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On the problem of the best circle to discontinuous groups in hyperbolic plane

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Abstract. The aim of this paper is to describe the largest inscribed circle in the fundamental domains of a discontinuous group in the Bolyai-Lobachevsky hyperbolic plane. We give some known basic facts related to the Poincare-Delone problem and the existence notion of the inscribed circle. We study the best circle of the group $G = [3, 3, 3, 3]$ with 4 rotational centers each of order 3. Using the Lagrange multiplier method, we would describe the characteristic of the best inscribed circle. The method could be applied to a more general case in $G = [3, 3, 3, \ldots, 3]$ with $l \geq 4$ rotational centers each of order 3, by more and more computations. We observed by a more geometric Theorem 2 that the maximum radius is attained by equalizing the angles at equivalent centers and the additional vertices with trivial stabilizers, respectively. Theorem 3 will close our arguments, where lemmas 3 and 4 play key roles.

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1. Introduction

The 17 crystal groups on Euclidean plane $\mathbb{E}^2$ have long been known (as an intuitive discovery of medieval Islamic art, e.g. the artistic mosaics of Alhambra in Granada, Spain). Delone (Delaunay) described the 46 types of their fundamental domains as late as in 1959, see [1]. In 1882 Poincare had already attempted to describe the analogous plane groups in Bolyai-Lobachevsky hyperbolic plane $\mathbb{H}^2$. A significant result of Macbeath was the description of an algebraic combinatorial classification of non-Euclidean plane crystallographic groups with compact quotient space by their signature, see [5] and [7].
In this paper, we would like to determine the best circle inscribed in the fundamental domain of a given discontinuous group in hyperbolic plane $\mathbb{H}^2$. This problem was actually raised by Prof. Molnár in [7] on the base of [3]. The fundamental domain for planar discontinuous groups and uniform tilings was studied by Lučić and Molnár in [3, 2]. The algorithm for classification of fundamental polygons for a given discontinuous group was also presented by Lučić, Molnár and Vasiljević in [4]. We are interested in the following theorem

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Theorem 1 (Lučić-Molnár, [3])**.** *Among all convex polygons in $\mathbb{E}^2$, $\mathbb{S}^2$, and $\mathbb{H}^2$ with given angles $\alpha_1, \alpha_2, \dots, \alpha_m, m \geq 3$, there exists up to similarity (for $\mathbb{E}^2$) and up to an isometry (for $\mathbb{S}^2$ and $\mathbb{H}^2$), respecting the order of the angles, exactly one polygon circumscribing a circle.*

That theorem will guarantee the existence of the inscribed circle in a fundamental domain for given angles. We shall determine the best circle, that is, the inscribed circle with the largest radius in fundamental domains determined by a discontinuous group in $\mathbb{H}^2$.

In this first section we study a typical case, the hyperbolic plane group $G = [3, 3, 3, 3]$ with 4 rotation centers of order 3 in $\mathbb{H}^2$, their fundamental domains and its representation in the tree graphs, see Figure 1. Basically, these tree graphs are topological images of the fundamental domain under the canonical projection mapping $\kappa : \mathbb{H}^2 \longrightarrow \mathbb{H}^2/G, \quad X \longmapsto \bar{X} := X^G$, or simply, κ identifies all points which form the same orbit by this group. By contrast, to obtain the topological fundamental domain, we imagine a scissor dissecting these tree graphs, and open it up (or unfold) through the fault to construct the pre-image fundamental domain. In Section 2 we shall consider the constrained optimum problem and apply the Lagrange multiplier method to find the solution. We will present sufficient conditions for the local maximum points through the second derivative method called the bordered Hessian criterion. In Section 3, we use the optimality condition based on Section 2 to determine the optimum incircle radius of the other type of G. Moreover, in Section 4, we also describe the optimum condition geometrically.

In Section 5, we develop the method to more general $G = [3, 3, 3, \dots, 3]$ of l rotational points, where $l \geq 4$. All types of fundamental domains are characterized combinatorially by a Diophantine equation system. Based on these constructions, we will show the global optimum of the inscribed circle radius of all fundamental domain types of G. We also provide an important fact as to the area of the fundamental domain of all types. Now, as a motivation, we begin with recalling the proof of the Theorem 1 in hyperbolic plane $\mathbb{H}^2$.

