum mechanics, these models allow us to use techniques of quantum mechanics to gain insight into quantum field theory [47]. Therefore, we will build the higher-order fractional formalism in (0+1) dimensions.

We follow the rationale of Ref. [21] where it is assumed that at least one stationary point for the fractional functional exists. We introduce the main notations, conventions and assumptions that underlie the remainder of the present work:

1. In the notation $t \to f(t)$, $t$ is a dummy variable.
2. Exactly, the same function can be written, for example $(\dot{q}, q, \tau) \to f(\dot{q}, q, \tau)$; $\dot{q}, q, \tau$ are here dummy variables.
3. For $(\dot{q}, q, \tau) \to f(\dot{q}, q, \tau)$, the partial derivative of $f$ with respect to the first argument is denoted by $\partial f/\partial \dot{q}$.
4. For a given scalar field $\phi$, $\partial_i \phi = \partial \phi / \partial q^i$, $i = 1, 2, \ldots, n$.
5. Einstein summation convention is applied throughout.
6. The analysis is carried out entirely at the classical level.
7. Following our previous work, we use in this paper the left-fractional Riemann-Liouville integral which is the most widely used definition of an integral of fractional order via an integral transform, defined as

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} f(\tau)(t-\tau)^{\alpha-1} \mathrm{d}\tau, \quad 0 < \alpha < 1\,. $$

8. Space-time variables and scalar fields are properly normalized as dimensionless observables.
9. No fractional-order derivatives will be introduced.
10. The Poincaré indices are denoted by $i, j = 0, 1, 2, 3$.
11. The Minkowski metric $\eta^{ij}$ has signature $(+, -, -, -)$ so that $\eta^{00} = +1$.
12. We work in units $\hbar = c = 1$.

To the best of our knowledge, this work represents the first attempt to apply the concept of FALVA to the Lagrangian procedures for higher-order field equations. We care that our contribution is planned to serve as a simple informal introduction and not a precise treatment of the topic. The paper is organized as follows: in Sec. 2, we review rapidly the basic concepts of FALVA in (0+1) dimensions (Problem 2.1 and Theorem 2.1). After that, in the same section, we derive the fourth- and third-order fractional Euler-Lagrange equations and we generalize our results to Lagrangian involving higher derivatives. The fractional canonical tensors are discussed in Sec. 3. The case in which the derivatives appear only in the invariant d'Alembertian operator is discussed within the same section and is illustrated by a simple example. The paper concludes in Sec. 4 with a brief summary of the main results and future challenge and perspectives.

## 2. *Brief overview of FALVA*

In 2005, the author introduced the one-dimensional FALVA problem as follows [13, 14]:

**Problem 2.1:** *Find the stationary points of the integral functional*

$$S\big[q(\bullet)\big] = \frac{1}{\Gamma(\alpha)} \int_a^t L(\dot{q}(\tau), q(\tau), \tau)(t-\tau)^{\alpha-1} \mathrm{d}\tau \,, \tag{1}$$

*under the initial condition $q(a) = q_a$, where $\dot{q} = \mathrm{d}q/\mathrm{d}\tau$, $\Gamma$ is the Euler gamma function, $0 < \alpha < 1$, $\tau$ is the intrinsic time, $t$ is the observer time, $t \neq \tau$, and the smooth Lagrangian function $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a $C^2$-function with respect to all its arguments.*

**Theorem 2.1:** *If $q(\bullet)$ are solutions to the previous problem, i.e., $q(\bullet)$ are critical points of the function (1), then $q(\bullet)$ satisfy the following Euler-Lagrange equations:*

$$\frac{\partial L(\dot{q}(\tau), q(\tau), \tau)}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\partial L(\dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}}\right) = \frac{1-\alpha}{t-\tau} \frac{\partial L(\dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}} \,. \tag{2}$$

To deal with the FALVA problem with higher-order derivatives, we first discuss, for convenience, the case of the Lagrangian of the type $L(\ddot{q}(\tau), \dot{q}(\tau), q(\tau), \tau)$.

Suppose we want to extremise the fractional functional

$$S\big[q(\bullet)\big] = \frac{1}{\Gamma(\alpha)} \int_a^t L(\dot{q}(\tau), q(\tau), \tau)(t-\tau)^{\alpha-1} \mathrm{d}\tau \,,$$

subject to the constraints $\mathcal{G}(\dot{q}(\tau), q(\tau), \tau) = 0$, where $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^k$ is a differentiable function. We may introduce the well-known Lagrange multipliers $\lambda : [a, b] \to \mathbb{R}^k$.

**Definition 2.1:** *The constrained fractional action integral is defined by*

$$S\big[q(\bullet), \lambda\big] = \frac{1}{\Gamma(\alpha)} \int_a^t \Big(L(\dot{q}(\tau), q(\tau), \tau) - \langle \lambda(\tau), \mathcal{G}(\dot{q}(\tau), q(\tau), \tau)\rangle\Big)(t-\tau)^{\alpha-1} \mathrm{d}\tau \,, \tag{3}$$

*where $\langle , \rangle$ is the dot product in $\mathbb{R}^k$.*

We may now derive the fractional Euler-Lagrange equation for a Lagrangian $L(\ddot{q}(\tau), \dot{q}(\tau), q(\tau), \tau)$ depending on the second derivatives of a $C^3$ function. This problem is the same as extremizing the fractional action with Lagrangian

$L(\dot{\mathcal{Y}}(\tau), \dot{q}(\tau), q(\tau), \tau)$, subject to the constraints $\mathcal{Y}(\tau) = \dot{q}(\tau)$. The modified fractional action is then written like

$$S[\mathcal{Y}, q(\bullet), \lambda] = \frac{1}{\Gamma(\alpha)} \int_a^t \Big(L(\dot{\mathcal{Y}}(\tau), \dot{q}(\tau), q(\tau), \tau) - \lambda(\tau)\big(\mathcal{Y} - \dot{q}(\tau)\big)\Big)(t-\tau)^{\alpha-1} \mathrm{d}\tau \,. \quad (4)$$

