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PARTIAL ALGEBRAIZATION AND CUT-OFF COULOMB POTENTIAL

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We have applied partial algebraization technique to the cut-off Coulomb potential $-Ze^2/(r+\beta)$. It has been found that for a spin j representation $2j+1$ exact solutions are obtained but they belong to excited states of potentials for different values of β. Degeneracies observed in the spectrum have been compared with exact numerical results.

1. Introduction

Recently the partial algebraic technique has been applied to a class of problems which are partially solvable[1−3]. (For a recent exposition see Karman and Oliver[4]). Relation between this technique and that of supersymmetric quantum mechanics has also been shown by Roy and Roychoudhury[5]. Attempts has been made to apply partial algebraization[6] to the nonpolynomial potential

$$V(x)=\frac{x^2}{1+gx^2}\,.$$

However this technique does not give any non-trivial results for the non-polynomial potential. This motivated us to see whether another important nonlinear type potential viz. the cut-off Coulomb potential $-Ze^2/(r+\beta)$ can be treated by this technique. We show that a hidden SU(2) symmetry exists for this potential and though the general formula for E_n for any n can be found explicitly independent of β, the restriction on β itself is such that for each β only one solution is obtained. However, we found that there is a Coulomb like degeneracy in those solutions by having recourse to numerical analysis.

2. Results

Before we cast the cut-off Coulomb potential problem in SU(2) symmetric form we briefly describe the method of partial algebraization. Given a Schrödinger equation

$$H\psi = E\psi\,, \tag{1}$$

we perform an imaginary gauge transformation on the wave function $\psi(x)$[7]

$$\psi(x) \to \psi(x)\mathrm{e}^{-f(x)}\,, \tag{2}$$

then

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\,, \tag{3}$$

$$H_G = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + A(x)\frac{\mathrm{d}}{\mathrm{d}x} + \Delta V\,, \tag{4}$$

where

$$\Delta V = V(x) + \frac{1}{2}A'(x) - \frac{1}{2}A^2(x)\,, \tag{5}$$

while

$$f(x) = \int A(x)\mathrm{d}x\,. \tag{6}$$

The gauge transformed eigenvalue equation reads

$$H_G\widetilde{\psi}(x) = E\widetilde{\psi}(x) \tag{7}$$

Next we consider a finite dimensional representation of the SU(2) group with spins. The generators of the group are

$$T^+ = 2j\xi - \xi^2\frac{\mathrm{d}}{\mathrm{d}\xi}$$

$$T^0 = -j\xi - \xi^2 \frac{\mathrm{d}}{\mathrm{d}\xi} \tag{8}$$

$$T^- = \frac{\mathrm{d}}{\mathrm{d}\xi}$$

The corresponding finite dimensional representation is

$$R^j = (1,\, \xi,\, \xi^2, \ldots,\, \xi^{2j})\,. \tag{9}$$

We choose the gauge such a way that H_G can be written as

$$H_G = \sum_{a,b\pm 0} C_{ab} T^a T^b + \sum_{a,b\pm 0} C_a T^a + \text{constant}, \tag{10}$$

where C_{ab} and C_a are numerical coefficients. Using (8), (10) can be written as

$$H_G = -\frac{1}{2} P_4(\xi) \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + P_3(\xi) \frac{\mathrm{d}}{\mathrm{d}\xi} + P_2(\xi)\,, \tag{11}$$

where $P_n(\xi)$ denotes at most a polynomial of degree n.

To bring (11) in Schrödinger like form we put

$$x = \int \mathrm{d}\xi P_4^{-1/2}(\xi) = F(\xi), \text{ say}, \tag{12}$$

then

$$H_G = -\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{P_4' + 4P_3}{4P_4^{1/2}} \frac{\mathrm{d}}{\mathrm{d}x} + P_2\,. \tag{13}$$

Now the basis can be chosen as

$$\{\bar{\psi}\} + (1,\, \xi,\, \xi^2, \ldots,\, \xi^{2j},\, \bar{\psi}_{2j+2},\, \bar{\psi}_{2j+3},\, \ldots) \tag{14}$$

where

$$\xi = F^{-1}(x)\,, \tag{15}$$

and $\bar{\psi}_{2j+k}$ is an arbitrary set of functions orthogonal to $(1, \xi, \ldots, \xi^{2j})$ with weight $\exp(-2f(x))$. Then H has the form

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \tag{16}$$

where H_1 is finite matrix and can be diagonalized.

