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SEMICLASSICAL THEORY OF SPONTANEOUS EMISSION OF RADIATION

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A nonrelativistic semiclassical theory of the radiation reaction interaction is developed. It is applied to the spontaneous emission of radiation from excited states of a hydrogen-like atom.

1. Introduction

One of the unresolved problems in quantum theory is the description of spontaneous emission of radiation from excited states using the semiclassical theory of radiation. At the roots of the problem is the difficulty of formulating the radiation reaction interaction, because it is responsible for the energy exchange between the particle and the electromagnetic (EM) field. In this work we would like to show a solution of this problem, i.e. how to incorporate the interaction in the nonrelativistic quantum equation. We test the idea on one example. The attempts along this line are not new. However, their aim was primarily to resolve the problem empirically [1-3]. The radiation reaction interaction causes the system to loose energy. Therefore, one only needs to incorporate the nonconservative forces into the quantum equations of motion. Although the approach is legitimate, the outcome is usually a result that has only a qualitative value.

The radiation reaction interaction is derived from the first principles in our approach. However, formulation of the interaction is based on modelling the system which consists of the EM field and charge. The main idea is to note that the probability density and the probability current of a particle play the role of the charge density and the charge current, however, only to a limited extent. These two quantities can be used for the calculation of the EM field, but if this field interacts back with the particle, one has to subtract its components which directly depend on the properties of the charge density and the current. These components are the "static" fields: the self repelling of charge due to the electrostatic force, and the self repelling of current due to the magnetostatic force. In other words, one only considers the forces which are due to the retardation effect.

This idea is not new. In fact, inclusion of the radiation field only in the description of the radiation reaction interaction is the basis of the quantum field theory (see e.g. Ref. 4). The differences are in certain crucial aspects which will be developed in Section 2. The equations, which will be derived, satisfy the energy conservation law, the point which is very important if they are to have physical meaning. Section 3 is devoted to the discussion of the energy balance in the system consisting of a particle and the EM field.

2. The theory

The set of coupled equations that describe the interaction of a particle and the EM field can be written in the form

$$\begin{aligned} \mathrm{i}\frac{\partial \psi}{\partial t} &= -\frac{1}{2}\left[\nabla - \mathrm{i}\vec{A}\right]^2 \psi + \Phi\psi + \frac{V}{mc^2}\psi, \\ \Delta\Phi - \frac{\partial^2 \Phi}{\partial t^2} &= -4\pi\alpha\,|\psi|^2, \\ \Delta\vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} &= -4\pi\alpha\left[\mathrm{Im}\,(\psi^*\nabla\psi) - \vec{A}\,|\psi|^2\right], \end{aligned} \tag{1}$$

where $\alpha = e^2/(\hbar c)$ is the fine structure constant. In the derivation of these equations use was made of a convenient scaling: $\vec{r}$ is the radial vector multiplied by κ and t the time multiplied by $c\kappa$, where $\kappa = mc/\hbar$ is the Compton wavenumber. Furthermore, the dimensionless potentials $\vec{A}$ and Φ are the true potentials multiplied by $e/(mc^2)$. Also, ψ is expressed in the units of $L^{-3/2}$. Hence we will define the dimensionless wave function by multiplying it by $\kappa^{-3/2}$.

In the set of equations (1) the current

$$\vec{J} = \frac{\hbar e}{m}\left[\psi^*\nabla\psi\right] - \frac{e^2}{mc}\vec{A}\,|\psi|^2 \tag{2}$$

and $|\psi|^2$ act as the source of the EM field, which in turn interacts back with the particle, i.e. this field is put back into the quantum equation.

The derivation of equations (1) is straightforward. However, one can show that the solution for ψ is nonphysical. This can be easily verified on a simple example, the stationary solutions for the hydrogen atom. In that case $|\psi|^2$ is time independent, and the equation for Φ has the solution

$$\Phi \;=\; \alpha \int \mathrm{d}^3 r' \, \frac{|\psi(\vec{r}\,')|^2}{|\vec{r}-\vec{r}\,'|}, \tag{3}$$

whilst $\vec{A}$ is zero. The time independent equation for ψ in (1) is then

$$E\psi \;=\; -\frac{1}{2}\Delta\psi + \alpha \int \mathrm{d}^3 r' \, \frac{|\psi(\vec{r}\,')|^2}{|\vec{r}-\vec{r}\,'|}\,\psi - \frac{\alpha}{r}\,\psi. \tag{4}$$

The long range behaviour of Φ is that of a point charge. Hence it will cancel the Coulomb potential of proton. Therefore, electron effectively moves in a field which is nonzero only in the close proximity of the proton, which is obviously in contradiction with the observations.

