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EULER-LAGRANGE SOLUTION FOR CALCULATING PARTICLE ORBITS IN GRAVITATIONAL FIELDS

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The derivation of particle equations of motion in gravitational fields in general relativity is done routinely via the use of covariant derivatives. Since the geodesic equations based on covariant derivatives are derived from the Euler-Lagrange equations in the first place, and since the Euler-Lagrange formalism is very intuitive, easy to derive with no mistakes, there is every reason to use them even for the most complicated situations. In the current paper we show the application of the lagrangian equations for various scenarios in general relativity. A special paragraph is dedicated to the radial motion. In textbooks, radial motion is given less attention than orbital motion, perhaps because solving the equations of motion is more difficult than in the case of orbital motion.

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1. *Introduction: the Lagrangian method applied to radial motion*

The pedagogical approach permeating through the paper is straightforward: derive the lagrangian from the metric, derive the Euler-Lagrange equations from the lagrangian and solve them. In the concluding paragraphs, four novel applications of the lagrangian method are presented. Firstly, we show the application for deriving the advancement of the perihelion of not only Mercury but also for Venus and Earth in a novel way by combining perturbation theory with the lagrangian approach. Secondly, we show how to calculate the length of a rod while in radial motion. While the paper is constructed around the case of gravitational fields described by the Schwarzschild metric, we demonstrate how to extend the algorithms to other metrics, like Reissner-Nordström or Kerr, for example. We show an application in the concluding paragraph. In the cases of Reissner-Nordström or Kerr metrics, the lagrangian method has a definite advantage since the Christoffel symbols are much more difficult to calculate than in the case of the Schwarzschild metric. We chose the case of the Reissner-Nordström metric because it is encountered in literature much less than the Schwarzschild solution and because finding the equations of motion for objects falling into or gravitating around a charged black hole are considerably more difficult than in the case of the Schwarzschild solution. We show how to use the lagrangian approach in solving this problem and we even solve the difficult problem of calculating the perihelion advancement for objects describing arbitrary orbits. We conclude by deriving the trajectories of light in the vicinity of a charged black hole. While radial motion is the easiest type of motion to describe in natural language, it turns out that its equations are far from trivial.

In order to find the equations of motion, we start with the Schwarzschild metric for the particular case of absence of rotation ($\mathrm{d}\theta = \mathrm{d}\varphi = 0$)

$$\mathrm{d}s^2 = \alpha \mathrm{d}t^2 - \frac{1}{\alpha}\mathrm{d}r^2, \qquad \alpha = 1 - \frac{2m}{r} = 1 - \frac{r_s}{r}\,, \tag{1}$$

where $m = GM/c^2 \ll 1$ and $r_s = 2GM/c^2$ is the Schwarzschild radius. For example, the Schwarzschild radius of the Earth is only 9 millimeters. From the metric we obtain:

a) the lagrangian

$$L = \alpha \frac{\mathrm{d}t^2}{\mathrm{d}s^2} - \frac{1}{\alpha}\frac{\mathrm{d}r^2}{\mathrm{d}s^2}\,. \tag{2}$$

b) from the lagrangian we obtain the Euler-Lagrange system of equations [1, 5],

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial t} = 0\,, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0\,, \tag{3}$$

and, respectively:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\alpha\frac{\mathrm{d}t}{\mathrm{d}s}\right) = 0\,, \qquad \alpha\frac{\mathrm{d}t}{\mathrm{d}s} = k\,, \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{-2\dot{r}}{\alpha}\right) - \dot{t}^2\frac{\mathrm{d}\alpha}{\mathrm{d}r} + \dot{r}^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{\alpha}\right)$$

$$= 2\left(-\frac{\ddot{r}}{\alpha} + \dot{r}^2\frac{\mathrm{d}\alpha/\mathrm{d}r}{\alpha^2}\right) - \dot{r}^2\frac{\mathrm{d}\alpha/\mathrm{d}r}{\alpha^2} - \dot{t}^2\frac{\mathrm{d}\alpha}{\mathrm{d}r} = -2\frac{\ddot{r}}{\alpha} + \left(\frac{\dot{r}^2}{\alpha^2} - \dot{t}^2\right)\frac{\mathrm{d}\alpha}{\mathrm{d}r}\,.$$

The over-dots signify derivative with respect to s. From the metric (1) we obtain:

$$\alpha\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = \frac{1}{\alpha}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2. \tag{5}$$

Substituting (5) into (4) we obtain

c) the equation of motion:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} + \frac{1}{2}\frac{\mathrm{d}\alpha}{\mathrm{d}r} = 0\,, \tag{6}$$

that is,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} = -\frac{m}{r^2}\,. \tag{7}$$

From (7) we can see that the acceleration increases as the radial coordinate decreases. In order to solve (7) we will need to resort to a lemma.