**Corollary 2.1**: *If $q(\bullet)$ are solutions to Eq. (3) subject to the constraints $\mathcal{Y}(\tau) = \dot{q}(\tau)$, then $q(\bullet)$ satisfy the following Euler-Lagrange equations for the Lagrangian $L(\ddot{q}(\tau), \dot{q}(\tau), q(\tau), \tau)$,*

$$\frac{\partial L(\ddot{q}(\tau), \dot{q}(\tau), q(\tau), \tau)}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L(\ddot{q}(\tau), \dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\frac{\partial L(\ddot{q}(\tau), \dot{q}(\tau), q(\tau), \tau)}{\partial \ddot{q}}\right)$$

$$= \frac{1-\alpha}{t-\tau}\left[\frac{\partial L(\ddot{q}(\tau), \dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}} - 2\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L(\ddot{q}(\tau), \dot{q}(\tau), q(\tau), \tau)}{\partial \ddot{q}}\right)\right]$$

$$- \frac{(1-\alpha)(2-\alpha)}{(t-\tau)^2} \frac{\partial L(\ddot{q}(\tau), \dot{q}(\tau), q(\tau), \tau)}{\partial \ddot{q}} \,. \quad (5)$$

For $\alpha = 1$, we find the standard Euler-Lagrange equations which correspond to the Lagrangian $L(\ddot{q}(\tau), \dot{q}(\tau), q(\tau), \tau)$. Equation (5) is fourth-order because in general the third term on its LHS involves a fourth-derivative term. However, when the second-order fractional Euler-Lagrange equations for $L(\dot{q}(\tau), q(\tau), \tau)$ are differentiated, one gets a third-order fractional equation as shown by the following simple calculation: first, by differentiating the fractional Euler-Lagrange equation for $L(\dot{q}(\tau), q(\tau), \tau)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L(\dot{q}(\tau), q(\tau), \tau)}{\partial q^j} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L(\dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}^j}\right)\right) - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1-\alpha}{t-\tau}\frac{\partial L(\dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}^j}\right) = 0 \,. \quad (6)$$

The above equation may be written as

$$\frac{\partial^2 L(\dot{q}(\tau), q(\tau), \tau)}{\partial q^j \partial q^i}\dot{q}^j + \frac{\partial^2 L(\dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}^j \partial q^i}\ddot{q}^j$$

$$- \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial^2 L(\dot{q}(\tau), q(\tau), \tau)}{\partial q^j \partial \dot{q}^i}\dot{q}^j + \frac{\partial^2 L(\dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}^j \partial \dot{q}^i}\ddot{q}^j\right) - \frac{1-\alpha}{(t-\tau)^2}\frac{\partial L(\dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}^j}$$

$$- \frac{1-\alpha}{t-\tau}\left(\frac{\partial^2 L(\dot{q}(\tau), q(\tau), \tau)}{\partial q^j \partial \dot{q}^i}\dot{q}^j + \frac{\partial^2 L(\dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}^j \partial \dot{q}^i}\ddot{q}^j\right) = 0 \,, \quad (7)$$

which is

$$\frac{\partial^2 L(\dot{q}(\tau), q(\tau), \tau)}{\partial q^j \partial q^i}\dot{q}^j + \frac{\partial^2 L(\dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}^j \partial q^i}\ddot{q}^j$$

$$+\frac{\partial^2 L(\dot{q}(\tau),q(\tau),\tau)}{\partial q^j\partial\dot{q}^i}\ddot{q}^j+\frac{\partial^3 L(\dot{q}(\tau),q(\tau),\tau)}{\partial q^k\partial q^j\partial\dot{q}^i}\dot{q}^k\dot{q}^j+\frac{\partial^3 L(\dot{q}(\tau),q(\tau),\tau)}{\partial \dot{q}^k\partial q^j\partial\dot{q}^i}\ddot{q}^k\dot{q}^j$$

$$+\frac{\partial^3 L(\dot{q}(\tau),q(\tau),\tau)}{\partial q^k\partial \dot{q}^j\partial\dot{q}^i}\dot{q}^k\ddot{q}^j+\frac{\partial^3 L(\dot{q}(\tau),q(\tau),\tau)}{\partial \dot{q}^k\partial \dot{q}^j\partial\dot{q}^i}\ddot{q}^k\ddot{q}^j+\frac{\partial^2 L(\dot{q}(\tau),q(\tau),\tau)}{\partial \dot{q}^j\partial\dot{q}^i}\dddot{q}^j$$

$$-\frac{1-\alpha}{t-\tau}\left(\frac{\partial^2 L(\dot{q}(\tau),q(\tau),\tau)}{\partial q^j\partial\dot{q}^i}\dot{q}^j+\frac{\partial^2 L(\dot{q}(\tau),q(\tau),\tau)}{\partial \dot{q}^j\partial\dot{q}^i}\ddot{q}^j\right)+\frac{\alpha-1}{(t-\tau)^2}\frac{\partial L(\dot{q}(\tau),q(\tau),\tau)}{\partial\dot{q}^j}=0\,. \tag{8}$$

This is the third-order fractional Euler-Lagrange equation. $i,j=1,\ldots,k$.

Equation (5) may be generalized to a Lagrangian involving higher derivatives $L(q^{(m)}(\tau),\ldots,\dot{q}(\tau),q(\tau),\tau)$ as follows [34–36].

**Problem 2.2:** *Find the stationary points of the integral functional*

$$S^m\big[q(\bullet)\big]=\frac{1}{\Gamma(\alpha)}\int\limits_a^t L(q^{(m)}(\tau),\ldots,\dot{q}(\tau),q(\tau),\tau)(t-\tau)^{\alpha-1}\mathrm{d}\tau\,, \tag{9}$$

*$m\geq 1$, under the initial condition $q^{(i)}(a)=q_a$, $i=0,1,2,\ldots,m$, where $q^{(i)}=\mathrm{d}q/\mathrm{d}\tau^i$, $\Gamma$ is the Euler gamma function, $0<\alpha\leq 1$, $\tau$ is the intrinsic time, $t$ is the observer time, $t\neq\tau$, and the smooth Lagrangian function $L:[a,b]\times\mathbb{R}^{n\times(m+1)}\to\mathbb{R}$ is a $C^{2m}$-function with respect to all its arguments.*