The nonphysical character of the set of equations (1) has its roots in the assumption that $|\psi|^2$ represents the charge density. This in fact is not entirely correct since its parts would repel each other through the electrostatic force, giving rise to the nonphysical results. Therefore, we want to exclude these "static" interactions from the equations by considering only the retardation forces. The most convenient way to select these components of the field is in the so called transverse gauge. In this gauge, the set of equations for the potentials are

$$\begin{aligned} \Delta\Phi \;&=\; -4\pi\alpha\,|\psi|^2 \\ \Delta\vec{A} - \frac{\partial^2\vec{A}}{\partial t^2} \;&=\; -4\pi\alpha\,\vec{J}_T \end{aligned} \tag{5}$$

where [5]

$$\vec{J}_T \;=\; \frac{1}{4\pi}\,\nabla\times\left[\nabla\times\int \mathrm{d}^3 r' \, \frac{\mathrm{Im}\,(\psi^*\nabla\psi) - \vec{A}|\psi|^2}{|\vec{r}-\vec{r}\,'|}\right]. \tag{6}$$

The equation for Φ is explicitly "static" and we can omit its contribution in the equation for ψ, and work only with $\vec{A}$, given by (5). The only retardation effects are, therefore, contained in the vector potential, which can be extracted by the method described in Appendix A. It is shown that the vector potential is given by

$$\vec{A} = \alpha\int \mathrm{d}^3 r' \left[\frac{\vec{J}(\vec{r}\,', t-R)}{R} - \frac{\vec{R}}{R^2}\,\rho(\vec{r}\,', t-R) - \frac{\vec{R}}{R^3}\left(\int\limits_0^{t-R} \mathrm{d}t'\,\rho(\vec{r}\,', t') - \int\limits_0^{t} \mathrm{d}t'\,\rho(\vec{r}\,', t')\right)\right] \tag{7}$$

where

$$\vec{J} = \mathrm{Im}\,(\psi^* \nabla \psi) - \vec{A}\,|\psi|^2 \qquad (8)$$
$$\rho = |\psi|^2 .$$

As shown in Appendix A, the leading component in $\vec{A}$, that does not depend on retardation is

$$\vec{A}^{\,(0)} = \frac{\alpha}{2} \int \mathrm{d}^3 r' \, \frac{\vec{J}(\vec{r}\,', t) + \hat{n} \left[\hat{n} \vec{J}(\vec{r}\,', t) \right]}{R} \qquad (9)$$

where $\hat{n} = \vec{R}/R$ and $\vec{R} = \vec{r} - \vec{r}\,'$. This component represents the "static" magnetic field, given by

$$\vec{H} = \nabla \times \vec{A}^{\,(0)} = \alpha \int \mathrm{d}^3 r' \, \vec{J}(\vec{r}\,', t) \times \frac{\vec{R}}{R^3} \qquad (10)$$

and is also omitted in the equation for ψ. Therefore, we define the vector potential which is due only to retardation

$$\vec{A}_{ret} = \vec{A} - \vec{A}^{\,(0)} . \qquad (11)$$

It can easily be shown that it is also transversal, i.e. $\nabla \vec{A}_{ret} = 0$. The potential (11) enters the equation for particle, and in this way the radiation reaction force is included in the dynamics. That equation is

$$\mathrm{i} \frac{\partial \psi}{\partial t} = -\frac{1}{2} \left[\nabla - \mathrm{i} \vec{A}_{ret} \right]^2 \psi + \frac{V}{mc^2} \psi \qquad (12)$$

where $\vec{A}_{ret}$ is defined by Eq. (11).

3. *Energy balance*

The equations (12), (11) and (7) describe the dynamics of a particle when the radiation reaction force is included. One important consequence of introducing this force is that the total energy of the system is conserved. Otherwise it is not the case. We will show that indeed the total energy of the system, i.e. the energy of the particle and the energy of the field, is conserved if the dynamics of the particle and the field is coupled through Eqs. (12) and (7). The total energy is

$$\frac{W}{mc^2} \equiv W = \mathrm{Re} \int \mathrm{d}^3 r \, \phi^* \left[-\frac{1}{2} \left(\nabla - \mathrm{i} \vec{A} \right)^2 + \frac{V}{mc^2} \right] \phi \, + \frac{1}{8\pi\alpha} \int \mathrm{d}^3 r \left[\left(\dot{\vec{A}} \right)^2 + \left(\nabla \times \vec{A} \right)^2 \right] \qquad (13)$$

where the dot designates the time derivative.