With these tools we now attempt to solve the cut-off Coulomb problem[8)] for which the Schrödinger equation is ($\hbar = m = c = 1$)

$$-\frac{1}{2}\phi''(r) - \frac{1}{2}\phi'(r) + V_E(r)\phi(r) = E\phi(r) \tag{17}$$

where we have written the radial part of the Schrödinger equation and

$$V_E = V(r) + \frac{l(l+1)}{2r^2} = -\frac{Ze^2}{r+\beta} + \frac{l(l+1)}{2r^2}\,. \tag{18}$$

(Henceforth we shall take $Ze^2 = 1$). Putting $\psi = r^{-1}\varphi(r)$, equation (16) reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\phi + (2E - 2V_E(r))\phi(r) = 0\,. \tag{19}$$

Now we choose the gauge $f(r)$ as (here we take $\xi = r$)

$$f(r) = ar - (l+1)\log r - \log(r+\beta)\,. \tag{20}$$

$A(r)$ is given by [see equation (5)]

$$A(r) = \frac{\mathrm{d}}{\mathrm{d}r}f(x) = a - \frac{l+1}{r} - \frac{1}{r+\beta} \tag{21}$$

$$\begin{aligned}\Delta V(r) &= V_E(r) + \frac{l+1}{r} - \frac{1}{2}\left(a - \frac{l+1}{r} - \frac{1}{r+\beta}\right)^2 + \frac{1}{2(r+\beta)^2}\\ &= V(r) + \frac{l+1}{r} - a - \frac{a^2}{2} + \left(a + \frac{l+1}{\beta}\right)\frac{1}{2(r+\beta)^2} - \frac{l+1}{r\beta}\end{aligned} \tag{22}$$

and finally

$$H_G = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \left(a - \frac{l+1}{r} - \frac{1}{r+\beta}\right)\frac{\mathrm{d}}{\mathrm{d}r} + \Delta V(r)\,. \tag{23}$$

Here we make a slight improvisation of the partial algebraization method following Ref. 9

$$\Omega_G = 2r(r+\beta)(E - H_g)\,, \tag{24}$$

then

$$\begin{aligned}\Omega_G = (r^2 + r\beta)\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \big(2ar^2 + 2ar\beta - 2r(l+1) - 2r - 2\beta(l+1)\big)\frac{\mathrm{d}}{\mathrm{d}r}\\ +r(2 - 2al - 4a) - 2a\beta(l+1) + 2(l+1)\,,\end{aligned} \tag{25a}$$

where we have taken

$$a^2 = -2E\,, \tag{25b}$$

to remove r^2 term from Ω_G. The eigenvalue equation now reads

$$\Omega_G \widetilde{\psi} = 0, \quad \text{where} \quad \widetilde{\psi} = \psi \mathrm{e}^{f(r)} \tag{25c}$$

We immediately see that Ω_G can be expressed in the form given by the r.h.s. of (9). Explicitly we write,

$$\Omega_G = AT_0^2 + DT^-T^0 + FT^-T^+ + GT^+ + H_cT^- + IT^0$$

$$\equiv (A-F)r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2} + Dr\frac{\mathrm{d}^2}{\mathrm{d}r^2} - Gr^2\frac{\mathrm{d}}{\mathrm{d}r} \tag{26}$$

$$+(A - 2jA + 2jF - 2F + I)r\frac{\mathrm{d}}{\mathrm{d}r} + (H_c - jD + D)\frac{\mathrm{d}}{\mathrm{d}r}$$

$$+2jGr + Aj^2 + 2jF - 2jGr - Ij \tag{27}$$

(H_c has nothing to do with H, the Hamiltonian).

Comparing with (25a) and equating the coefficients of $r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}$, $r^2\frac{\mathrm{d}}{\mathrm{d}r}$ etc., separately, one gets 7 simultaneous equations in A, F, G etc. Solving them we get (for $j \neq 0$)

$$A = \frac{2j}{j+1} + \frac{2(l+1)(1-a\beta)}{j(j+1)} - \frac{2a\beta - 2(l+2)}{j+1} \tag{28}$$

$$D = \beta \tag{29}$$

$$G = 2a = \frac{2}{2j+l+2} \tag{30}$$

$$H_c = \frac{\beta}{2l+j+1} \tag{31}$$

$$F = \frac{j-1}{j+1} + \frac{2(l+1)(1-a\beta}{j(j+1)} - \frac{2a\beta - 2(l+2)}{j+1)} \tag{32}$$

$$I = \frac{2j^+ 2j - 2}{j+1} + \frac{2(l+1)(1-a\beta)}{j(j+1)} - \frac{2(j+2)(a\beta - 1 - 2)}{j+1} \tag{33}$$

From (22b) and (30),

$$-E = \frac{1}{2(2j + l + 2)^2} \tag{34}$$

Though E is apparently independent of β, it depends on β through l. Because, as will be shown below, if we fix l, β is also fixed. Below we discuss several cases.