**Theorem 2.2:** *If $q(\bullet)$ are solutions to the problem 2.2, i.e., if $q(\bullet)$ are critical points of the function (9), then $q(\bullet)$ satisfy the following higher-order Euler-Lagrange equations in (0+1) dimensions:*

$$\sum_{i=0}^{m}(-1)^i\frac{\mathrm{d}^i}{\mathrm{d}\tau^i}\partial_{i+2}L(q^{(m)}(\tau),\ldots,\dot{q}(\tau),q(\tau),\tau)$$

$$=\frac{1-\alpha}{t-\tau}\sum_{i=1}^{m}i(-1)^{i-1}\frac{\mathrm{d}^{i-1}}{\mathrm{d}\tau^{i-1}}\partial_{i+2}L(q^{(m)}(\tau),\ldots,\dot{q}(\tau),q(\tau),\tau)$$

$$+\sum_{k=2}^{m}\sum_{i=2}^{k}(-1)^{i-1}\frac{\Gamma(i-\alpha+1)}{(t-\tau)^i\Gamma(1-\alpha)}\binom{k}{k-i}\frac{\mathrm{d}^{k-i}}{\mathrm{d}\tau^{k-i}}\partial_{k+2}L(q^{(m)}(\tau),\,...,\dot{q}(\tau),q(\tau),\tau). \tag{10}$$

Here $\partial_i L$ denotes the partial derivatives of $L(\bullet,\bullet,\,...,\bullet,)$ with respects to its $i$th argument. In the particular case when $m=1$, problem 2.2 reduces to the problem 2.1.

## 3. Fractional field theories

Our main aim now is to deal with field theories. We start with a Lagrangian function of a scalar field $\phi$ and of its first $m$ derivatives, i.e.,

$$\delta L = \frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial \not{\partial}_\beta \phi}\delta(\not{\partial}_\beta \phi) + \ldots + \frac{\partial L}{\partial \not{\partial}_{\beta_1}\phi \cdots \not{\partial}_{\beta_m}\phi}\delta(\not{\partial}_{\beta_1}\cdots\not{\partial}_{\beta_m}\phi)\,. \tag{11}$$

We shall assume for simplicity that $m = N$.

**Definition 3.1:** *The fractional problems for the calculus of variations of the field theory in (0+1)-dimensions are defined by*

$$S^m\big[q(\bullet)\big] = \frac{1}{\Gamma(\alpha)}\int_a^t L\big(\partial^{(m)}\phi(\bullet),\ldots,\partial\phi(\bullet),\phi(\bullet)\big)(t-\tau)^{\alpha-1}\,\mathrm{d}\tau\,, \tag{12}$$

*where the admissible paths are smooth functions $\phi : \Omega \subset \mathbb{R} \to M$, satisfying given Dirichlet boundary conditions on $\partial\Omega$. The Lagrangian function is supposed to be sufficiently smooth with respect to all its arguments.*

**Theorem 3.1:** *If $\phi(\bullet)$ are solutions to the action (12), i.e., if $\phi(\bullet)$ are critical points of the function (12), then $\phi(\bullet)$ satisfy the following higher-order Euler-Lagrange equations,*

$$\sum_{i=0}^{m}(-1)^i\frac{\mathrm{d}^i}{\mathrm{d}\tau^i}\partial_{i+2}L\big(\partial^{(m)}\phi(\bullet),\ldots,\partial\phi(\bullet),\phi(\bullet)\big)$$

$$= \frac{1-\alpha}{t-\tau}\sum_{i=1}^{m} i(-1)^{i-1}\frac{\mathrm{d}^{i-1}}{\mathrm{d}\tau^{i-1}}\partial_{i+2}L\big(\partial^{(m)}\phi(\bullet),\ldots,\partial\phi(\bullet),\phi(\bullet)\big)$$

$$\sum_{k=2}^{m}\sum_{i=2}^{k}(-1)^{i-1}\frac{\Gamma(i-\alpha+1)}{(t-\tau)^i\Gamma(1-\alpha)}\binom{k}{k-i}\frac{\mathrm{d}^{k-i}}{\mathrm{d}\tau^{k-i}}\partial_{k+2}L\big(\partial^{(m)}\phi(\bullet),...,\partial\phi(\bullet),\phi(\bullet)\big)\,. \tag{13}$$

**Remark 3.1:** In reality, FALVA is accompanied with a non-conservative total energy as a new expression appears which also depends on the fractional order $\alpha$. Therefore, the Noether's theorem is violated and ceased to be valid. However, it was recently shown that it is still possible to obtain a Noether-type theorem which covers conservative and nonconservative dynamical systems simultaneously [36–39].

We discuss the special case where the Lagrangian is $L\big(\square^{(N)}\phi(\bullet), ..., \square\phi(\bullet), \phi(\bullet)\big)$, i.e. $\phi^{(0)} = 0$. Here $\square = \eta^{ij}\partial_i\partial_j$. The corresponding fractional Euler-Lagrange equations are obtained using the previous arguments as follows,

$$\sum_{i=0}^{m}(-1)^i \frac{\mathrm{d}^i}{\mathrm{d}\tau^i}\partial_{i+2}L\big(\square^{(N)}\phi(\bullet), \dots, \square\phi(\bullet), \phi(\bullet)\big)$$

$$= \frac{1-\alpha}{t-\tau}\sum_{i=1}^{m} i(-1)^{i-1}\frac{\mathrm{d}^{i-1}}{\mathrm{d}\tau^{i-1}}\partial_{i+2}L\big(\square^{(N)}\phi(\bullet), \dots, \square\phi(\bullet), \phi(\bullet)\big)$$

$$+\sum_{k=2}^{m}\sum_{i=2}^{k}(-1)^{i-1}\frac{\Gamma(i-\alpha+1)}{(t-\tau)^i\Gamma(1-\alpha)}\binom{k}{k-i}\frac{\mathrm{d}^{k-i}}{\mathrm{d}\tau^{k-i}}\partial_{k+2}L\big(\square^{(N)}\phi(\bullet), ..., \square\phi(\bullet), \phi(\bullet)\big). \quad (14)$$

Similarly, it is still possible to obtain a Noether-type theorem which covers conservative and nonconservative field theories simultaneously. Here $\xi_1 = t$ and $x_1 = \tau$. Let us illustrate by the following example.