By taking the time derivative of the total energy, we want to show that $dW/dt = 0$. It can be shown that

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1}{4\pi\alpha}\int \mathrm{d}\Omega \left[\dot{\vec{A}} \times \nabla \times \vec{A}\right] R_M^2 \tag{14}$$

where R_M is the radius of a large sphere which encloses the system of the particle and the field. The integral is zero for $R_M > t$ and, therefore, within this time interval the total energy within the sphere is conserved. On the other hand, when $t > R_M$ the time derivative (14) is nonzero and always negative, as it will be shown, and so the energy flows out of the space where the system is confined. This energy can only come at the expense of energy of the particle (it is assumed that V is time independent) which indicates emission of radiation by particle.

The precise form of the field in the radiation region, i.e. on the surface of the sphere, is obtained from $\vec{A}$ by taking $r \to \infty$. In this limit

$$\dot{\vec{A}} \approx -\frac{\alpha}{r}\int \mathrm{d}^3r' \left[\dot{\vec{J}}(\vec{r}', t - r + \hat{n}\vec{r}') - \hat{n}\left(\hat{n}\dot{\vec{J}}(\vec{r}', t - r + \hat{n}\vec{r}')\right)\right] \tag{15}$$

and

$$\nabla \times \vec{A} \approx \frac{\alpha}{r}\hat{n} \times \int \mathrm{d}^3r' \,\dot{\vec{J}}(\vec{r}', t - r + \hat{n}\vec{r}') \tag{16}$$

where $\hat{n} = \vec{r}/r$. From these two estimates we obtain the Poynting vector

$$\vec{P} = -\frac{1}{4\pi\alpha}\dot{\vec{A}} \times \nabla \times \vec{A} \approx \frac{\alpha}{r^2}\hat{n}\left[\int \mathrm{d}^3r' \,\dot{\vec{J}}_N(\vec{r}', t - r + \hat{n}\vec{r}')\right]^2 \tag{17}$$

where

$$\vec{J}_N = \vec{J} - \hat{n}(\hat{n}\vec{J}) \tag{18}$$

is the perpendicular component of the current. The Poynting vector is always positive (in the direction of $\hat{n}$), and indeed the energy flows out from the sphere.

The intensity of radiation as a function of the frequency of the radiation field is of particular interest. We calculate the Fourier transform

$$\vec{J}_\omega = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}t\, e^{\mathrm{i}\omega t}\int \mathrm{d}^3r' \,\dot{\vec{J}}_N(\vec{r}', t - r + \hat{n}\vec{r}') \tag{19}$$

or

$$\vec{J}_\omega = -\frac{\mathrm{i}\omega}{\sqrt{2\pi}}e^{\mathrm{i}\omega r}\int_0^\infty \mathrm{d}t\, e^{\mathrm{i}\omega t}\int \mathrm{d}^3r' \, e^{-\mathrm{i}\omega\hat{n}\vec{r}'}\,\vec{J}_N(\vec{r}', t) \tag{20}$$

The intensity of radiation in the frequency interval $d\omega$ is then

$$I_\omega = |\vec{J}_\omega|^2. \tag{21}$$

4. Hydrogen-like atom

We apply the above theory for the analysis of the spontaneous emission of radiation from excited states in the hydrogen-like atoms. When specified for an electron in the field of a nucleus with N protons, the equation (12) becomes

$$\mathrm{i}\frac{\partial\psi}{\partial t} \;=\; -\frac{1}{2}\left[\nabla-\mathrm{i}\vec{A}_{ret}\right]^2\psi-\frac{N\alpha}{r}\psi \tag{22}$$

where $\vec{A}$ refers to the retardation potential (11). We can introduce the additional useful scaling by defining $\vec{u} = N\alpha\vec{r}$ and $\tau = N^2\alpha^2 t$. Then the equation for ψ becomes

$$\mathrm{i}\frac{\partial\psi}{\partial t} \;=\; -\frac{1}{2}\Delta\psi+\frac{\mathrm{i}}{N}\,\vec{A}\nabla\psi-\frac{1}{r}\psi \tag{23}$$