Lemma

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}s}{\mathrm{d}r}\right)^{-2}. \tag{8}$$

Proof:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right) = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)\frac{\mathrm{d}r}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}s}{\mathrm{d}r}\right)^{-1}\left(\frac{\mathrm{d}s}{\mathrm{d}r}\right)^{-1}$$

$$= -\left(\frac{\mathrm{d}s}{\mathrm{d}r}\right)^{-2}\frac{\mathrm{d}^2 s}{\mathrm{d}r^2}\left(\frac{\mathrm{d}s}{\mathrm{d}r}\right)^{-1} = -\left(\frac{\mathrm{d}s}{\mathrm{d}r}\right)^{-3}\frac{\mathrm{d}^2 s}{\mathrm{d}r^2} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}s}{\mathrm{d}r}\right)^{-2}. \tag{9}$$

Applying the lemma, Eq. (7) becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}s}{\mathrm{d}r}\right)^{-2} = -\frac{2m}{r^2}\,. \tag{10}$$

With the notation $y = \left(\frac{\mathrm{d}s}{\mathrm{d}r}\right)^{-2}$, Eq. (10) becomes

$$\frac{\mathrm{d}y}{\mathrm{d}r} = -\frac{2m}{r^2}\,, \tag{11}$$

with the immediate solution:

$$y = \frac{2m}{r} - \frac{2m}{r_0}\,, \tag{12}$$

where $r_0 = r(0)$. On the other hand, $y = \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2$, so (12) reduces to

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{\frac{2m}{r} - \frac{2m}{r_0}}\,. \tag{13}$$

From (13), we can see that the proper speed increases as the radial distance decreases. Finally, we are ready to obtain the equation of motion by solving (13) through variable separation:

$$\frac{\mathrm{d}r}{\sqrt{2m/r - 2m/r_0}} = \mathrm{d}s\,, \tag{14}$$

$$s\sqrt{\frac{2m}{r_0}} = r_0 \operatorname{arctg}\sqrt{\frac{r}{r_0 - r}} - \sqrt{r(r_0 - r)}\,. \tag{15}$$

Unfortunately, expression (15) is a transcendental equation in r, so we cannot obtain r as a symbolic function of the proper time s. Yet, as we will see later in this paper, the information is very valuable in solving other classes of problems.

2. A different approach for radial motion

We can determine the proper and coordinate speed for radial motion with a slightly different approach. From (1)

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = \alpha^2\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \alpha\,, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{k^2 - \alpha}\,, \tag{16}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} = \frac{-\mathrm{d}\alpha/\mathrm{d}s}{2\sqrt{k^2-\alpha}} = \frac{-1}{2\sqrt{k^2-\alpha}}\frac{2m}{r^2}\frac{\mathrm{d}r}{\mathrm{d}s} = -\frac{m}{r^2}\,. \tag{17}$$

Using (1) and (3), the coordinate speed is

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \alpha^2 - \alpha\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2, \qquad \frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\alpha^2 - \alpha\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2} = \sqrt{\alpha^2 - \frac{\alpha^3}{k^2}}\,. \tag{18}$$

From (18), we get the coordinate acceleration

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right) = \frac{2\alpha - 3\alpha^2/k^2}{2\sqrt{\alpha^2 - \alpha^3/k^2}}\frac{\mathrm{d}\alpha}{\mathrm{d}s}\frac{\alpha}{k} = \frac{\alpha}{k}\frac{2\alpha - 3\alpha^2/k^2}{2(\alpha/k)\sqrt{k^2-\alpha}}\frac{2m}{r^2}\frac{\mathrm{d}r}{\mathrm{d}s}$$

$$= \frac{2\alpha - 3\alpha^2/k^2}{\sqrt{k^2-\alpha}}\frac{m}{r^2}\sqrt{k^2-\alpha} = \frac{m}{r^2}\alpha\left(2 - \frac{3\alpha}{k^2}\right). \tag{19}$$

k can be determined by setting the condition that the coordinate (or proper) speed is zero when the particle is dropped from radial distance r_0 towards the mass M,

$$0 = \frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{k^2 - \alpha(r_0)}\,, \qquad k = \sqrt{\alpha(r_0)} = \sqrt{1 - \frac{2m}{r_0}}\,, \tag{20}$$

$$0 = \frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\alpha^2(r_0) - \frac{\alpha^3(r_0)}{k^2}}\,, \qquad k = \sqrt{\alpha(r_0)}\,. \tag{21}$$