Proof of Theorem 1 (for a hyperbolic case). Given p is a polygon with given angles $\alpha_1, \alpha_2, \dots, \alpha_m \in (0, \frac{\pi}{2})$, near vertices $A_1, A_2, \dots, A_m$, which is circumscribed around a circle $k(X, x)$. Let $B_1, B_2, \dots, B_m$ be the set of points of tangency of p and k, such that the angles $B_m X B_1$, $B_1 X B_2$, …, B_{m-1} are equal to $\beta_1, \beta_2, \dots, \beta_m$. Then, $\beta_1 + \beta_2 + \cdots + \beta_m = 2\pi$.

By applying trigonometry to the rectangular central triangles XA_iB_i we obtain the formula (in $\mathbb{H}^2$):

$$\cos\left(\frac{\alpha_i}{2}\right) = \cosh x \sin\left(\frac{\beta_i}{2}\right). \tag{1}$$

Therefore,

$$\frac{\cos\left(\frac{\alpha_1}{2}\right)}{\sin\left(\frac{\beta_1}{2}\right)} = \frac{\cos\left(\frac{\alpha_2}{2}\right)}{\sin\left(\frac{\beta_2}{2}\right)} = \cdots = \frac{\cos\left(\frac{\alpha_m}{2}\right)}{\sin\left(\frac{\beta_m}{2}\right)} = \cosh x,$$

for a factor $\cosh x > 1$ is necessary for p in $\mathbb{H}^2$.

The existence of x and also β_i, such that $\sum_i \beta_i = 2\pi$, can be shown as follows.

Consider $\cos\left(\frac{\alpha_1}{2}\right), \cos\left(\frac{\alpha_2}{2}\right), \ldots, \cos\left(\frac{\alpha_m}{2}\right)$. From (1), we have

$$\beta_i = 2\sin^{-1}\left(\frac{\cos\left(\frac{\alpha_i}{2}\right)}{\cosh x}\right).$$

Now, consider the following continuous function:

$$S(x) = \left(\sum_i^m 2\sin^{-1}\left(\frac{\cos\left(\frac{\alpha_i}{2}\right)}{\cosh x}\right)\right) - 2\pi \qquad x \in (0, \infty)$$

$$S(0) = \left(\sum_i^m 2\sin^{-1}\left(\cos\left(\frac{\alpha_i}{2}\right)\right)\right) - 2\pi = (m-2)\pi - (\alpha_1 + \alpha_2 + \cdots + \alpha_m) > 0$$

$$(\text{since } \alpha_1 + \cdots + \alpha_m < (m-2)\pi \text{ on } \mathbb{H}^2)$$

We choose x_0, such that $\cosh x_0 > \frac{1}{\sin\left(\frac{2\pi}{m}\right)}$.

$$\begin{aligned} S(x_0) &= \left(\sum_i^m 2\sin^{-1}\left(\frac{\cos\left(\frac{\alpha_i}{2}\right)}{\cosh x_0}\right)\right) - 2\pi \\ &< \left(\sum_i^m 2\sin^{-1}\left(\cos\left(\frac{\alpha_i}{2}\right)\sin\left(\frac{2\pi}{m}\right)\right)\right) - 2\pi \\ &< \left(\sum_i^m 2\sin^{-1}\left(\sin\left(\frac{2\pi}{m}\right)\right)\right) - 2\pi = 0. \end{aligned}$$

We see that the function S changes sign in $[0, x_0]$. Since S is continuous, by the intermediate value theorem, there is a value $r \in [0, x_0]$, such that $S(r) = 0$. In other words, $\left(\sum_i^m 2\sin^{-1}\left(\frac{\cos\left(\frac{\alpha_i}{2}\right)}{\cosh r}\right)\right) = 2\pi$. Hence, the inscribed circle radius is $x = r$, with the corresponding central angles β_i satisfying $\beta_1 + \beta_2 + \cdots + \beta_m = 2\pi$ □

1.1. The hyperbolic plane group $G = [3, 3, 3, 3]$

As a typical example, the group $G = [3, 3, 3, 3]$ contains exactly 4 rotational centers each of order 3 on a topological sphere. The tree surface graphs from $G = [3, 3, 3, 3]$ are presented in Figure 1. There are 5 types of graphs that represent the fundamental domains of G. We could construct fundamental domains based on these tree graphs. The complete corresponding fundamental domains are given sketchily in Figure 2.