where the notation $\vec{r}$ and t was introduced for $\vec{u}$ and τ, respectively. In the equation we neglected A^2. In this scaling, the vector potential is

$$\vec{A}=\int \mathrm{d}^3r'\left[N^2\alpha^2\,\frac{\vec{J}(\vec{r}\,',t-N\alpha R)}{R}-N\alpha\frac{\vec{R}}{R^2}\rho(\vec{r}\,',t-N\alpha R)-\right. \tag{24}$$

$$\left.N^2\alpha^2\,\frac{\vec{J}(\vec{r}\,',t)+\hat{n}[\hat{n}\vec{J}(\vec{r}\,',t)]}{2R}-\frac{\vec{R}}{R^3}\left(\int\limits_0^{t-N\alpha R}\mathrm{d}t'\,\rho(\vec{r}\,',t'\,)-\int\limits_0^{t}\mathrm{d}t'\,\rho(\vec{r}\,',t'\,)\right)\right]$$

where $\vec{R} = \vec{r} - \vec{r}\,'$ and $\hat{n} = \vec{R}/R$. The density is $\rho = |\psi|^2$, whilst in the current we neglect the term with $\vec{A}$.

In this work we assume that the retardation effects are small, and so the integrand in the vector potential can be expanded in powers of R. From (24) we note that this is equivalent to expanding $\vec{A}$ in the powers of $N\alpha$ (R is of the order of unity), and the leading term is (see Appendix A)

$$\vec{A} \;=\; -\frac{2}{3}N^3\alpha^3\int \mathrm{d}^3r'\dot{\vec{J}}(\vec{r}\,',t). \tag{25}$$

The expansion parameter $N\alpha$ can be used to estimate the neglected terms in the equation for ψ and in the current. The terms in the kinetic energy operator are estimated as: $\Delta\approx\alpha^2$, $\vec{A}\nabla\approx N^3\alpha^4$ and $A^2\approx N^6\alpha^6$, and for small N (in our study the maximal N is about 100) the term A^2 can indeed be neglected. In the current we have similar estimates: $\mathrm{Im}(\psi^*\nabla\psi)\approx N\alpha$ and $\vec{A}|\psi|^2\approx N^3\alpha^3$, and again $\vec{A}$ can be neglected. However, this approximation is less favourable than the neglecting of A^2.

The equation (23) is solved by expanding ψ in the series

$$\psi \;=\; \sum_{n,l,m} C_{n,l,m}(t)\,\phi_{n,l}(r)Y_{l,m}(\Omega) \tag{26}$$

where $\phi_{n,l}Y_{l,m}$ are the eigenfunctions for electron when no radiation reaction force is included. The radial functions are

$$\phi_{n,l}(r) = \frac{2}{n^2}\sqrt{\frac{(n-l-1)!}{(n+l)!}}\left(\frac{2r}{n}\right)^l e^{-r/n} L_{n-l-1}^{2l+1}\left(\frac{2r}{n}\right) \tag{27}$$

and the eigenvalues

$$E_n = -\frac{1}{2n^2}. \tag{28}$$

When the expansion (26) is replaced in (23) we obtain a set of coupled equations for the coefficients

$$\mathrm{i}\dot{C}_{n,l,m} = -\frac{1}{2n^2}C_{n,l,m} + \frac{\mathrm{i}}{N}\sum_{n',l',m'} C_{n',l',m'}M_{n,l,m;n',l',m'}(t) \tag{29}$$

where the matrix elements M are defined by

$$M_{n,l,m;n',l',m'} = \int \mathrm{d}^3r\, \phi_{n,l}Y^*_{l,m}\vec{A}\nabla\phi_{n',l'}Y_{l',m'}. \tag{30}$$

In our analysis we make a simplifying assumption: the initial conditions for $C_{n,l,m}$ are nonzero for only $m = 0$. It can be shown that in this case there is no coupling between the coefficients with different m, and hence we can set $m = 0$ in (29). The matrix elements M for this simplifying assumption are

$$M_{n,l;n',l'} = \sum_{\substack{n,l\\n',l'}} \frac{\mathrm{d}}{\mathrm{d}t}\left[C^*_{n,l}C_{n',l'} - C_{n,l}C^*_{n',l'}\right] S_{n,l;n',l'} \tag{31}$$

where $S_{n,l;n',l'}$ are time independent coefficients.