Given (21), the coordinate acceleration becomes

$$a = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{m}{r^2}\alpha\left(2 - \frac{3\alpha}{k^2}\right) = \frac{m}{r^2}\alpha\left(2 - \frac{3\alpha(r)}{\alpha(r_0)}\right) = -\frac{m}{r^2}\left(1 - \frac{2m}{r}\right)\left(3\frac{1-2m/r}{1-2m/r_0} - 2\right). \tag{22}$$

If the particle is dropped from infinity, (22) becomes

$$a = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{m}{r^2}\left(1 - \frac{2m}{r}\right)\left(1 - \frac{6m}{r}\right). \tag{22a}$$

The proper speed (16) is

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{k^2 - \alpha} = \sqrt{\alpha(r_0) - \alpha(r)} = \sqrt{\frac{2m}{r} - \frac{2m}{r_0}}. \tag{23}$$

We can see that we have re-derived expression (13) through the new method. Finally, the coordinate s peed (in units of $c = 1$) is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\alpha^2 - \frac{\alpha^3}{k^2}} = \alpha\sqrt{1 - \frac{\alpha(r)}{\alpha(r_0)}} = \left(1 - \frac{2m}{r}\right)\sqrt{1 - \frac{1-2m/r}{1-2m/r_0}}\,. \tag{24}$$

3. *Classical treatment of unidimensional radial motion*

The same problem, in Newtonian formulation, for the case of unidimensional radial motion reduces to the equation of motion:

$$m\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GMm}{r^2}\,. \tag{25}$$

It is interesting to note that GR and Newtonian mechanics produce exactly the same equation of motion. Equation (25) gives us the tool for determining when two bodies of radiuses r_1 and r_2 and masses M and m will collide after starting from rest at locations x_1 and respectively x_2 separated by the initial distance $D = x_1 - x_2$. We would need to solve the system of differential equations:

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = -\frac{GM}{(x_1 - x_2)^2}\,, \qquad \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = +\frac{Gm}{(x_1 - x_2)^2}\,, \tag{26}$$

with initial conditions

$$x_1(0) = D\,, \qquad x_2(0) = 0\,, \qquad \left.\frac{\mathrm{d}x_1}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}x_2}{\mathrm{d}t}\right|_{t=0} = 0\,, \tag{27}$$

and find out the time when $x_1 - x_2 = r_1 + r_2$ (i.e., when the two masses touch) by solving a transcendental equation in t. The system gets easily reduced to a single equation by subtracting the two equations:

$$\frac{\mathrm{d}^2(x_1 - x_2)}{\mathrm{d}t^2} = -\frac{G(M+m)}{(x_1 - x_2)^2}\,. \tag{28}$$

From (7), we know that equation (28) has the general solution:

$$t\sqrt{\frac{2G(M+m)}{D}} = D\,\mathrm{arctg}\sqrt{\frac{x_1 - x_2}{D - (x_1 - x_2)}} - \sqrt{(x_1 - x_2)\big(D - (x_1 - x_2)\big)}\,. \tag{29}$$

At the time of collision, $x_1 - x_2 = r_1 + r_2$, so

$$t = \frac{D^{3/2}}{\sqrt{2G(M+m)}}\left(\mathrm{arctg}\sqrt{\frac{r_1 + r_2}{D - (r_1 + r_2)}} - \sqrt{\frac{(r_1 + r_2)\big(D - (r_1 + r_2)\big)}{D}}\right). \tag{30}$$

Now, we can see that the transcendental Eq. (15) proved instrumental in finding the "time to collision" for the unidimensional classical problem.

4. *Generalization to arbitrary planar orbits*

In the case of arbitrary planar orbits characterized by constant θ (that is, $\mathrm{d}\theta = 0$), we start with the Schwarzschild metric,

$$\mathrm{d}s^2 = \alpha \mathrm{d}t^2 - \frac{1}{\alpha}\mathrm{d}r^2 - (r\mathrm{d}\varphi)^2. \tag{31}$$

The lagrangian associated with the metric (31) is:

$$L = \alpha\frac{\mathrm{d}t^2}{\mathrm{d}s^2} - \frac{1}{\alpha}\frac{\mathrm{d}r^2}{\mathrm{d}s^2} - r^2\frac{\mathrm{d}\varphi^2}{\mathrm{d}s^2}\,. \tag{32}$$