In fact, we discuss the case where $L = \frac{1}{2}\square\phi\square\phi - \frac{1}{2}\mu^4\phi^2$, $\mu$ is a real parameter [48]. The corresponding equations of motion are

$$\square\square\phi + \frac{1-\alpha}{t-\tau}\square\phi - \mu^4\phi = 0\,. \quad (15)$$

Using the Fourier development for the scalar field $\phi$, i.e., $\phi(x) = \int \mathrm{d}k\, \mathrm{e}^{-ikx}\widetilde{\phi}(k)$, we obtain easily

$$\left(k^4 + \frac{1-\alpha}{t-\tau}k^2 - \mu^4\right)\widetilde{\phi}(k) = 0\,, \quad (16)$$

and therefore

$$k_+^2 = \frac{1}{2}\left[\frac{1-\alpha}{\tau-t} + \sqrt{\left(\frac{1-\alpha}{\tau-t}\right)^2 + 4\mu^4}\right] \quad (17)$$

$$k_-^2 = \frac{1}{2}\left[\frac{1-\alpha}{\tau-t} - \sqrt{\left(\frac{1-\alpha}{\tau-t}\right)^2 + 4\mu^4}\right] \quad (18)$$

The term $(1-\alpha)/(\tau-t)$can be identified to a decaying mass with positive square mass $m^2 > 0$ $(0 < \alpha < 1)$. Hence, Eqs. (17) and (18) take the form

$$k_+^2 = \frac{1}{2}\left[m^2 + \sqrt{m^4 + 4\mu^4}\,\right], \quad (19)$$

$$k_-^2 = \frac{1}{2}\left[m^2 - \sqrt{m^4 + 4\mu^4}\right]. \tag{20}$$

At very large times $m^2 \to 0$ and therefore $k_+^2 \to \mu^2$ (normal particle) and $k_-^2 \to -\mu^2$ (tachyon), in agreement with the standard case. At very early times, $m^2 \gg \mu^2$ and, therefore, $k_+^2 \approx m^2$ and $k_-^2 \approx 0$. Note that the mass $m$ may be attributed to the Hubble mass $H$ up to a certain order if the difference in times $\tau - t$ is identified as the cosmic time, i.e., $H = \beta/T$, $T = \tau - t$. $\beta$ is a real and positive parameter to be determined from astrophysical observations. Hence, $m = (1-\alpha)H/\beta$. This fractional mass appears within the context of quantum field theory with quantum corrections. Recent observations adopt the conservative bounds $0.85 \lesssim H_0 T_0 \lesssim 1.91$ [49]. Here $T = \tau - t$. As a result, at very early times $m^2 \gg H^2$ and, consequently, $k_+ \approx (1-\alpha)^2 H^2/\beta^2$ and $k_- \approx 0$. In summary, at the early epoch of time, the field is dominated by normal and massless particles. During the growth of time, massless particles turn into tachyons and this is highly interesting as it may have interesting consequences on cosmology and the dark energy problem [50].

However, at very large time, one may also attend to have weak-mass particles and not massless particles. For this, we propose the following problem.

**Problem 3.1:** *Find the stationary points of the integral exponential-like functional*

$$S\big[q(\bullet)\big] = \frac{1}{\Gamma(\alpha)} \int_a^t L(\dot{q}(\tau), q(\tau), \tau)(\mathrm{e}^t - \mathrm{e}^\tau)^{\alpha-1} \mathrm{d}\tau\,, \tag{21}$$

*under the initial condition* $q(a) = q_a$, $0 < \alpha \leq 1$, *where the smooth Lagrangian function* $L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *is a* $C^2$*-function with respect to all its arguments.*

**Theorem 3.2:** *If* $q(\bullet)$ *are solutions to the problem 3.1, then* $q(\bullet)$ *satisfy the following Euler-Lagrange equations*

$$\frac{\partial L(\dot{q}(\tau), q(\tau), \tau)}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\partial L(\dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}}\right) = \frac{(1-\alpha)\mathrm{e}^\tau}{\mathrm{e}^t - \mathrm{e}^\tau}\,\frac{\partial L(\dot{q}(\tau), q(\tau), \tau)}{\partial \dot{q}}\,. \tag{22}$$

**Proof:** We may write $q_k(\tau) = q_k^0(\tau) + \sigma_k(\tau)$, where $q_k^0(\tau)$ is the minimum solution and $\sigma_k(\tau)$ describes the deviation of $q_k(\tau)$ from the minimum path $q_k^0(\tau)$. Replacing into the the action (1) gives

$$S = \frac{1}{\Gamma(\alpha)} \int_a^t L(\dot{q}_k^0(\tau) + \dot{\sigma}_k(\tau), q_k^0(\tau) + \sigma_k(\tau), \tau)(\mathrm{e}^t - \mathrm{e}^\tau)^{\alpha-1} \mathrm{d}\tau\,,$$

Performing Taylor expansion to first order in $\dot{\sigma}_k(\tau)$ and $\sigma_k(\tau)$ yields

$$S = \frac{1}{\Gamma(\alpha)}\left[\int_a^t \left\{L\big(\dot{q}_k^0(\tau), q_k^0(\tau), \tau\big) + \frac{\partial L}{\partial \dot{q}_k}\dot{\sigma}_k(\tau) + \frac{\partial L}{\partial q_k}\sigma_k(\tau)\right\}(\mathrm{e}^t - \mathrm{e}^\tau)^{\alpha-1}\mathrm{d}\tau\right].$$

Integrating the term in $\sigma_k(t)$ by parts gives without difficulty

$$S = \frac{1}{\Gamma(\alpha)} \left[ \int_a^t L\big(\dot{q}_k^0(\tau), q_k^0(\tau), \tau\big)(\mathrm{e}^t - \mathrm{e}^\tau)^{\alpha-1} \mathrm{d}\tau \right]$$

$$-\frac{1}{\Gamma(\alpha)} \int_a^t \sigma(\tau) \left[ (\mathrm{e}^t - \mathrm{e}^\tau)^{\alpha-1} \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\partial L}{\partial \dot{q}_k} + (1-\alpha)\mathrm{e}^\tau \frac{\partial L}{\partial \dot{q}_k} (\mathrm{e}^t - \mathrm{e}^\tau)^{\alpha-2} - \frac{\partial L}{\partial q_k} (\mathrm{e}^t - \mathrm{e}^\tau)^{\alpha-1} \right] \mathrm{d}\tau \,,$$

and we get the required result. ■

The originality in this approach concerns the occurrence of the new decaying $(1-\alpha)\mathrm{e}^\tau(\mathrm{e}^t - \mathrm{e}^\tau)^{-1}$ holding the properties,

$$\lim_{\tau\to\infty,\ t\neq\tau} (1-\alpha)\frac{\mathrm{e}^\tau}{\mathrm{e}^t - \mathrm{e}^\tau} = 1-\alpha \,, \tag{23}$$

$$\lim_{\tau\to 0,\ t\neq\tau} (1-\alpha)\frac{\mathrm{e}^\tau}{\mathrm{e}^t - \mathrm{e}^\tau} = \frac{1-\alpha}{\mathrm{e}^t - 1} \,. \tag{24}$$