The set equations (29) for the coefficients $C_{n,l}$ takes the form where the first derivatives $\dot{C}_{n,l}$ are on both sides of equations. The complex coefficients can be written $C_{n,l} = C^R_{n,l} + iC^I_{n,l}$. Then Eqs. (29) separate into the real and imaginary parts. One obtains then a set of equations for $C^R_{n,l}$ and $C^I_{n,l}$, which have a general matrix form

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}\begin{pmatrix} \dot{C}^R \\ \dot{C}^I \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \tag{32}$$

where $A_{i,j}$ and B_i are matrices that are also functions of C^R and C^I. The derivatives of the coefficients are now easily calculated from the matrix equation (32), thus enabling numerical solution of the set.

In our study we assume that no external forces act on the electron (except that of the nucleus). Our object is to investigate properties of the spontaneous decay of excited states. These states will be selected by the appropriate choice of the coefficients $C_{n,l}$ at t=0. However, one should avoid pure states. For example, the choice $C_{2,1} = 1$, and all other coefficients being zero at t=0 means that initially the electron is in a pure state (n=2; l=1). In our theory such states are stable because the current is zero, or it does not change in time, and hence they will not radiate. However, a small perturbation, i.e. making several coefficients $C_{n,l}$ nonzero, other than the one which corresponds to the pure state, leads to instability because the electron radiates and looses the energy. Therefore, pure states (except, perhaps, few) are in fact metastable states.

The model which is used to simulate excitation has two drawbacks. It is inconsistent with the assumption in the theory that at t=0 the current is zero. This fact will be neglected, although, as the result, one should expect small deviations from the energy conservation law. The second drawback is that the choice of initial coefficients is somewhat arbitrary, and therefore the results may not be as general as one would wish them to be. On the other hand, there are so many various ways of excitation of the electron so that it is difficult to decide on the initial "physical" set of $C_{n,l}$.

A typical example of an excited state is given by the parameters: $C_{3,1} = 1$, $C_{4,0} = 0.1$ and $C_{4,1} = 0.1$. The basis functions which were included in solving (29), contained all the states up-to-and-including the n=4 state. The increase in the number of the basis functions beyond this n does not significantly, alter the results. We calculate the average energy of the electron, defined by

$$E \; = \; \mathrm{Re}\left[\mathrm{i}\int \mathrm{d}^3 r\, \psi^* \dot{\psi}\right] \tag{33}$$

and the integral over the current (this quantity is essential for the calculation of spectrum from (20))

$$\vec{J} \; = \; \int \mathrm{d}^3 r'\, \vec{J}(\vec{r}\,', t). \tag{34}$$

The results are shown in Fig. 1 for the nuclear charge N=100. This, relatively large N, was taken for practical reasons: the case N=1, the hydrogen atom, is very difficult to analyze because calculation is very time consuming. The result is quite unexpected: it appears that the electron does not immediately radiate energy, but stays excited for a long time, and then in a relatively short interval releases the energy in the form of a pulse (the ground state energy of electron is -0.5, in our units). This can be observed in the time dependence of the current, which is represented by the envelope of its oscillations. Because there are about 15000 oscillations of the current in the time interval shown in Fig. 1, they are represented only symbolically. The mechanism of emission is contrary to what is normally assumed (exponential decay). This suggests that the exponential decay curves are in fact not typical of the emission phenomenon, but that the energy is released in pulses. Because of that the lifetime of the excited state is now easily associated with the maximum of the current. Several calculations were made for various values of N, and it was found that the lifetime is very well parametrized by the function

$$\tau = \frac{A}{N^4}. \tag{35}$$

Using this function we were able to extrapolate our results to N=1 giving for the lifetime of this excitation $\tau \approx 5.47 \times 10^{-9}$ s, which is in a good agreement with $\tau \approx 6.25 \times 10^{-9}$ s obtained from Fig. 3.5 of Ref. 1.

