Printed ISSN 1330–0008
Online ISSN 1333–9125
CD ISSN 1333–8390
CODEN FIZAE4

A COMBINATION METHOD FOR CONSTRUCTING THE EXPLICIT EXACT SOLUTIONS OF THE GENERALIZED BBM-BURGERS EQUATION

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Received 23 February 2009; Revised manuscript received 12 April 2011
Accepted 9 February 2011 Online 2 May 2011

In view of the analysis of characteristics of the generalized Burgers equation, generalized BBM equation and generalized BBM-Burgers equation, a combination method is developed to construct the explicit exact solutions of the generalized BBM-Burgers equation by combining the solutions of the generalized Burgers equation and the generalized BBM equation. As a result, many explicit exact solutions of the generalized BBM-Burgers equation are successfully derived.

PACS numbers: 02.30.Jr, 03.65.Ge UDC 532.59

Keywords: combination method, generalized BBM-Burgers equation, explicit exact solution

1. Introduction

Construction of the explicit exact solutions of nonlinear evolution equations (NEEs), by using different methods, is the goal for many researchers. Due to the complexity of nonlinear systems, finding the explicit exact solutions for a real nonlinear physical model equation is often difficult. Fortunately, a vast variety of powerful methods for seeking the explicit exact solutions of NEEs have been proposed and developed. Among them are the hyperbolic tangent function expansion method [1−5], the homogeneous balance method [6, 7], the trial function method [8−12], the auxiliary equation method [13−16], and so on. Nevertheless, no unified method exists for solving NEEs. As a consequence, it is still a very challenging task to explore more powerful and efficient methods to solve NEEs.

In the present paper, by analyzing the features of the generalized Burgers equation, generalized BBM equation and generalized BBM-Burgers equation, we present a combination method to construct the explicit exact solutions of the generalized

BBM-Burgers equation from those of the generalized Burgers equation and generalized BBM equation. By applying the method, we obtain many explicit exact solutions of the generalized BBM-Burgers equation.

2. *Solutions to the generalized BBM-Burgers equation by linear combination*

The generalized BBM-Burgers equation reads

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^2 u}{\partial x \partial t} = 0\,, \tag{1}$$

where α and β are arbitrary real constants with $\alpha\beta \neq 0$.

Apparently, the generalized BBM-Burgers equation (1) can be thought of as the combination of the following generalized Burgers equation and generalized BBM equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} = 0\,, \tag{2}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^2 u}{\partial x \partial t} = 0\,. \tag{3}$$

By observing the above three equations, it is readily seen that the generalized BBM-Burgers equation (1), the generalized Burgers equation (2) and the generalized BBM equation (3) are related: These three equations are all of the same nonlinear term $\left(u\frac{\partial u}{\partial x}\right)$, whereas the linear terms of Eq. (1) $\left(\alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^2 u}{\partial x \partial t}\right)$ are equal to the sum of those of Eq. (2) and Eq. (3). Therefore, we can construct the solutions of the generalized BBM-Burgers Eq. (1) by the linear combination of the solutions to the generalized Burgers Eq. (2) and generalized BBM Eq. (3). That is to say, we may suppose that Eq. (1) has the following ansatz solution

$$u = a_0 + a_1 u_{\mathrm{B}} + a_2 u_{\mathrm{BM}}\,, \tag{4}$$

where a_0, a_1 and a_2 are constants to be determined later, while u_{B} is the solution of Eq. (2) and u_{BM} the solution of Eq. (3). We call Eq. (4) the combination formula to find the solutions of the generalized BBM-Burgers equation (1). In what follows, let us explore the explicit exact solutions of Eq. (1) by the combination formula (4).