The lagrangian (32) is the generalization for the more particular lagrangian (2). Likewise, the generalization of the Euler-Lagrange equations is

$$-2\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot r}{\alpha}\right) - \dot t^2\frac{\mathrm{d}\alpha}{\mathrm{d}r} + \dot r^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{\alpha}\right) + 2r\dot\varphi^2 = -2\frac{\ddot r}{\alpha} + \left(\frac{\dot r^2}{\alpha^2} - \dot t^2\right)\frac{\mathrm{d}\alpha}{\mathrm{d}r} + 2r\dot\varphi^2 = 0, \tag{33}$$

$$\alpha\frac{\mathrm{d}t}{\mathrm{d}s} = k\,, \tag{34}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(r^2\dot\varphi) = 0\,, \qquad r^2\dot\varphi = h\,. \tag{35}$$

From the general equation of motion (33), we can obtain interesting particular cases.

a. For circular orbits, $r = R$, $\dot{r} = 0$, so

$$-\dot{t}^2 \frac{\mathrm{d}\alpha}{\mathrm{d}r} + 2r\dot{\varphi}^2 = 0\,, \tag{36}$$

meaning that

$$\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \frac{2m}{r^2} = 2r\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2\,, \tag{37}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \sqrt{\frac{m}{r^3}}\,. \tag{38}$$

Inserting (38) back into the metric (31), we obtain

$$\mathrm{d}s^2 = \left(1 - \frac{3m}{r}\right)\mathrm{d}t^2, \tag{39}$$

with the immediate consequence

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \sqrt{\frac{m}{r^3}}\frac{1}{\sqrt{1-3m/r}}\,. \tag{40}$$

Thus, we have recovered a well known equation of the mechanics describing circular orbits.

b. For radial orbits, $\mathrm{d}\varphi = 0$, so, the Euler-Lagrange Eq. (33) reduces to

$$-2\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{r}}{\alpha}\right) - \dot{t}^2\frac{\mathrm{d}\alpha}{\mathrm{d}r} + \dot{r}^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{\alpha}\right) = -2\frac{\ddot{r}}{\alpha} + \left(\frac{\dot{r}^2}{\alpha^2} - \dot{t}^2\right)\frac{\mathrm{d}\alpha}{\mathrm{d}r} = 0\,. \tag{41}$$

If we add to the above the metric (31), we obtain

$$\mathrm{d}s^2 = \alpha \mathrm{d}t^2 - \frac{1}{\alpha}\mathrm{d}r^2. \tag{42}$$

From (41) and (42), we obtain the equation of motion

$$-2\ddot{r} - \frac{2m}{r^2} = 0\,, \tag{43}$$

that is, we recovered Eq. (7).

c. For arbitrary planar orbits, the Euler-Lagrange equation is given by (33). Coupled with the general metric (31) the equation reduces to

$$\ddot{r} = -\frac{m}{r^2} + \left(1 - \frac{3m}{r}\right)r\dot{\varphi}^2. \tag{44}$$

The above is very interesting since it allows recovering the previous answers to both the radial and circular orbit situations. Indeed, $\dot{\varphi} = 0$ implies $\ddot{r} = -m/r^2$, and $\dot{r} = 0$ implies

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \sqrt{\frac{m}{r^3}} \frac{1}{\sqrt{1 - 3m/r}} \,.$$

Thus we have recovered Eq. (40).

5. *The derivation of the advancement of Mercury perihelion via perturbation theory*

In this paragraph we will combine the lagrangian approach with perturbation theory in producing a novel solution to the advancement of the perihelion of not only Mercury, but also of Venus, Earth and Mars. Using the Euler-Lagrange Eq. (35) Eq. (44) can be simplified to

$$\ddot{r} + \frac{m}{r^2} = \frac{h^2}{r^3} - 3m\frac{h^2}{r^4} \,. \tag{45}$$

Using the substitution

$$u(\varphi) = \frac{1}{r(\varphi)} \,, \tag{46}$$

with the immediate consequences:

$$\dot{r} = -\frac{\dot{u}}{u^2} = -r^2 \frac{\mathrm{d}u}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}s} = -h\frac{\mathrm{d}u}{\mathrm{d}\varphi} \,, \qquad \ddot{r} = -h\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2}\frac{\mathrm{d}\varphi}{\mathrm{d}s} = -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} \,. \tag{47}$$

Equation (45) transforms into

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{m}{h^2} + 3mu^2. \tag{48}$$