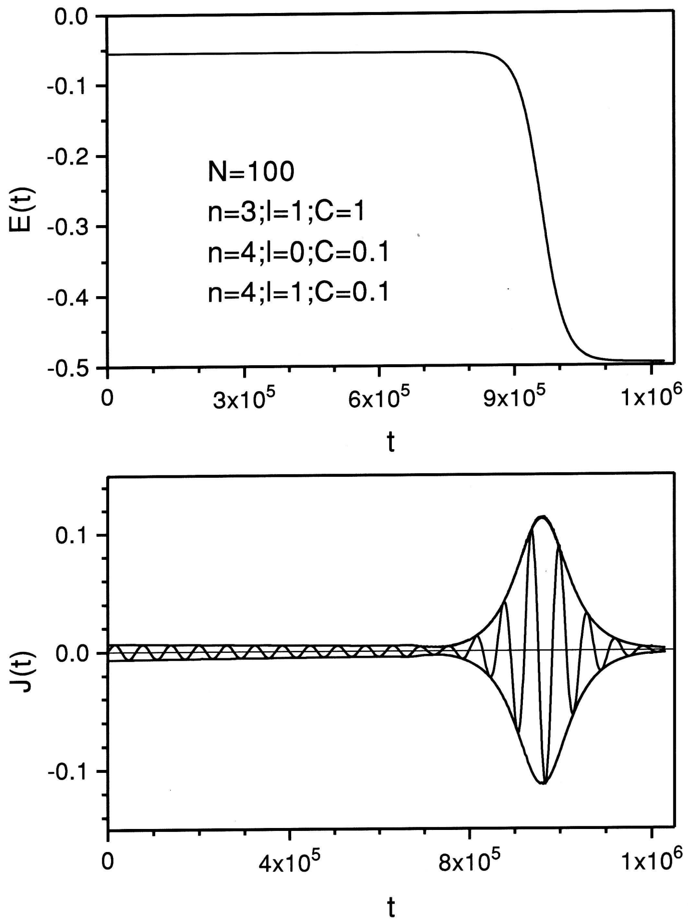

Fig. 1. Time dependence of the average energy E and of the current J of the excited state. The initial values of parameters of the excited state are shown in figure. The current is represented by the envelope of its oscillations, which are symbolically shown by an oscillating curve. In reality there are about 15000 oscillations of the current in the time interval shown.

The frequency spectrum of radiation, for this example, is shown in Fig. 2, where we notice three lines. The frequency of these lines corresponds to various energy differences of the states. They are also shown in the figure. The most dominant is the line for the $(3, 1) \to (1, 0)$ transition. The intensities can be interpreted as the transition probabilities, but here they play the role of the Fourier components in the current.

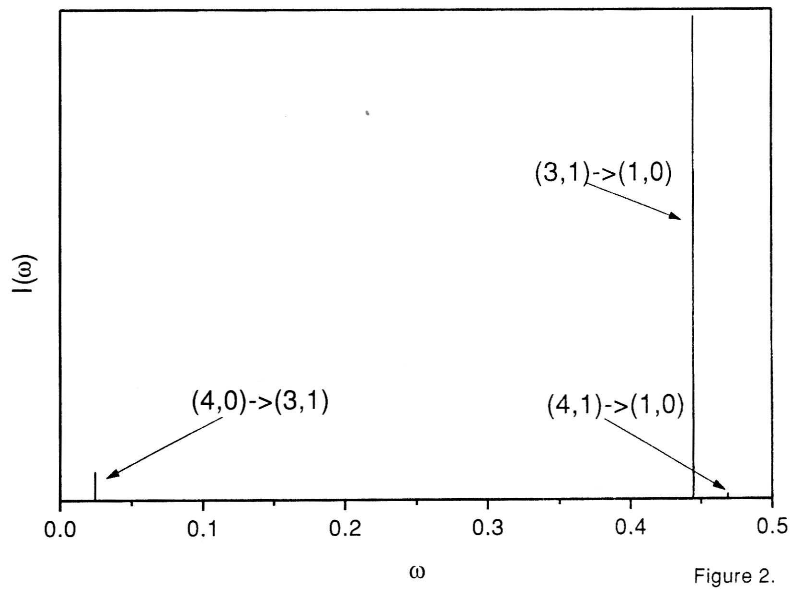

Fig. 2 The current shown in Fig. 1 produces radiation with three spectral lines shown in this figure. The position of the lines correspond to the energy differences for the indicated transitions.

5. *Conclusion*

In this work we showed how to treat dynamics of a charged particle if we want the radiation reaction interaction to be included. The theory has been applied to the decay phenomenon in the hydrogen-like atoms. However, a much more elaborate study of the emission mechanism is required, which would also take into account the excitation process, but even in this example the theory showed several features. First, it gives a reasonable description of the decay process, with one interesting prediction: the radiation is emitted in the form of pulses rather than being described by the exponential decay. It gives sensible answers for the lifetime of excited states, which are comparable with the values cited in the literature (Ref. 1, Fig. 3.5). On the practical side, the theory is quite straightforward in applications, with one surprising feature: it is very stable with regard to the numerical solution. These practical features enable easy extension of the theory to describe various other processes, e.g. electron in a plane EM wave and collision phenomena.