Let us illustrate by discussing again the case when $L = \frac{1}{2}\Box\phi\Box\phi - \frac{1}{2}\mu^4\phi^2$, $\mu \in \mathbb{R}$. The corresponding equations of motion are accordingly

$$\Box\Box\phi + (1-\alpha)\frac{\mathrm{e}^\tau}{\mathrm{e}^t - \mathrm{e}^\tau}\Box\phi - \mu^4\phi = 0 \,. \tag{25}$$

Making use of the Fourier development $\phi(x) = \int \mathrm{d}k\, \mathrm{e}^{-ikx}\widetilde{\phi}(k)$, we obtain straightforwardly,

$$\left( k^4 + \frac{(1-\alpha)\mathrm{e}^\tau}{\mathrm{e}^t - \mathrm{e}^\tau} k^2 - \mu^4 \right) \widetilde{\phi}(k) = 0 \,, \tag{26}$$

and consequently:

$$k_+^2 = \frac{1}{2} \left[ \frac{(1-\alpha)\mathrm{e}^\tau}{\mathrm{e}^t - \mathrm{e}^\tau} + \sqrt{\left( \frac{(1-\alpha)\mathrm{e}^\tau}{\mathrm{e}^t - \mathrm{e}^\tau} \right)^2 + 4\mu^4} \right] \tag{27}$$

$$k_-^2 = \frac{1}{2} \left[ \frac{(1-\alpha)\mathrm{e}^\tau}{\mathrm{e}^t - \mathrm{e}^\tau} - \sqrt{\left( \frac{(1-\alpha)\mathrm{e}^\tau}{\mathrm{e}^t - \mathrm{e}^\tau} \right)^2 + 4\mu^4} \right] \tag{28}$$

The term $(1-\alpha)\mathrm{e}^\tau/(\mathrm{e}^t - \mathrm{e}^\tau)$ can again be identified to a fractionally decaying mass with positive square mass $m^2 > 0$, $(- < \alpha < 1)$. Amazingly in this approach, the mass $m^2$ depends on the observer time at the origin of time. At very large times

$m^2 \to (1-\alpha)$, i.e., weakly massive particles, in contrast to the first case which yields massless particles. Therefore

$$k_+^2 = \frac{1}{2}\left[(1-\alpha) + \sqrt{(1-\alpha)^2 + 4\mu^4}\,\right], \tag{29}$$

$$k_-^2 = \frac{1}{2}\left[(1-\alpha) - \sqrt{(1-\alpha)^2 + 4\mu^4}\,\right], \tag{30}$$

which for $\mu = 0$, give $k_+^2 = 1-\alpha$ (massive particles) and $k_-^2 = 0$ (massless particles), in contrast to the first discussed case. The tachyons disappear in this approach, in particular at very large time, what is quite interesting. The main difference between Eqs. (29, 30) and (19, 20) concern the behavior of the positive and negative modes at very large time. In the first Riemann-Liouville fractional approach, $k_+^2 \to \mu^2$ (massive particle) and $k_+^2 \to -\mu^2$ (tachyon) at very large time, while in the second fractional exponential approach, $k_+^2 > \mu^2$ and $k_-^2 < -\mu^2$. This may have interesting consequences in high energy physics. Work in this direction is under progress.

At the end, we want, as previously, to extremise the fractional functional (21) subject to the constraints $\mathcal{G}(\dot q(\tau), q(\tau), \tau) = 0$ where $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^k$ is again a differentiable function by introducing the Lagrange multipliers $\lambda : [a, b] \to \mathbb{R}^k$.

**Definition 3.2:** *The constrained fractional action integral in $\mathbb{R}^k$ is defined by*

$$S\big[q(\bullet),\lambda\big] = \frac{1}{\Gamma(\alpha)}\int_a^t \Big(L(\dot q(\tau), q(\tau), \tau) - \langle \lambda(\tau), \mathcal{G}(\dot q(\tau), q(\tau), \tau)\rangle\Big)(\mathrm{e}^t - \mathrm{e}^\tau)^{\alpha-1}\mathrm{d}\tau\,. \tag{31}$$

Our main aim now is to derive the fractional Euler-Lagrange equations for the Lagrangian $L(\ddot q(\tau), \dot q(\tau), q(\tau), \tau)$ depending on the second derivatives of a $C^3$ function subject to the constraints $\mathcal{Y}(\tau) = \dot q(\tau)$. The modified fractional action is afterward written like

$$S\big[\mathcal{Y}, q(\bullet),\lambda\big] = \frac{1}{\Gamma(\alpha)}\int_a^t \Big(L(\dot{\mathcal{Y}}(\tau), \dot q(\tau), q(\tau), \tau) - \lambda(\tau)\big(\mathcal{Y} - \dot q(\tau)\big)\Big)(\mathrm{e}^t - \mathrm{e}^\tau)^{\alpha-1}\mathrm{d}\tau\,. \tag{32}$$

**Corollary 3.1:** *If $q(\bullet)$ are solutions to the integral exponential-like functional (32) subject to the constraints $\mathcal{Y} = \dot q(\tau)$, then $q(\bullet)$ satisfy the following Euler-Lagrange equations for the Lagrangian $L(\ddot q(\tau), \dot q(\tau), q(\tau), \tau)$,*

$$\frac{\partial L(\ddot q(\tau), \dot q(\tau), q(\tau), \tau)}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L(\ddot q(\tau), \dot q(\tau), q(\tau), \tau)}{\partial \dot q}\right) + \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\frac{\partial L(\ddot q(\tau), \dot q(\tau), q(\tau), \tau)}{\partial \ddot q}\right)$$

$$= \frac{(1-\alpha)\mathrm{e}^\tau}{\mathrm{e}^t - \mathrm{e}^\tau}\left[\frac{\partial L(\ddot q(\tau), \dot q(\tau), q(\tau), \tau)}{\partial \dot q} - 2\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L(\ddot q(\tau), \dot q(\tau), q(\tau), \tau)}{\partial \ddot q}\right)\right]$$

$$-(1-\alpha)(2-\alpha)\left[\frac{\mathrm{e}^{\tau}}{\mathrm{e}^{t}-\mathrm{e}^{\tau}}+(2-\alpha)\frac{\mathrm{e}^{2\tau}}{(\mathrm{e}^{t}-\mathrm{e}^{\tau})^{2}}\right]\frac{\partial L(\ddot{q}(\tau),\dot{q}(\tau),q(\tau),\tau)}{\partial\ddot{q}}. \tag{33}$$