Equation (48) is nothing but the Kepler's first law from Newtonian mechanics,

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{m}{h^2} \,, \tag{49}$$

with the added relativistic perturbation of $+3mu^2$. Now, we know the solution for (49) is

$$u(\varphi) = \frac{m}{h^2}(1 - e\cos\varphi) \,, \tag{50}$$

or, expressed in terms of $r = r(\varphi)$

$$r(\varphi) = \frac{r_c}{1 - e\cos\varphi}\,, \tag{51}$$

where for $e < 1$ (51) represents the parametric equation of an ellipse in which case $r_c = h^2/m = r(\pi/2)$ is the radial distance from the focus to the ellipse. Armed with the solution for classical mechanics equation (49) we can now attempt to solve the GR equation (48) by applying perturbation theory. An appropriate solution is

$$u(\varphi) = \frac{m}{h^2}\big(1 - e\cos(\Omega\varphi)\big). \tag{52}$$

In order for (52) to be a solution for (48) it must satisfy the condition

$$\frac{3m^2}{h^2}\big(1 + e^2\cos^2(\Omega\varphi)\big) - \frac{6m^2}{h^2}e\cos(\Omega\varphi) = -e(1-\Omega^2)\cos(\Omega\varphi)\,. \tag{53}$$

Since

$$\frac{m^2}{h^2} = \frac{m}{h^2/m} = \frac{r_s/2}{r_c} \ll 1\,, \tag{54}$$

it follows that (53) is satisfied if

$$\frac{6m^2}{h^2} = 1 - \Omega^2, \tag{55}$$

that is,

$$\Omega = \sqrt{1 - \frac{6m^2}{h^2}} \approx 1 - \frac{3m^2}{h^2}\,. \tag{56}$$

Rindler [1] produced a similar explanation, but his derivation relies on a less rigorous series of multiple approximations. Thus, the solution for the GR Eq. (48) is

$$u(\varphi) = \frac{m}{h^2}\left[1 - e\cos\left(\left(1 - \frac{3m^2}{h^2}\right)\varphi\right)\right], \tag{57}$$

$$r(\varphi) = \frac{r_c}{1 - e\cos\big((1 - 3m^2/h^2)\varphi\big)}\,. \tag{58}$$

The solution (58) agrees with the Newtonian solution for $m = 0$, thus giving us a high level of confidence that it is correct.

When $0 < \varphi < 2\pi$, $0 < (1 - 3m^2/h^2)\varphi < 2\pi - 6\pi m^2/h^2$, that is the orbit "misses" its closure [2] by $6\pi m^2/h^2$ per revolution, resulting into a precession phenomenon seen in Fig. 1.

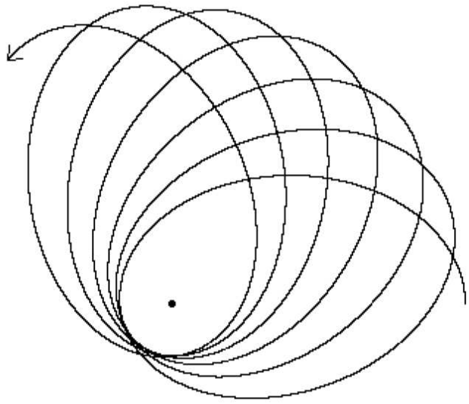

Fig. 1. Precession of orbits in gravitational field.

The precession per revolution is a direct function of the term $m^2/h^2 = r_s/(2r_c)$, while the overall observed precession per century is also a function of the number of revolutions per century. From Table 1, we can see that physics "conspires" in such a fashion that, for our solar system, Mercury is by far the best candidate for observing the precession given that it has not only the highest precession per revolution but also the largest number of revolutions per century.

TABLE 1. Perihelion precession of the inner solar planets.

Planet	r_c (10^6 km)	Precession per revolution	Revolutions per century	Precession per century (arcsec)
Mercury	55.443	0.1034	414.9378	42.9195
Venus	108.1947	0.0530	162.6016	8.6186
Earth	149.5568	0.03835	100	3.8335
Mars	225.9289	0.0254	53.1915	1.3502

Over the centuries, the astronomers have observed that the precession of Mercury perihelion is actually a much larger number (5600 arcsecond/century). Of the total 5600 arcsecond/century, 5557 can be accounted for by Newtonian mechanics, leaving the balance of 43 arcsecond/century to be explained by the disparity between the Newtonian equation (49) and its relativistic counterpart (48). It was Einstein [1] who explained the disparity between the Newtonian calculations and the observed values. Given the advancements in modern measuring devices, we can today not only account for the advancement of Mercury perihelion but also for the advancements for Venus and Earth [3–4].