1.1.1. Type-5 fundamental domain

We would like to find the best inscribed circle in the fundamental domain of the above hyperbolic plane group G. We are first focused on the type-5 fundamental domain. Since this type has the most edges, we guess that the largest circle radius would be attained in this type. This tree graph on the sphere is the surface diagram of the conjectured optimal fundamental domain of G=[3,3,3,3] given by

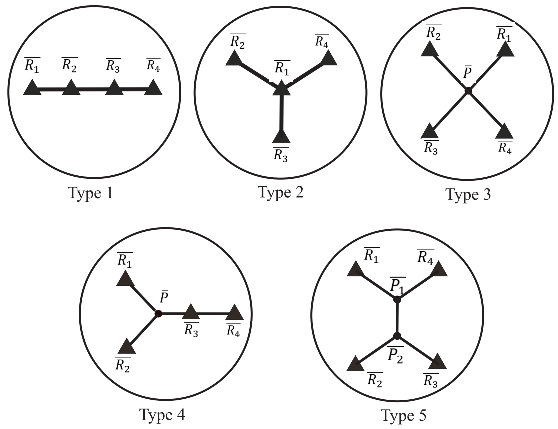

Figure 1: *All together: 5 types of tree surface graphs of fundamental domains for* $G = [3,3,3,3]$ *on a sphere*

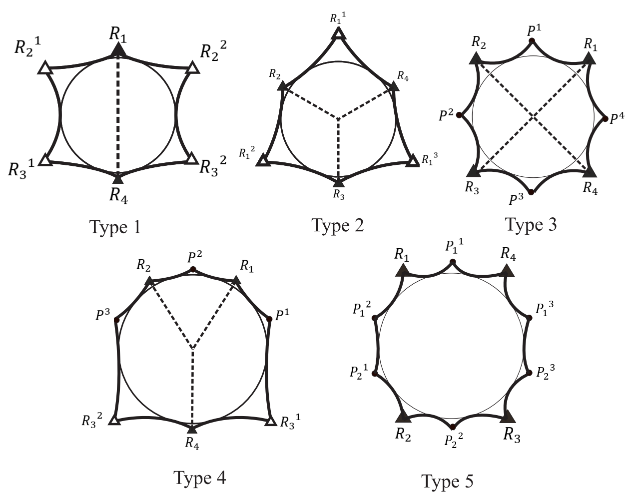

Figure 2: *All together: 5 types of sketchy fundamental domains for* $G = [3,3,3,3]$

its Conway-Macbeath signature. This diagram is a tree graph on a topological elastic sphere with the given 3-centers as $4 = m$ vertices, each of valence (degree) 1 and $2 = y$ additional vertices, each of valence 3. Then imagine a pair of scissors we take, and cut the sphere along this tree graph to obtain a topological domain with the later metrical properties. Then the number of vertices is $6 = v$, and the number of edges is $5 = e$. The criterion of a tree $v = e + 1$ is fulfilled. We get a fundamental polygon of $m*1+y*3 = 10$ vertices (and sides), as in Figure 3. To give more details, see Figure 3, we dissect the tree graph of type-5 through directions: $\bar{P}_1 \to \bar{R}_1 \to \bar{P}_1 \to \bar{P}_2 \to \bar{R}_2 \to \bar{P}_2 \to \bar{R}_3 \to \bar{P}_2 \to \bar{P}_1 \to \bar{R}_4 \to \bar{P}_1$. Then we denote the future angles $\alpha_1, \alpha_2, \alpha_6$ at vertex $\bar{P}_1$ and $\alpha_3, \alpha_4, \alpha_5$ at vertex $\bar{P}_2$, Figure 3. We construct the fundamental domain by opening up the dissected elastic tree surface graph. As a result, we obtain a type-5 fundamental domain as shown in figures 2-4.

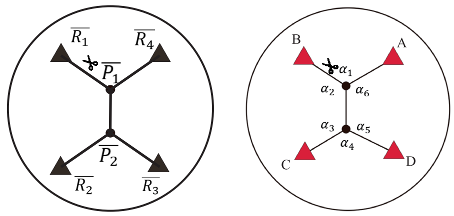

Figure 3: *Type-5 tree surface graph of $G = [3,3,3,3]$ is dissected by a pair of scissors with orientation $\bar{P}_1 \to \bar{R}_1 \to \bar{P}_1 \to \bar{P}_2 \to \bar{R}_2 \to \bar{P}_2 \to \bar{R}_3 \to \bar{P}_2 \to \bar{P}_1 \to \bar{R}_4 \to \bar{P}_1$*