From Ref. [17], we are told that the generalized Burgers equation (2) admits the following solution

$$u_{\mathrm{B}} = 2\alpha k \tanh(k\xi) + c - 1\,, \tag{5}$$

and that the generalized BBM equation (3) admits the following solution

$$u_{\mathrm{BM}} = -12c\beta k^2 \mathrm{sech}^2(k\xi) + c - 1 + 4c\beta k^2\,, \tag{6}$$

where

$$\xi = x - ct\,. \tag{7}$$

Based on Eq. (5) and Eq. (6) and using Eq. (4), we assume that Eq. (1) has the following ansatz solution

$$\begin{aligned} u &= a_0 + a_1(2\alpha k \tanh(k\xi) + c - 1) + a_2(-12c\beta k^2 \mathrm{sech}^2(k\xi) + c - 1 + 4c\beta k^2) \\ &= b_0 + b_1 \tanh(k\xi) + b_2 \tanh^2(k\xi)\,, \end{aligned} \tag{8}$$

where

$$b_0 = a_0 + a_1(c-1) + a_2(c-1) - 8a_2c\beta k^2, \quad b_1 = 2a_1\alpha k \quad \text{and} \quad b_2 = 12a_2c\beta k^2.$$

Substituting Eq. (8) into Eq. (1) engenders the following set of algebraic equations for the unknown constants b_0, b_1 and b_2,

$$b_1k - b_1ck + 2b_1ck^3\beta + b_0b_1k + 2b_2k^2\alpha = 0\,, \tag{9}$$

$$-2b_1k^2\alpha + b_1^2k + 2b_2k - 2b_2ck + 16b_2k^3\beta + 2b_0b_2k = 0\,, \tag{10}$$

$$-b_1k + b_1ck - 8b_1ck^3\beta - b_0b_1k - 8b_2k^2\alpha + 3b_1b_2k = 0\,, \tag{11}$$

$$2b_1k^2\alpha - b_1^2k - 2b_2k + 2b_2ck - 40b_2k^3\beta - 2b_0b_2k + 2b_2^2k = 0\,, \tag{12}$$

$$6b_1ck^3\beta + 6b_2k^2\alpha - 3b_1b_2k = 0\,, \tag{13}$$

$$24b_2k^3\beta - 2b_2^2k = 0\,. \tag{14}$$

Solving the above set of algebraic equations with the aid of Mathematica, we get the following results:

$$b_0 = c - 1 - 12c\beta k^2, \qquad b_1 = \frac{12\alpha k}{5}, \qquad b_2 = 12c\beta k^2, \qquad k = \pm\frac{\alpha}{10c\beta}\,. \tag{15}$$

Putting Eq. (15) into Eq. (8), we obtain the following solution of the generalized BBM-Burgers equation (1)

$$u = c - 1 - 12c\beta k^2 + \frac{12\alpha k}{5}\tanh(k\xi) + 12c\beta k^2\tanh^2(k\xi)\,. \tag{16}$$

From Ref. [17], we are also told that the generalized Burgers equation (2) has the solution $2\alpha k \coth(k\xi) + c - 1$ and that the generalized BBM equation (3) has the solution $12c\beta k^2 \mathrm{csch}^2(k\xi) + c - 1 + 4c\beta k^2$.

Similarly, we assume that Eq. (1) has the following ansatz solution

$$u = a_0 + a_1(2\alpha k \coth(k\xi) + c - 1) + a_2(12c\beta k^2 \mathrm{csch}^2(k\xi) + c - 1 + 4c\beta k^2)$$

$$= b_0 + b_1 \coth(k\xi) + b_2 \coth^2(k\xi)\,, \tag{17}$$

where

$$b_0 = a_0 + a_1(c-1) + a_2(c-1) - 8a_2 c\beta k^2, \qquad b_1 = 2a_1\alpha k, \qquad b_2 = 12a_2 c\beta k^2.$$

Utilizing the same procedure as before, we find that the generalized BBM-Burgers equation (1) has the following solution

$$u = c - 1 - 12c\beta k^2 + \frac{12\alpha k}{5}\coth(k\xi) + 12c\beta k^2 \coth^2(k\xi)\,. \tag{18}$$

From Ref. [17], we are still told that the generalized Burgers equation (2) admits the solution $-2\alpha k \tan(k\xi) + c - 1$ and that the generalized BBM equation (3) admits the solution $12c\beta k^2 \sec^2(k\xi) + c - 1 - 4c\beta k^2$.