6. *Application to calculating the length of a rod in radial fall*

Let's assume that we are asked to find the length of a rod of proper length L as calculated from the perspective of a distant Schwarzschild observer. Now, our observer has read reference [6] and understands that there are several ways of operationally determining the length of an object in motion. So, the observer decides to drop the rod from a distance r_0 and he decides to set a "trap" at location $r_1 < r_0$. By calculating the time interval Δt between the leading end of the rod and the trailing end of the rod passing through the "trap" set at r_1 and by knowing the coordinate speed at the same point, our observer can determine the length of the moving rod. We will assume throughout this chapter that the rod is Born-rigid, so it is not distorted by tidal forces, that is, all its points travel at the same speed. The coordinate speed is variable along the trajectory and, at location r_1, according to what we derived in Eq. (14), it is

$$v\big|_{r=r_1} = \left(1 - \frac{2m}{r_1}\right)\sqrt{1 - \frac{1-2m/r_1}{1-2m/r_0}}\,. \tag{59}$$

The time for the leading end of the rod to reach location r_1 is

$$t_{\mathrm{lead}} = \int\limits_{r_1}^{r_0} \mathrm{d}t\,, \tag{60}$$

while the time for the trailing end of the rod is

$$t_{\mathrm{trail}} = \int\limits_{r_1}^{r_0+L} \mathrm{d}t\,. \tag{61}$$

Thus, the elapsed time for the rod to pass through the "speed trap" at location r_1 is

$$\Delta t = \int\limits_{r_1}^{r_0+L} \mathrm{d}t - \int\limits_{r_1}^{r_0} \mathrm{d}t = \int\limits_{r_0}^{r_0+L} \mathrm{d}t\,. \tag{62}$$

From (14) we also know that

$$\mathrm{d}t = \frac{\mathrm{d}r}{\left(1 - \dfrac{2m}{r}\right)\sqrt{1 - \dfrac{1-2m/r}{1-2m/r_0}}}\,. \tag{63}$$

We are now ready to calculate the length of the rod, as it passes through r_1,

$$\Delta l = v\Delta t = \left(1 - \frac{2m}{r_1}\right)\sqrt{\frac{1}{r_1} - \frac{1}{r_0}} \int_{r_0}^{r_0+L} \frac{\mathrm{d}r}{(1-2m/r)\sqrt{1/r - 1/r_0}} \,. \tag{64}$$

The above represents one operational way of determining the length of the falling rod. Now, the astute observer, who has read Geroch's book [6], may decide to apply a different operational definition in determining the rod's length, such as marking both ends of the rod at the same coordinate time. So, the observer decides to find out where the trailing end of the rod is when the leading end has reached r_1. Assume that this is at the radial location r, where

$$t_{\mathrm{trail}} = \int_{r}^{r_0+L} \mathrm{d}t \,. \tag{65}$$

In this case, the coordinate length of the rod is $r - r_1$, where r is the solution of the integral equation

$$t_{\mathrm{trail}} = t_{\mathrm{lead}} \qquad \int_{r}^{r_0+L} \mathrm{d}t = \int_{r_1}^{r_0} \mathrm{d}t \,. \tag{66}$$

The above equation can be further simplified in two steps. Firstly, we reduce it to

$$\int_{r_1}^{r} \mathrm{d}t = \int_{r_0}^{r_0+L} \mathrm{d}t \,. \tag{67}$$

Now, the RHS is a constant, independent of r and the LHS is a polynomial in r. In the second step, we notice that for r, $r_1 \gg 2m$, we have

$$\frac{1}{(1-2m/r)\sqrt{2m/r - 2m/r_0}} \approx \left(1 + \frac{2m}{r}\right)\sqrt{\frac{r}{2m}} = \sqrt{\frac{r}{2m}} + \sqrt{\frac{2m}{r}} \,. \tag{68}$$

Thus, equation (67) reduces to a simple algebraic equation in r,

$$\sqrt{2mr} + \frac{r}{3}\sqrt{\frac{r}{2m}} = \sqrt{2mr_1} + \frac{r_1}{3}\sqrt{\frac{r_1}{2m}} - \sqrt{2mr_0}\frac{r_0}{3}\sqrt{\frac{r_0}{2m}} + \sqrt{2m(r_0+L)} + \frac{r_0+L}{3}\sqrt{\frac{r_0+L}{2m}} \,. \tag{69}$$

We presented just two different modes of determining the length of a moving rod, the reader can decide on his/her own operational way of determining the length since several more ways can be found in the literature [6].