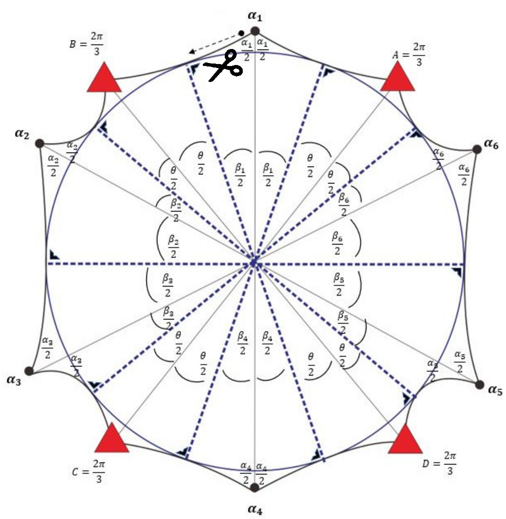

Figure 4: *Type-5 fundamental domain of $G = [3,3,3,3]$. Imagine also later on for $G = [3,3,3,\dots,3]$ (l-times).*

We have some metrical properties as presented in equation system (2)-(11).

$$\cos\left(\frac{\alpha_1}{2}\right) = \cosh x \ \sin\left(\frac{\beta_1}{2}\right), \tag{2}$$

$$\cos\left(\frac{\alpha_2}{2}\right) = \cosh x \ \sin\left(\frac{\beta_2}{2}\right), \tag{3}$$

$$\cos\left(\frac{\alpha_3}{2}\right) = \cosh x \ \sin\left(\frac{\beta_3}{2}\right), \tag{4}$$

$$\cos\left(\frac{\alpha_4}{2}\right) = \cosh x \ \sin\left(\frac{\beta_4}{2}\right), \tag{5}$$

$$\cos\left(\frac{\alpha_5}{2}\right) = \cosh x \ \sin\left(\frac{\beta_5}{2}\right), \tag{6}$$

$$\cos\left(\frac{\alpha_6}{2}\right) = \cosh x \ \sin\left(\frac{\beta_6}{2}\right), \tag{7}$$

$$\cos\left(\frac{\pi}{3}\right) = \cosh x \ \sin\left(\frac{\theta}{2}\right), \tag{8}$$

where

$$\sum_{i=1}^{6} \beta_i \ + \ 4\theta = 2\pi, \tag{9}$$

$$\alpha_1 + \alpha_2 + \alpha_6 = 2\pi, \tag{10}$$

$$\alpha_3 + \alpha_4 + \alpha_5 = 2\pi. \tag{11}$$

From these 10 equations, we treat equation (8) as an equation that provides the objective function f, that is, we form $f = \cosh(x) = \frac{\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\theta}{2}\right)}$ and we want to find the best value of radius x, i.e., f is maximal. But, there are some conditions, i.e equations (2)-(7), (9)-(11), which should be satisfied. Therefore, we face a constrained extremum problem. We shall describe the so-called Lagrange multiplier method to deal with this problem in Section 2. Now we shall motivate our approach. We want to find the best value of radius x, meaning the maximum value of x with the constraint above. We shall consider equation (8) as a candidate for our objective function as follows:

$$x = f(\theta) = \cosh^{-1}\left(\frac{1}{2\sin(\frac{\theta}{2})}\right).$$

One could formally reduce the conditions above by substituting all of the constraints to f. Then we have $f = f(\alpha_1, \ldots, \alpha_6, \beta_1, \ldots, \beta_6, \theta)$, where the natural domain of f (a subset of $\mathbb{R}^{13}$) is determined by the remaining constrains. We first study the very specific case (a regular case), where all vertices have the same interior angles $2\pi/3$. This case provides our conjectured optimum.

1.2. Very specific case (regular case)

We consider a specific case, the so-called regular case, by setting $\alpha_1 = \alpha_2 = \cdots = \alpha_6$. Constraints (10), (11) and (9) impose the vertex angles are equal, $\frac{2\pi}{3}$. This choice also implies that the central angles are also equal to θ, and that they satisfy the following inequality (the triangle condition in the hyperbolic plane):

$$\frac{\theta}{2} + \frac{1}{2}\frac{2\pi}{3} + \frac{\pi}{2} < \pi, \text{ then } \theta < \frac{\pi}{3}.$$

We just have one equation for solving the radius x, $\cos\left(\frac{\pi}{3}\right) = \cosh(x)\sin\left(\frac{\theta}{2}\right)$. Therefore, $x = \cosh^{-1}\left(\frac{1}{2\sin\left(\frac{\theta}{2}\right)}\right)$. The value x is depends only on the central angle θ.