**Proof:** In fact, we may follow the standard steps and we write the variation $\delta S^{m}\big[q(\bullet)\big]$ like

$$\delta S^{m}\big[q(\bullet)\big]=\frac{1}{\Gamma(\alpha)}\int\limits_{a}^{t}\left(\sum_{i=0}^{m}\partial_{i+2}L\ \delta q^{(i)}\right)(\mathrm{e}^{t}-\mathrm{e}^{\tau})^{\alpha-1}\mathrm{d}\tau\,,$$

where $\delta q^{(i)}\in C^{2m}\big([a,b];\mathbb{R}\big)$, $i=1,2,\ldots,m$ are the variations of $q^{(i)}$ with $\delta q^{(i)}(a)=0$. By performing the integration by parts, we obtain

$$\text{for } m=1,\quad \delta S\Big[q(\bullet)\Big]=\frac{1}{\Gamma(\alpha)}\int\limits_{a}^{t}\left(\partial_2 L-\frac{\mathrm{d}}{\mathrm{d}\tau}\partial_3 L-\frac{(1-\alpha)\mathrm{e}^{\tau}}{\mathrm{e}^{t}-\mathrm{e}^{\tau}}\partial_3 L\right)(\mathrm{e}^{t}-\mathrm{e}^{\tau})^{\alpha-1}\ \delta q\mathrm{d}\tau\,,$$

$$\text{for } m=2,\quad \delta S^{2}\Big[q(\bullet)\Big]=\frac{1}{\Gamma(\alpha)}\int\limits_{a}^{t}\left(\partial_2 L-\frac{\mathrm{d}}{\mathrm{d}\tau}\partial_3 L+\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\partial_4 L\right)-\left(\frac{(1-\alpha)\mathrm{e}^{\tau}}{\mathrm{e}^{t}-\mathrm{e}^{\tau}}\left(\partial_3 L-2\frac{\mathrm{d}}{\mathrm{d}\tau}\partial_4 L\right)\right.$$

$$\left.-(1-\alpha)(2-\alpha)\left[\frac{\mathrm{e}^{\tau}}{\mathrm{e}^{t}-\mathrm{e}^{\tau}}+\frac{(2-\alpha)\mathrm{e}^{2\tau}}{(\mathrm{e}^{t}-\mathrm{e}^{\tau})^{2}}\right]\partial_4 L\right)(\mathrm{e}^{t}-\mathrm{e}^{\tau})^{\alpha-1}\ \delta q\mathrm{d}\tau\,,$$

and in general, we get the required results. Here $\partial_i L$ represents the partial derivative of the Lagrangian with respect to its $i$-th argument, $i\in\mathbb{N}$. ■

**Remark 3.2:** For $\alpha=1$, we find the standard Euler-Lagrange equations which corresponds to the Lagrangian $L(\ddot{q}(\tau),\dot{q}(\tau),q(\tau),\tau)$.

**Theorem 3.3:** *If $q(\bullet)$ are solutions to the integral exponential-like functional (21), then the third-order fractional Euler-Lagrange equation is*

$$\frac{\partial^2 L(\dot{q}(\tau),q(\tau),\tau)}{\partial q^j\partial q^i}\dot{q}^j+\frac{\partial^2 L(\dot{q}(\tau),q(\tau),\tau)}{\partial \dot{q}^j\partial q^i}\ddot{q}^j$$

$$+\frac{\partial^2 L(\dot{q}(\tau),q(\tau),\tau)}{\partial q^j\partial \dot{q}^i}\ddot{q}^j+\frac{\partial^3 L(\dot{q}(\tau),q(\tau),\tau)}{\partial q^k\partial q^j\partial \dot{q}^i}\dot{q}^k\dot{q}^j+\frac{\partial^3 L(\dot{q}(\tau),q(\tau),\tau)}{\partial \dot{q}^k\partial q^j\partial \dot{q}^i}\ddot{q}^k\dot{q}^j$$

$$+\frac{\partial^3 L(\dot{q}(\tau),q(\tau),\tau)}{\partial q^k\partial \dot{q}^j\partial \dot{q}^i}\dot{q}^k\ddot{q}^j+\frac{\partial^3 L(\dot{q}(\tau),q(\tau),\tau)}{\partial \dot{q}^k\partial \dot{q}^j\partial \dot{q}^i}\ddot{q}^k\ddot{q}^j+\frac{\partial^2 L(\dot{q}(\tau),q(\tau),\tau)}{\partial \dot{q}^j\partial \dot{q}^i}\dddot{q}^j$$

$$-\frac{(1-\alpha)\mathrm{e}^{\tau}}{\mathrm{e}^{t}-\mathrm{e}^{\tau}}\left(\frac{\partial^2 L(\dot{q}(\tau),q(\tau),\tau)}{\partial q^j\partial \dot{q}^i}\dot{q}^j+\frac{\partial^2 L(\dot{q}(\tau),q(\tau),\tau)}{\partial \dot{q}^j\partial \dot{q}^i}\ddot{q}^j\right)$$

$$-(1-\alpha)\frac{\mathrm{e}^{\tau}\mathrm{e}^{t}}{(\mathrm{e}^{t}-\mathrm{e}^{\tau})^2}\frac{\partial L(\dot q(\tau),q(\tau),\tau)}{\partial \dot q^j}=0\,. \tag{34}$$

**Proof:** By differentiating the fractional Euler-Lagrange equation (22) for $L(\dot q(\tau),q(\tau),\tau)$, we get straightforwardly,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L(\dot q(\tau),q(\tau),\tau)}{\partial q^j}-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L(\dot q(\tau),q(\tau),\tau)}{\partial \dot q}\right)\right)-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{(1-\alpha)\mathrm{e}^{\tau}}{\mathrm{e}^{t}-\mathrm{e}^{\tau}}\frac{\partial L(\dot q(\tau),q(\tau),\tau)}{\partial \dot q}\right)=0. \tag{35}$$