1. $j = 0$

For this case solutions (28), (32) and (33) will be inconsistent unless

$$a\beta = 1 \tag{35}$$

or

$$\beta = \frac{1}{a} = l + 2 \quad \text{from Eq. (30))}$$

As expected there is only one solution for a fixed l which is given by

$$\widetilde{\psi} = C_0 \tag{36}$$

(C_0 being a constant), or

$$\psi = C_0 r^{l+1}(r + \beta)\mathrm{e}^{-r/\beta} \tag{37}$$

and

$$E = -\frac{a^2}{2} = -\frac{1}{2(l + 2)^2} \tag{38}$$

2. $j = \frac{1}{2}$

Here β can not be determined from the set of equations (28) to (34). However for $i = 1/2$, $\tilde{\psi}$ will be at most a linear function of r and we follow the method of Ref. 7 to determine $\tilde{\psi}$. We write

$$\widetilde{\psi} = C_1(r - r_0)\,. \tag{39}$$

To find r_0 we put (39) in (25c) and equate powers of r^2, r etc. to zero separately. After some elementary algebra we get

$$a = \frac{1}{l + 3} \tag{40}$$

$$r_0 = (a - 1/\beta)^{-1} \tag{41}$$

and

$$(l + 2)a^2\beta^2 - 3a\beta(l + 2) + (2l + 3) = 0\,. \tag{42}$$

From (42)

$$a\beta = \frac{3(l+2) \pm (9(l+2)^2 - 4(2l+3)(l+2))^{1/2}}{2(l+2)} . \tag{43}$$

Hence for same l we get two different values of β. However, since r_0 will have two different signs for these different values of β, the eigenvalue will be of ground state

Table 1.

l	β	E_0 (ground state)	E_1^*
0	2.0	$-.1230000$	$-.0544741$
1	3.0	$-.0555556$	$-.0310546$
2	4.0	$-.0312500$	$-.0199477$
3	5.0	$-.0200000$	$-.0138710$
4	6.0	$-.0138881$	$-.0101969$
5	7.0	$-.0102041$	$-.0078093$
6	8.0	$-.0078125$	$-.0061712$
7	9.0	$-.0061728$	$-.0049992$

*Numerical results.

Table 2.

l	β	r_0	E_1^*	E_1	E_2^*
0	1.902	5.1762		$-.05551$	
	(7.098)	(-5.1962)			
1	2.945	11.16515	$-.056013$	$-.03125$	$-.01996$
	(9.0551)	(-7.1652)			
2	3.964	19.1421	$-.03125$	$-.02000$	$-.013873$
	(11.0355)	(-9.1421)			
3	4.9751	29.1246	$-.02003$	$-.01388$	$-.01020$
	(13.0241)	(-11.1246)			
4	5.9815	41.11048	$-.01390$	$-.0102$	$-.00781$
	(15.0185)	(-13.11088)			
5	6.9857	55.0999	$-.01021$	$-.00781$	$-.00617$
	(17.01447)	(-15.0997)			

*Numerical results.

for a particular potential and will be the excited state for another potential. Hence SU(2) symmetry does not give two states for the same potential contrary to the claim made in the original papers on partial algebraization[1−3]. Table 2 gives the eigenvalues obtained from the present work together with the numerical values for E_0 and E_2 states (the numbers within brackets are for negative values of r_0). It may be noted that our exact solutions for $j = 0$ and $j = 1/2$ completely agree with those obtained from supersymmetric quantum mechanics technique[19] which again supports the result of Ref. 4 for $N = 0$ and $N = 1$, respectively. However for $N > 1$ our technique can be carried on to higher values of j. We give below the results for $j = 1$.

3. $j = 1$
Here

$$E = -\frac{1}{(l+4)^2}$$

We take $\psi(r) = (r - r_0)(r - r_1)$.

We apply the method of Ref. 9 to determine r_0 and r_1. Elimination of r_0 and r_1 gives a cubic equation in β. Though in principle the equation could be solved analytically, we present below only a set of numerical solutions. The signs of r_0 and r_1 show that the energy eigenvalues are those of the first excited states.

In Table 3 energy eigenvalues for $l = 0$ to $l = 3$ are presented for the case $j = 1$ for a particular set of β values. Similar analysis can produce exact results for the case $j = \frac{3}{2}, 2$, etc., but then β has to be solved numerically.

Table 3.

l	β	r_0	r_1	Energy
0	6.6108	8.9069	−4.7372	−0.625
1	8.7617	16.8375	−6.8875	−.04
2	10.8364	26.7687	−8.9492	−.02777
3	12.8745	38.7024	−10.9820	−.020408

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DJELOMIČNA ALGEBRIZACIJA I ODREZANI COULOMBOV POTENCIJAL

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Primijenili smo tehniku djelomične algebrizacije na odrezani Coulombov potencijal $-Ze^2/(r+\beta)$. Nađeno je da se za spin j dobiva $2j+1$ točnih rješenja, koja pripadaju pobuđenim stanjima potencijala za različite vrijednosti β. Opažene degeneracije u spektru uspoređene su s točnim numeričkim rezultatima.