Appendix A: Vector potential in the transverse gauge

The vector potential satisfies the equation

$$\Delta \vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} = -4\pi\alpha \vec{J}_T. \qquad (A1)$$

In the transverse gauge the current is given by

$$\vec{J}_T = \frac{1}{4\pi} \nabla \times \left[\nabla \times \int \mathrm{d}^3 r' \, \frac{\vec{J}\,(\vec{r}\,', t)}{|\vec{r} - \vec{r}\,'\,|} \right]. \qquad (A2)$$

We use the scaling of Section 2. Equation A1 has the solution

$$\vec{A}(\vec{r}', t) = \alpha \int \mathrm{d}^3 r' \, \mathrm{d}t' \, G_{ret}(\vec{r} - \vec{r}\,', t - t'\,) \, \vec{J}_T(\vec{r}\,', t'\,) \qquad (A3)$$

where G_{ret} is the retarded Green function which can be put in the form

$$G_{ret}(\vec{R}, \tau) = \frac{1}{2\pi R} \int \mathrm{d}\omega \, e^{-\mathrm{i}\omega\tau + \mathrm{i}\omega R}. \qquad (A4)$$

The solution of (A1)

$$\vec{A} = \frac{\alpha}{8\pi^2} \nabla \times \nabla \times \int \mathrm{d}^3 r' \int \mathrm{d}t' \, \vec{j}(\vec{r}\,', t'\,) \int_{-\infty}^{\infty} \mathrm{d}\omega \, e^{-\mathrm{i}\omega(t - t'\,)} \int \mathrm{d}^3 r'' \, \frac{e^{\mathrm{i}\omega|\vec{r} - \vec{r}\,''|}}{|\vec{r} - \vec{r}\,''|\,|\vec{r}\,'' - \vec{r}\,'|} \qquad (A5)$$

where by a suitable manipulation the operator $\nabla \times \nabla \times$ was taken out of all the spatial integrations, and now refers to the coordinates of $\vec{r}$.

The integral over $\vec{r}\,''$ has the analytic solution, but it is not finite because the upper limit of r'' is infinite. Therefore, we will restrict this volume integral to the sphere of a large radius R_M. Then its value is

$$\int \mathrm{d}^3 r'' .... = \frac{4\pi}{R\omega^2} \left[e^{\mathrm{i}\omega R} - 1 - \mathrm{i} \sin(\omega R) e^{\mathrm{i}\omega R_M} \right] \qquad (A6)$$

where $\vec{R} = \vec{r} - \vec{r}\,'$. In the next step we evaluate the integral over ω, which is finite and has the value

$$\int_{-\infty}^{\infty} \mathrm{d}\omega .... = -\frac{8\pi^2}{R} \left[(t - t' - R)\Theta(t - t' - R) - (t - t'\,)\Theta(t - t'\,) + \right. \qquad (A7)$$

$$\frac{t-t'-R_M+R}{2}\Theta(t-t'-R_M+R)-\frac{t-t'-R_M-R}{2}\Theta(t-t'-R_M-R)]$$

where $\Theta(z)=0$ for $z<0$ and $\Theta(z)=1$ for $z\geq 0$. The vector potential is now

$$\vec{A}=-\alpha\nabla\times\nabla\times\int\frac{\mathrm{d}^3r'}{R}\left[\frac{1}{2}\int\limits_{-\infty}^{t-R_M+R}\mathrm{d}t'\,(t-t'-R_M+R)\,\vec{J}(\vec{r}\,',t')-\right. \qquad (A8)$$

$$\left.\frac{1}{2}\int\limits_{-\infty}^{t-R_M-R}\mathrm{d}t'\,(t-t'-R_M-R)\,\vec{J}(\vec{r}\,',t')+\int\limits_{-\infty}^{t-R}\mathrm{d}t'\,(t-t'-R)\,\vec{J}(\vec{r}\,',t')-\int\limits_{-\infty}^{t}\mathrm{d}t'\,(t-t')\,\vec{J}(\vec{r}\,',t')\right].$$