7. *Charged black holes*

In (1916) Reissner [7] and in 1918 Nordström [8] derived independently the metric that represents the static solution to the Einstein field equations in empty space, which corresponds to the gravitational field of a charged, non-rotating, spherically symmetric body of mass M and charge Q. Finding the equations of motion for objects falling into or gravitating around a charged black hole is considerably more difficult than in the case of the Schwarzschild solution. We will show how to use the lagrangian approach in solving this problem and we will even solve the difficult problem of calculating the perihelion advancement for objects describing arbitrary orbits. The Reissner-Nordström metric is given by

$$\mathrm{d}s^2 = \alpha' \mathrm{d}t^2 - \frac{1}{\alpha'}\mathrm{d}r^2 - r^2 \mathrm{d}\varphi^2, \tag{70}$$

where

$$\alpha' = 1 - \frac{2m}{r} + \frac{r_Q^2}{r^2}\,, \tag{71}$$

and

$$r_Q^2 = \frac{GQ^2}{4\pi\epsilon_0 r^4}\,. \tag{72}$$

The Euler-Lagrange equations are

$$-2\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{r}}{\alpha'}\right) - \dot{t}^2\frac{\mathrm{d}\alpha'}{\mathrm{d}r} + \dot{r}^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{\alpha'}\right) + 2r\dot{\varphi}^2 = 0\,, \qquad \alpha\frac{\mathrm{d}t}{\mathrm{d}s} = k\,, \qquad r^2\dot{\varphi} = h\,. \tag{73}$$

d. For circular orbits we obtain:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \sqrt{\frac{m}{r^3} - \frac{r_Q^2}{r^4}}\,. \tag{74}$$

Inserting (74) back into the metric (70), we obtain

$$\mathrm{d}s^2 = \left(1 - \frac{3m}{r} + \frac{2r_Q}{r^2}\right)\mathrm{d}t^2, \tag{75}$$

with the immediate consequence

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \sqrt{\frac{m}{r^3} - \frac{r_Q^2}{r^4}}\sqrt{\frac{1}{\sqrt{1 - 3m/r + r_Q^2/r^2}}}\,. \tag{76}$$

Thus we obtained a very elegant result showing that circular orbits for charged black holes can be obtained by applying a charge-dependent correction to the solution for neutral black holes.

e. For radial orbits, the Euler-Lagrange equation reduces to

$$-2\frac{\ddot{r}}{\alpha'}+\left(\frac{\dot{r}^2}{\alpha'^2}-\dot{t}^2\right)\frac{\mathrm{d}\alpha'}{\mathrm{d}r}=0\,. \tag{77}$$

We add to the above the fact that the metric (70) reduces to

$$\mathrm{d}s^2=\alpha'\mathrm{d}t^2-\frac{1}{\alpha'}\mathrm{d}r^2. \tag{78}$$

From (41) and (42), we obtain the equation of motion

$$-2\ddot{r}-\frac{\mathrm{d}a'}{\mathrm{d}r}=-2\ddot{r}-\frac{2m}{r^2}+\frac{2r_Q^2}{r^3}=0\,, \tag{79}$$

that is, we recovered Eq. (7). with the charge-related perturbation $2r_Q^2/r^3$.

f. For arbitrary planar orbits, the equation reduces to

$$\ddot{r}=-\frac{m}{r^2}+\frac{r_Q^2}{r^3}+\left(1-\frac{3m}{r}+\frac{2r_Q^2}{r^3}\right)r\dot{\varphi}^2. \tag{80}$$

The computation for the advancement of the perihelion becomes more complicated since the starting point is now the equation

$$\ddot{r}+\frac{m}{r^2}-\frac{r_Q^2}{r^3}=\frac{h^2}{r^3}-3m\frac{h^2}{r^4}+\frac{2h^2r_Q^2}{r^5}\,. \tag{81}$$

Using again the substitution $u(\varphi)=1/r(\varphi)$ and neglecting the term in r^5, we obtain

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2}+u=\frac{m}{h^2}+3mu^2-\frac{r_Q^2}{h^2}u\,, \tag{82}$$

an equation very similar to (48). We can proceed by considering the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2}+u=\frac{m}{h^2}\,, \tag{83}$$

with the perturbation $3mu^2-r_Q^2/h^2u$. As an alternative, we can start from

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2}+\left(1+\frac{r_Q^2}{h^2}\right)u=\frac{m}{h^2}\,, \tag{84}$$

with the known solution

$$u(\varphi)=\frac{1}{\sqrt{1+r_Q^2/h^2}}\frac{m}{h^2}+e\cos\left(\varphi\sqrt{1+\frac{r_Q^2}{h^2}}\right)\,, \tag{85}$$

and to apply the perturbation $3mu^2$. Either way, the perturbation approach that we developed in (52) – (56) bears fruit since the computation of the perihelion advancement becomes a simple algebraic exercise.