Similarly, we assume that Eq. (1) has the following ansatz solution

$$u = a_0 + a_1(-2\alpha k \tan(k\xi) + c - 1) + a_2(12c\beta k^2 \sec^2(k\xi) + c - 1 - 4c\beta k^2)$$

$$= b_0 + b_1 \tan(k\xi) + b_2 \tan^2(k\xi)\,, \tag{19}$$

where

$$b_0 = a_0 + a_1(c-1) + a_2(c-1) + 8a_2 c\beta k^2, \qquad b_1 = -2a_1\alpha k, \qquad b_2 = 12a_2 c\beta k^2.$$

Employing the same procedure as before, we find that the generalized BBM-Burgers equation (1) has the following solution

$$u = c - 1 + 12c\beta k^2 - \frac{12\alpha k}{5}\tan(k\xi) + 12c\beta k^2 \tan^2(k\xi)\,, \tag{20}$$

where k satisfies the constraint condition $k = \pm\sqrt{-1}\;\alpha/(10c\beta)$.

From Ref. [17], we are still told that the generalized Burgers equation (2) admits the solution $2\alpha k \cot(k\xi) + c - 1$ and that the generalized BBM equation (3) admits the solution $12c\beta k^2 \csc^2(k\xi) + c - 1 - 4c\beta k^2$.

Similarly, we assume that Eq. (1) has the following ansatz solution

$$u = a_0 + a_1(2\alpha k \cot(k\xi) + c - 1) + a_2(12c\beta k^2 \csc^2(k\xi) + c - 1 - 4c\beta k^2)$$

$$= b_0 + b_1 \cot(k\xi) + b_2 \cot^2(k\xi)\,, \tag{21}$$

where

$$b_0 = a_0 + a_1(c-1) + a_2(c-1) + 8a_2c\beta k^2, \qquad b_1 = 2a_1\alpha k, \qquad b_2 = 12a_2c\beta k^2.$$

Using the same procedure as before, we find that the generalized BBM-Burgers equation (1) has the following solution

$$u = c - 1 + 12c\beta k^2 + \frac{12\alpha k}{5}\cot(k\xi) + 12c\beta k^2\cot^2(k\xi)\,, \tag{22}$$

where k satisfies the constraint condition $k = \pm\sqrt{-1}\ \alpha/(10c\beta)$.

Similarly, in view of other solutions of the generalized Burgers equation and generalized BBM equation, we may construct other solutions, even new ones, of the generalized BBM-Burgers equation by the combination formula (4). Here we omit them for the reason of brevity.

3. *Conclusions and remarks*

In summary, by analyzing the features of the generalized Burgers equation, generalized BBM equation and generalized BBM-Burgers equation, we put forward a combination method to construct the explicit exact solutions of the generalized BBM-Burgers equation from those of the generalized Burgers equation and generalized BBM equation. By employing the method, we obtain many explicit exact solutions to the generalized BBM-Burgers equation. The key idea of our approach is to construct solutions of NEEs by combining solutions to two or more basic distinct NEEs. Therefore, the problem for solving nonlinear partial differential equations is significantly simplified. Although it does not seem that our approach is reasonable in theory, it is efficient and feasible in practice for solving this kind of NEEs. More importantly, our approach provides a novel line that one may construct solutions of some unknown NEEs by using those of some known NEEs, as is very important and significant to scientific study. The topic discussed by us may be interesting and fascinating in the solution theory of nonlinear partial differential equations, and the results obtained by us could supplement to the existing literature as new examples of exact solutions for the generalized BBM-Burgers equation. Furthermore, we are convinced that our technique may be used to search for the explicit exact solutions of other NEEs with the similar features stated above. We leave this to future investigation.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant No. 10672053).

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METODA SLAGANJA ZA NALAŽENJE EKSPLICITIH TOČNIH RJEŠENJA POOPĆENE BBM-BURGERSOVE JEDNADŽBE

Na osnovi analiza značajki poopćene Burgersove jednadžbe, poopćene BBM jednadžbe i poopćene Burgers-BBM jednadžbe, razvijamo metodu slaganja za nalaženje eksplicitnih i točnih rješenja Burgers-BBM jednadžbe slaganjem rješenja poopćene Burgersove jednadžbe i poopćene BBM jednadžbe. Postižemo mnoga eksplicitna i točna rješenja poopćene Burgers-BBM jednadžbe.