The left-hand side of the above equation is

$$\frac{\partial^2 L(\dot q(\tau),q(\tau),\tau)}{\partial q^j\partial q^i}\dot q^j+\frac{\partial^2 L(\dot q(\tau),q(\tau),\tau)}{\partial \dot q^j\partial q^i}\ddot q^j-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial^2 L(\dot q(\tau),q(\tau),\tau)}{\partial q^j\partial \dot q^i}\dot q^j\right)$$

$$-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial^2 L(\dot q(\tau),q(\tau),\tau)}{\partial \dot q^j\partial \dot q^i}\ddot q^j\right)-(1-\alpha)\frac{\mathrm{e}^{\tau}\mathrm{e}^{t}}{(\mathrm{e}^{t}-\mathrm{e}^{\tau})^2}\frac{\partial L(\dot q(\tau),q(\tau),\tau)}{\partial \dot q^j}$$

$$-(1-\alpha)\frac{\mathrm{e}^{\tau}}{\mathrm{e}^{t}-\mathrm{e}^{\tau}}\left(\frac{\partial^2 L(\dot q(\tau),q(\tau),\tau)}{\partial q^j\partial \dot q^i}\dot q^j+\frac{\partial^2 L(\dot q(\tau),q(\tau),\tau)}{\partial \dot q^j\partial \dot q^i}\ddot q^j\right)=0\,,$$

which is

$$\frac{\partial^2 L(\dot q(\tau),q(\tau),\tau)}{\partial q^j\partial q^i}\dot q^j+\frac{\partial^2 L(\dot q(\tau),q(\tau),\tau)}{\partial \dot q^j\partial q^i}\ddot q^j$$

$$+\frac{\partial^2 L(\dot q(\tau),q(\tau),\tau)}{\partial q^j\partial \dot q^i}\ddot q^j+\frac{\partial^3 L(\dot q(\tau),q(\tau),\tau)}{\partial q^k\partial q^j\partial \dot q^i}\dot q^k\dot q^j+\frac{\partial^3 L(\dot q(\tau),q(\tau),\tau)}{\partial \dot q^k\partial q^j\partial \dot q^i}\ddot q^k\dot q^j$$

$$+\frac{\partial^3 L(\dot q(\tau),q(\tau),\tau)}{\partial q^k\partial \dot q^j\partial \dot q^i}\dot q^k\ddot q^j+\frac{\partial^3 L(\dot q(\tau),q(\tau),\tau)}{\partial \dot q^k\partial \dot q^j\partial \dot q^i}\ddot q^k\ddot q^j+\frac{\partial^2 L(\dot q(\tau),q(\tau),\tau)}{\partial \dot q^j\partial \dot q^i}\dddot q^j$$

$$-\frac{(1-\alpha)\mathrm{e}^{\tau}}{\mathrm{e}^{t}-\mathrm{e}^{\tau}}\left(\frac{\partial^2 L(\dot q(\tau),q(\tau),\tau)}{\partial q^j\partial \dot q^i}\dot q^j+\frac{\partial^2 L(\dot q(\tau),q(\tau),\tau)}{\partial \dot q^j\partial \dot q^i}\ddot q^j\right)$$

$$-(1-\alpha)\frac{\mathrm{e}^{\tau}\mathrm{e}^{t}}{(\mathrm{e}^{t}-\mathrm{e}^{\tau})^2}\frac{\partial L(\dot q(\tau),q(\tau),\tau)}{\partial \dot q^j}=0\,. \qquad \blacksquare$$

The generalization of the previous problems and results for the case of higher-order derivatives, i.e. $L(\partial^{(m)}\phi(\bullet),\ldots,\partial\phi(\bullet),\phi(\bullet))$ where $\phi:\Omega\subset\mathbb{R}\to M$ satisfies given Dirichlet boundary conditions on $\partial\Omega$, are under progress.

## 4. *Conclusions*

The present work represents a new application of the fractional problems of the calculus of variations to the quantum field theory in (0+1) dimensions which is equivalent to quantum mechanics. We derived the fourth- and third-order fractional Euler-Lagrange equations and we generalized our results to the case in which the Lagrangian contains higher-order derivatives. To deal with quantum field theory in (0+1) dimensions, we choose the particular case in which the derivatives appear only in the invariant d'Alembertian operator. As a simple application, we discussed the case where the Lagrangian is $L = (1/2)(\Box\phi\Box\phi - \mu^4\phi^2)$. In the standard case, the field is dominated by massless particles and tachyons at any epoch of times. In the fractional approach, the field is dominated at the early epoch of time by normal and massless particles. In the growth of time, massless particles turn into tachyons and it is the author's speculation that it might have interesting consequences in cosmology and dark energy problem. The fractional field theory is still an open problem under development. The exponentially fractional integral approach introduced here requires more study and elaboration and work in this direction is under progress. We expect they will open up in the future a new stimulating research area in fractional field theory and provide us with a powerful tool to understand many fundamental problems in the area of high energy physics and physics of curved spacetime [51]. Contemporaneous research efforts are needed to confirm or falsify, develop or disprove the fractional dynamics discussed here including our preliminary findings.

*Acknowledgements*

The author would like to thank the anonymous referees for their useful comments and valuable suggestions.

References

[1] K. B. Oldham, J. Spanier, *The Fractional Calculus*, Acad. Press, New York, London (1974).

[2] S. Samko, A. Kilbas and O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, New York (1993).

[3] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons Inc., New York (1993).

[4] I. Podlubny, *An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications*, Academic Press, New York, London (1999).

[5] R. Hilfer, Editor, *Applications of Fractional Calculus in Physics*, Word Scientific Publishing Co., New Jersey, London, Hong Kong (2000).

[6] J. A. Tenreiro Machado, Editor, *Nonlinear Dynamics*, Special Issue *Fractional Order Systems*, Kluwer, vol. **29**, no. 1 – 4, July 2002.

[7] *Signal Processing*, Elsevier, Special Issue *Fractional Signal Processing and Applications*, vol. **83**, Issue 11, Nov. (2003) p. 2285.