8. Light bending by charged black holes

Light bending can be calculated starting from the fact that the light path is null,

$$0 = \alpha \mathrm{d}t^2 - \frac{1}{\alpha}\mathrm{d}r^2 - r^2\mathrm{d}\varphi^2. \tag{86}$$

So

$$\mathrm{d}r^2 = \alpha^2\mathrm{d}t^2 - \alpha r^2\mathrm{d}\varphi^2. \tag{87}$$

To the above we add the Euler-Lagrange equations (34) and (35),

$$\alpha\dot{t} = k\,, \tag{88}$$

$$r^2\dot{\varphi} = h\,. \tag{89}$$

Combining (85) with (86) and (87), we obtain immediately

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = k^2 - h^2\frac{\alpha}{r^2}\,. \tag{90}$$

Differentiating Eq. (88) with respect to r, we obtain a simpler equation,

$$2\frac{\mathrm{d}^2r}{\mathrm{d}s^2} = \frac{h^2}{r^3}\left(2\alpha - r\frac{\mathrm{d}\alpha}{\mathrm{d}r}\right). \tag{91}$$

For the case of uncharged black holes, $\alpha = 1 - 2m/r$, and using the notation $u(\varphi) = 1/r(\varphi)$, Eq. (89) reduces to [1]

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = 3mu^2. \tag{92}$$

For the case of charged black holes, $\alpha = 1 - 2m/r + r_Q^2/r^2$, and the equation becomes

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = 3mu^2 - 2r_Q^2u^3. \tag{93}$$

The solution of Eq. (91) is the superposition of the solutions of Eq. (90) and the solution of

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = -2r_Q^2u^3. \tag{94}$$

The solution of Eq. (90) is

$$u = C \sin\varphi + \frac{3mC^2}{2}\left(1 + \frac{\cos 2\varphi}{3}\right), \tag{95}$$

where $C = 1/R$ and R is the effective radius. The solution of Eq. (92) is

$$u = -\frac{3A}{8}\varphi\cos\varphi + \frac{A}{32}\sin 3\varphi\,, \tag{96}$$

where $A = -2r_Q^2C^3$. When $r \to \infty$, $u \to 0$ and $\varphi \to \varphi_\infty$, so

$$0 = C\varphi_\infty + 2mC^2 + \frac{9r_Q^2C^3}{16}\varphi_\infty\,, \tag{97}$$

resulting in

$$\varphi_\infty \approx -\frac{2m}{R}\left(1 - \frac{9r_Q^2}{16R^2}\right). \tag{98}$$

The total deflection angle is

$$\vartheta = 2|\varphi_\infty| \approx \frac{4m}{R}\left(1 - \frac{9r_Q^2}{16R^2}\right). \tag{99}$$

Comparing (97) with the deflection by an uncharged black hole [1], we can conclude that the charge contributes an additional effect of $-\dfrac{4m}{R}\dfrac{9r_Q^2}{16R^2}$.

9. Conclusion

We have shown the derivation of the equations of motion from the Schwarzschild metric via the Euler-Lagrange formalism. In the process, we have fully developed the equations describing radial motion, a subject much less developed in literature than the orbital motion. In the case of orbital motion, we have produced a derivation that is more rigorous and which entails fewer approximations than the one that can be found in Rindler [1]. While the pedagogical approach was constructed around the case of gravitational fields described by the Schwarzschild metric, it became easy to extend the algorithms to other, more difficult metrics, like the Reissner-Nordström metric.

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EULER-LAGRANGEOVO RJEŠAVANJE STAZA ČESTICA U GRAVITACIJSKOM POLJU

Izvođenje jednadžbi gibanja čestica u gravitacijskom polju u općoj teoriji relativnosti obično se zasniva na kovarijantnim derivacijama. Kako su geodezijske jednadžbe zasnovane na kovarijantnim derivacijama i prvotno se izvode iz Euler-Lagrangeovih jednadžbi, i budući da je Euler-Lagrangeov formalizam vrlo poimljiv, lako izvodljiv s malo pogrešaka, mnogo je razloga da se rabi pa i u najzamršenijim zadacima. U ovom se radu primjenjuju Lagrangeove jednadžbe u nizu zadaća iz opće relativnosti. Poseban odjeljak posvećen je radijalnom gibanju. U udžbenicima se tome posvećuje malo pažnje, vjerojatno s toga što je rješavanje radijalnih jednadžbi gibanja teže nego za gibanja u stazi.