Since the sum of all central angles should be 2π, it follows that $10\theta = 2\pi$, and then $\theta = \frac{\pi}{5}$. Now we can directly compute the exact value of x in this specific case.

$$x \approx 1.061275061$$

The area A of a circle disc with radius x in the hyperbolic plane is given by:

$$A = 4\pi \sinh^2\left(\frac{x}{2}\right), \text{ in our case } A \approx 3.883222071.$$

Furthermore, the density d is described by division of the area of the circle and that of the fundamental polygon. We could compute directly that the area of the polygon is $\frac{4}{3}\pi$, a characteristic invariant for group $G = [3, 3, 3, 3]$. In our calculation, we found that $d \approx 0.9270509814$. Our conjecture is that this regular case would give the best circle, i.e. largest one inscribed in the fundamental polygon of group $G = [3, 3, 3, 3]$. We would like to investigate this conjecture by studying some possible situations. Let us have a conditional extremum problem. First, we use the tools in multivariate calculus, the so-called Lagrange multiplier method.

2. The Lagrange multiplier method

This method is based on the system of equations in the previous section, equations (2)-(11). We formulate the following conditional extremum problem. From these 10 equations we set the function f from the constraint equation (8) and some constraints g_i, h, h_j from 9 remaining equations. We would like to find the maximum of radius x. From equation (8), we have $\cosh(x) = \frac{\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\theta}{2}\right)}$. Since cosh is a monotonic increasing function for $x > 0$, to maximize x we just maximize $\cosh(x)$. We take $f(\alpha_1, \ldots \alpha_6, \beta_1, \ldots, \beta_6, \theta) = \frac{\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\theta}{2}\right)}$ for the objective function.

We formulate the constraints by setting a subtraction of the expression in equations (2)-(7) from the expression in equation (8), i.e., $\frac{\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\theta}{2}\right)} - \frac{\cos\left(\frac{\alpha_i}{2}\right)}{\sin\left(\frac{\beta_i}{2}\right)} = 0$, for $i = 1, .., 6$. Since $\alpha_i, \beta_i, \theta$ are the variables, half of them representing angles of rectangular triangles, we can restrict their value in $[0, \pi]$.
Therefore, we treat our problems in region $[0, \pi]^{13} \subset \mathbb{R}^{13}$. For convenience, we also write the tuple $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \theta) = \boldsymbol{X}$, as an element in $[0, \pi]^{13} \subset \mathbb{R}^{13}$. The set of constraints is described by the following system.

$$\begin{aligned}
g_i =& \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\beta_i}{2}\right) - \cos\left(\frac{\alpha_i}{2}\right)\sin\left(\frac{\theta}{2}\right) = 0,\\
&\text{where } \cos\frac{\pi}{3} = \frac{1}{2}, \text{ and } i = 1, 2, \ldots, 6\\
h =& \sum_{i=1}^{6} \beta_i + 4\theta - 2\pi = 0,\\
h_1 =& \alpha_1 + \alpha_2 + \alpha_6 - 2\pi = 0,\\
h_2 =& \alpha_3 + \alpha_4 + \alpha_5 - 2\pi = 0.
\end{aligned}$$

The complete construction of our constrained extremum problem is described as follows:

$$\text{Maximize } f(\boldsymbol{X}) = \frac{1}{2\sin\left(\frac{\theta}{2}\right)}, \text{ subject to the above constraints.}$$

2.1. The compactness of constrained region

We consider the constrained region S. The compactness of S could help us to guarantee the existence of maximum (and minimum) of f in S. Consider $g_i(\boldsymbol{X}) = \frac{1}{2}\sin\left(\frac{\beta_i}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\alpha_i}{2}\right)$ for $i = 1, \ldots 6$ are bounded continuous functions. Therefore, $g_i^{-1}(0)$, the inverse images of 0, a closed set, under the continuous function are also closed in $\mathbb{R}^{13}$. One could see that $h^{-1}(0)$, $h_1^{-1}(0)$, $h_2^{-1}(0)$ are also closed in $\mathbb{R}^{13}$. With this compactness assumption, we note that f is bounded in S; then f has a maximum and a minimum in S. We have obtained that our conjectured point X_0 satisfies the necessary condition for the local maximum of the constrained extremum problem. We need to observe further whether this point is really a local maximum point. We apply the second derivative test called the bordered determinant criterion test in [6, 8].