The limits of the integral in t' are determined by the physical circumstances. In our analysis we will assume that the current $\vec{J}$ is zero for $t<0$, and since M is large, the vector potential is

$$\vec{A}=-\alpha\nabla\times\nabla\times\int\frac{\mathrm{d}^3r'}{R}\left[\int\limits_{0}^{t-R}\mathrm{d}t'\,(t-t'-R)\,\vec{J}(\vec{r}\,',t')-\int\limits_{0}^{t}\mathrm{d}t'\,(t-t')\,\vec{J}(\vec{r}\,',t')\right]. \qquad (A9)$$

By applying the operator $\nabla\times\nabla\times$, we obtain a more explicit form for the potential which is a relatively complicated function of the current $\vec{J}$. However, by using the transformation

$$\sum_m\partial'_m\left(J_m\frac{\vec{R}}{R^3}\right)=\frac{\vec{R}}{R^3}\nabla'\vec{J}+3\frac{\vec{R}}{R^5}\left(\vec{R}\vec{J}\right)-\frac{\vec{J}}{R^3} \qquad (A10)$$

where ∂'_m designates derivative with respect to the m-th component of $\vec{r}\,'$, and noting that the current vanishes outside a certain region, we obtain for $\vec{A}$

$$\vec{A}=\alpha\int\mathrm{d}^3r'\left[\frac{\vec{J}(\vec{r}\,',t-R)}{R}-\frac{\vec{R}}{R^3}\int\limits_{0}^{t-R}\mathrm{d}t'\,(t-t')\dot{\rho}(\vec{r}\,',t')+\frac{\vec{R}}{R^3}\int\limits_{0}^{t}\mathrm{d}t'\,(t-t')\dot{\rho}(\vec{r}\,',t')\right] \qquad (A11)$$

where the dot designates derivative with respect to the time and $\rho=|\psi|^2$. In the derivation of (A11) we also used the continuity equation

$$\nabla\vec{j}+\dot{\rho}=0. \qquad (A12)$$

By partial integration in time we obtain another form for the vector potential

$$\vec{A}=\alpha\int\mathrm{d}^3r'\left[\frac{\vec{j}(\vec{r}\,',t-R)}{R}-\frac{\vec{R}}{R^2}\rho(\vec{r}\,',t-R)-\frac{\vec{R}}{R^3}\left(\int\limits_{0}^{t-R}\mathrm{d}t'\,\rho(\vec{r}\,',t')-\int\limits_{0}^{t}\mathrm{d}t'\,\rho(\vec{r}\,',t')\right)\right]. \qquad (A13)$$

We distinguish two regions in the potential depending on the value of r. In the interaction region $r \approx r'$, whilst in the radiation region $r >> r'$. In the radiation region the potential is analyzed by expanding $\vec{A}$ in the powers of r^{-1}. However, in the interaction region one would expand it in the series of R. The latter expansion is appropriate in the circumstances when the current is nearly static during the time it takes the light to travel the distance across the potential. In this case we get for the vector potential

$$\vec{A} \approx \alpha \int \mathrm{d}^3 r' \left[\frac{\vec{J} + \hat{n}(\hat{n}\vec{J})}{2R} - \frac{2}{3}\dot{\vec{J}} + \frac{R}{8}\left(3\,\ddot{\vec{J}} - \hat{n}(\hat{n}\ddot{\vec{J}})\right) + ... \right] \tag{A14}$$

where we used the continuity equation (A12), and generalization of the identity (A10). The orders of "retardation" in (A14) are distinguished according to the order of time derivative of the current. Thus the "static" term is

$$\vec{A}^{(0)} \;=\; \frac{\alpha}{2} \int \mathrm{d}^3 r' \, \frac{\vec{J}(\vec{r}\,', t) + \hat{n}[\hat{n}\vec{J}(\vec{r}\,', t)]}{R} \tag{A15}$$

and the first order correction is

$$\vec{A}^{(1)} \;=\; -\frac{2\alpha}{3} \int \mathrm{d}^3 r' \dot{\vec{J}}(\vec{r}\,', t). \tag{A16}$$

It is important to note that each order of "retardation" satisfies the transversality condition, i.e.

$$\nabla \vec{A}^{(n)} \;=\; 0 \tag{A17}$$

which is an important property of the solution of the radiation reaction problems.

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POLUKLASIČNA TEORIJA SPONTANE EMISIJE ZRAČENJA

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Razvijena je nerelativistička poluklasična teorija radijacijske sile zračenja. Primjenjena je na opis spontane emisije zračenja s pobuđenog stanja atoma poput vodikovog.