[8] *Dynamics*, Kluwer, Special Issue *Fractional Derivatives and Their Applications*, vol. **38**, n. 1-4, pp. 1-489, Dec. 2004.

[9] *Signal Processing*, Elsevier, Special Issue *Fractional Calculus Applications in Signals and Systems*, vol. **86**, Issue 10, Oct. 2006, p.2503.

[10] Journal of Vibration and Control, Sage Publ., Special Issue *Fractional Differentiation and its Applications*, vol. **14**, No. 9-10 (2008) p. 1253.

[11] ASME Journal of Computational and Nonlinear Dynamics, Special Issue *Discontinuous and Fractional Dynamical Systems*, vol. **3**, Issue 2, April 2008.

[12] O. P. Agrawal, J. Math. Anal. Appl. **272** (2006) 368.

[13] R. A. El-Nabulsi, Fizika A **14**, 4 (2005) 289.

[14] R. A. El-Nabulsi, Int. J. Appl. Math **17**, 3, (2005) 299.

[15] R. A. El-Nabulsi, Int. J. Geom. Meth. Mod. Phys. **5**, 6 (2008) 863.

[16] R. A. El-Nabulsi, Int. J. Geom. Meth. Mod. Phys. **6**, 1 (2009) 25.

[17] R. A. El-Nabulsi, Int. J. Geom. Meth. Mod. Phys. **6**, 6 (2009) 841.

[18] R. A. El-Nabulsi, Int. J. Geom. Meth. Mod. Phys. **1**, 7 (2010) 92.

[19] R. A. El-Nabulsi, Mod. Phys. Lett. B **23**, 28 (2009) 3369.

[20] R. A. El-Nabulsi, *Complexified Dynamical Systems From real Fractional Actionlike with Time-Dependent Fractional Dimension on Multifractal Sets*, Invited contribution to The 3rd Int. Conf. Complex Systems and Applications, University of Le Havre, Le Havre, Normandy, France, June 29 – July 02, 2009.

[21] R. A. El-Nabulsi and D. F. M. Torres, J. Math. Phys. **49** (2008) 053521.

[22] R. A. El-Nabulsi and D. F. M. Torres, Math. Meth. Appl. Sci. **30**, 15 (2007) 1931.

[23] R. A. El-Nabulsi, Chaos, Solitons and Fractals **42** (2009) 52.

[24] R. A. El-Nabulsi, Chaos, Solitons and Fractals **42** (2009) 84.

[25] R. A. El-Nabulsi, Chaos, Solitons and Fractals **41** (2009) 2262.

[26] R. A. El-Nabulsi, Chaos, Solitons and Fractals **42** (2009) 2614.

[27] R. A. El-Nabulsi, Chaos, Solitons and Fractals **42** (2009) 2294.

[28] R. A. El-Nabulsi, Chaos, Solitons and Fractals **42** (2009) 2384.

[29] R. A. El-Nabulsi, Fizika B **17**, 3 (2008) 369.

[30] R. A. El-Nabulsi, Fizika A **16**, 3 (2007) 137.

[31] R. A. El-Nabulsi, Fizika A **17**, 2 (2008) 71.

[32] R. A. El-Nabulsi, Centr. Europ. J. Phys. (in press).

[33] R. A. El-Nabulsi, I. A. Dzenite and D. F. M. Torres, *Fractional Action Functional in Classical and Quantum Field Theory*, Scientific Proc. Riga Technical University, 48th International Thematic Issue: Boundary Field Problems, (2006) P. 189.

[34] G. S. F. Frederico and D. F. M. Torres, Int. Math. Forum, **3**, 10 (2008) 479.

[35] G. S. F. Frederico and D. F. M. Torres, J. Math. Anal. Appl. **334**, 2 (2007) 834.

[36] G. S. F. Frederico and D. F. M. Torres, Int. J. Appl. Math. **19**, 1 (2006) 97.

[37] G. Ivan, M. Ivan and D. Opris, *Fractional Euler-Lagrange and Fractional Wong Equations for Lie Algebroids*, Proc. 4th Int. Colloquium Mathematics and Numerical Physics, October 6-8, 2006, Bucharest, Romania, pp. 73-80.

[38] G. S. F. Frederico and D. F. M. Torres, WSEAS Trans. Math., Issue **1**, 7, 2008, 6.

[39] G. S. F. Frederico and D. F. M. Torres, *Fractional Conservation Laws in Optimal Control Theory*, FDA's06-2nd Workshop on Fractional Differentiation and its Applications, 19 – 21 July (2006), Porto, Portugal (arXiv: math.OC: 0603598).

[40] G. S. F. Frederico and D. F. M. Torres, Int. J. Ecol. Econ. Stat. **9**, F07 (2007) 74.

[41] G. Jumarie, Chaos, Solitons and Fractals **32** (3) (2007) 969.

[42] G. Jumarie, J. Appl. Math and Computing **23** (1-2) (2007) 215.

[43] M. Zaganescu, *Some Applications of Fractional Integration and Differentiation in Quantum Mechanics and Field Theory*, Preprint U.T.F.T. 5/82, Univ. Timisoara (1982).

[44] R. Herrmann, arXiv: 0708.2262.

[45] R. Herrmann, nucl-th/0508033.

[46] R. N. Mohapatra, *Unification and Supersymmetry*, *The Frontiers of Quark-Lepton Physics*, Springer-Verlag, Berlin (2002).

[47] A. D. Boozer, Eur. Journ. Phys. **28** (2007) 729.

[48] C. G. Bollini and J. J. Giambiagi, *Lagrangian Procedures for Higher Order Field Equations*, Rev. Braz. Fisica **17** (1987) 14.

[49] A. Abate and O. Lahav, arXiv:0805.3160.

[50] P. Q. Hung, astro-ph/0612245.

[51] R. A. El-Nabulsi, Comm. Theor. Phys. **52**, 2 (2009) (in press).

## RAZLOMNE JEDNADŽBE POLJA VIŠEG REDA U (0+1) DIMENZIJI I FIZIKA IZA STANDARDNOG MODELA

Raspravljam o Lagrangeovom postupku za obradu razlomnih jednadžbi polja višeg reda u (0+1) dimenziji koji zasnivamo na razlomnim djelatnim varijacijskim zadacima s izvodima višeg reda. Istražujem slučaj kada se izvodi javljaju samo u invarijantnom d'Alembertovom operatoru. Nalaze se zanimljivi ishodi.