3. Other fundamental domain types: finding global maximum

Based on our analysis on the type-5 of the fundamental domains for $G = [3, 3, 3, 3]$, we obtain the largest radius $x \approx 1.061275061$. We need to compare it with the largest radius reached on other fundamental domains of types 1, 2, 3, 4. (figures 1-2). The analogous methods, Lagrange multiplier and bordered determinant are applied to the cases of types 3 and 4 when they have independent parameters raised by the additional point, while the fundamental domains of types 1 and 2 have only fixed vertex angles. The equation system could be solved immediately by some appropriate substitutions.

1. **Type-1**
   The constructed fundamental domain of this type has no additional point. It just contains two rotational centers R_1, R_4 and two rotational centers R_2, R_3 that appear twice, see figures 1 and 2. The angles on the rotational vertices R_1 and R_4 are equal to $\frac{2\pi}{3}$. While the angles on vertices that appeared twice R_2^1, R_2^2, R_3^1, R_3^2 are half of its original, i.e., $\frac{2\pi}{6}$. We derive the following system of equations:

$$\cos\left(\frac{1}{2}\cdot\frac{2\pi}{3}\right) = \cosh x \sin\left(\frac{\theta_1}{2}\right), \quad \cos\left(\frac{1}{2}\cdot\frac{2\pi}{6}\right) = \cosh x \sin\left(\frac{\theta_2}{2}\right)$$
$$2\theta_1 + 4\theta_2 = 2\pi$$

   Basically, this equation system has only fixed parameters. Using some appropriate substitutions, we conclude that the value of radius x is given by:

$$x = \cosh^{-1}\left(\frac{3}{2}\right) \approx 0.962423$$

2. **Type-2**
   In this type, we have rotational center R_1 that appears three times on the fundamental domain. If we derive a single equation similar to type-1, we get the numerical value $x \approx 0.927539$.

3. **Type-3**
   This type has a single additional point on the tree graph in Figure 1. The corresponding fundamental domain is given in Figure 2. On that fundamental domain, the additional point P appears four times, namely P^1, P^2, P^3, and P^4. We denote the angles near P^1, P^2, P^3, P^4 by α_1, α_2, α_3, α_4 and their corresponding central angles by β_1, β_2, β_3, β_4, respectively. It is interesting to see that the value of x depends on α_1, α_2, α_3, α_4. Inspired by research on the type-5 in Section 2, we can formulate a set of constraints and find the maximum value of $\cosh x = \frac{1}{2\sin\left(\frac{\theta}{2}\right)}$. Finally, this equation system could be solved for x, that is, $x \approx 1.031718$.

4. **Type-4**
   In this type, the vertex R_3 appears twice on the fundamental domain, see Figure 2, while the additional point P is copied three times, namely P^1, P^2, P^3. We denote the angles near P^1, P^2, P^3 by $\alpha_1, \alpha_2, \alpha_3$ and their corresponding central angles by $\beta_1, \beta_2, \beta_3$, respectively. We have the equation system of the constraints and the corresponding approximated value of x is about 1.011595.

The summary of all largest possible inscribed circle radii on each type of fundamental domain is presented in the following tables. According to the comparison of the

Type	Largest radius
1	0.962423
2	0.927539
3	1.031718
4	1.011595
5	1.061275

Table 1: *The largest inscribed circle radius comparison*

largest radii given in Table 1, the largest radius of all types is attained on type-5, namely $x \approx 1.061275$. Based on the exploration of the constrained optimum problem, it could be conjectured that optimum conditions might happen whenever the corresponding parameters are equal. This intuition could be shown in the next section.

4. Geometric argument: Conclusion to $[3, 3, 3, 3]$

According to the approach of the Lagrange multiplier method and the bordered determinant criterion, it could be concluded that the maximum possible inscribed circle radius is attained whenever the corresponding independent vertex angles are equal. The following theorem will state this more intuitively and geometrically simpler: **Whenever we equalize the corresponding angles at $G-$equivalent vertices, the radius will increase**. This could be further developed to the proof of necessary conditions for the general problem for arbitrary cocompact plane group G (as conjectured by the authors of [4]).

Theorem 2. *If we exchange two angles in Theorem 1, say α_1 and α_2, both to $\frac{\alpha_1+\alpha_2}{2}$ in a given configuration with fixed radius x, so $\cosh x$, then for the corresponding central angles β_1 and β_2 their arithmetic mean $\frac{\beta_1+\beta_2}{2}$ increases. So, changing only α_1, α_2 to $\frac{1}{2}(\alpha_1+\alpha_2)$, the inscribed circle will have a bigger radius in the procedure.*

Before proving this theorem, we first discuss a Jensen-type inequality of Lučić-Molnar by [3] for $\mathbb{H}^2$, which is described in the following lemma.

Lemma 1. *The function $\beta : (0, \frac{\pi}{2}) \ni \alpha \mapsto \beta(\alpha) \in (0, \frac{\pi}{2})$, as above, given by $\sin(\beta(\alpha)) = \frac{\cos\alpha}{\cosh x}$, with fixed x and $\cosh x$, is concave (from below).*

Proof. (By communication with Prof. Emil Molnár.) As we look at formulas in Theorem 1, $\cos\left(\dfrac{\alpha}{2}\right) = \cosh x \sin\left(\dfrac{\beta}{2}\right)$ is the crucial relation (with fixed $\cosh x > 1$) to define central angles $\beta_i(\alpha_i)$ of α_i, $(i = 1, \ldots, m \geq 3)$ as function $\beta(\alpha)$ of α. Let us start with

$$\sin(\beta(\alpha)) = \frac{\cos\alpha}{\cosh x}, \quad (0 < \alpha < \frac{\pi}{2}) \tag{12}$$

By differentiating both sides by α, we obtain

$$\frac{d}{d\alpha}(\sin(\beta(\alpha))) = \frac{d}{d\alpha}\left(\frac{\cos\alpha}{\cosh x}\right),$$

which leads to

$$\begin{aligned} \cos(\beta(\alpha))\frac{d\beta(\alpha)}{d\alpha} &= -\frac{\sin\alpha}{\cosh x} \\ \frac{d\beta(\alpha)}{d\alpha} &= \frac{1}{\cosh x}\left(-\frac{\sin\alpha}{\cos(\beta(\alpha))}\right). \end{aligned} \tag{13}$$

We differentiate again $\frac{d\beta(\alpha)}{d\alpha}$ by α

$$\frac{d^2}{d\alpha^2}(\beta(\alpha)) = \frac{d}{d\alpha}\left(\frac{d\beta(\alpha)}{d\alpha}\right) = \frac{1}{\cosh x}\left(\frac{\sin(\beta(\alpha))(\sin\alpha)\frac{d\beta(\alpha)}{d\alpha} - \cos(\beta(\alpha))\cos\alpha}{\cos^2(\beta(\alpha))}\right)$$

Using the facts $\dfrac{1}{\cosh^2 x} = 1 - \tanh^2 x$ and $\dfrac{1}{\cos^2\beta(\alpha)} = 1 + \tan^2\beta(\alpha)$, and also substituting equations (12), (13), we obtain

$$\begin{aligned} \frac{d^2}{d\alpha^2}(\beta(\alpha)) &= -\frac{1}{\cosh^2 x}\left(\frac{\tan(\beta(\alpha))\sin^2\alpha + \cos(\beta(\alpha))\cos\alpha\cosh x}{\cos^2\beta(\alpha)}\right) \\ &= -\frac{1}{\cosh^2 x}\left(\frac{\tan^2(\beta(\alpha)) + 1}{\tan(\beta(\alpha))}\right)\left(\tan^2(\beta(\alpha))\sin^2\alpha + \cos^2\alpha\right) < 0 \quad \square \end{aligned}$$

Thus $\beta(\alpha)$ is a concave (from below) function. $\square$

Proof of Theorem 2. By Lemma 1, and because sine is a monotone increasing function in $(0, \frac{\pi}{2})$,

$$\sin\left(\beta\left(\frac{\alpha_1+\alpha_2}{2}\right)\right) > \frac{\sin(\beta(\alpha_1)) + \sin(\beta(\alpha_2))}{2} = \left(\frac{\sin\beta_1 + \sin\beta_2}{2}\right)$$

holds as a Jensen-type inequality (The graph of the function is over the segment $(\alpha_1;\beta_1)$ $(\alpha_2;\beta_2)$ in midpoint $\left(\frac{\alpha_1+\alpha_2}{2}\right)$, then this stands every point of the segment). Then the sum would be $\sum_i \beta_i > 2\pi$. **To equalize it again**, by the procedure in Theorem 1 with previous angles and two times $\frac{1}{2}(\alpha_1+\alpha_2)$ instead of α_1 and α_2, $\cosh x$ and x are to be chosen greater. In our local optimal cases where every possible equality has been reached, such increase in of x ($\cosh x$) by choosing $\left(\frac{\alpha_1+\alpha_2}{2}\right)$ is not possible, so x cannot increase in such a way. A comparison of these local optima serves the optimum since the existence has already been guaranteed by compactness of the domain of variables. □

5. Generalization to $G = [3, 3, 3 \ldots, 3]$ of $l \geq 4$ rotational centers each of order 3

Finally, we shall see that group $G = [3, 3, 3, \ldots, 3]$, with the l-times rotational center of order 3, $l \geq 4$. The largest inscribed circle radius could be attained by equalizing the angles corresponding to as many additional vertices as possible. We follow the following propositions in [2, 7] to study this general construction.

Proposition 1. *For the number w of additional points of an orbifold tree there holds*

$$w \leq 2\alpha g + l - q - 2$$

If n is the number of edges (and vertices) of a fundamental domain of a plane group G, then (with some exceptions if the domain is unique) there holds $n_{\min} \leq n \leq n_{\max}$, where

$$n_{\min} = 2\alpha g \text{ if } l = q = 0, \quad \text{or} \quad n_{\min} = q_0 + \left(\sum_{k=1}^{q} l_k\right) + 2\alpha g + 2l + 2q - 2, \text{ otherwise}$$

and

$$n_{\max} = \left(\sum_{k=1}^{q} l_k\right) + 6\alpha g + 4l + 5q - 6,$$

where $\alpha = 2$ if the orbifold is orientable and $\alpha = 1$, otherwise, and q_0 is the number of boundary components containing no dihedral corner. Moreover, for a given G there exist fundamental domains with $n_{\min}$ and $n_{\max}$ edges.

We study that in our cases $G = [3, 3, 3, \ldots, 3]$ above the l-rotational center are embedded into a topological sphere, i.e., $g = 0$. Since it is an orientable surface, $\alpha = 2$. Moreover, it has no boundary component, $q = q_0 = 0$. Applying these conditions to the proposition, we have Lemma 1 as follows:

Lemma 2. *In $G = [3, 3, 3, \ldots, 3]$ of l-rotational centers of order 3, $l \geq 4$, there are a possible number of additional points w that are bounded as follows:*

$$0 \leq w \leq l - 2, \tag{14}$$

Furthermore, the possible number n of sides (and vertices) of the fundamental polygon is given by:

$$2l - 2 \leq n \leq 4l - 6$$

Finally, we give the last theorem of this paper, namely the maximum radius of the inscribed circle into the fundamental domain of $G = [3, 3, 3, \ldots, 3]$.

Theorem 3. *Let $G = [3, 3, 3, \ldots, 3]$ be a group with l-rotational centers of order 3, $l \geq 4$. The largest inscribed circle radius in its fundamental domain is realized when $l-2$ additional points are given, and their corresponding vertex angles are equalized. Furthermore, the inscribed circle radius x is given by the formula:*

$$x = \cosh^{-1}\left(\frac{1}{2\sin\left(\frac{\pi}{4l-6}\right)}\right), \quad \textit{for all} \;\; l = 4, 5, 6, \ldots$$

We need some preparations to prove this theorem. We divided our discussion into the following three subsections with additional information.

5.1. On combinatorial structure to the tree graph of $G = [3, 3, 3, \ldots, 3]$

Firstly, the tree graph on the topological sphere for the corresponding fundamental domain can be obtained completely through the algorithm in [4], as indicated previously. Particularly, in this case, $G = [3, 3, 3, \ldots, 3]$, the tree graphs can be represented by the set of solutions for a "linear Diophantine equation system".
Let A_i be the number of rotational centers that have degree i in the tree surface graph, i.e., they have i edges connected. Hence, the total number of all A_i should be l, $\sum_{i=1}^{l-1} \mathrm{A}_i = l$. Note that the maximum possible degree of a rotational center is $l - 1$.
Again, let B_j be the number of additional points whose degree is j in the tree graph. The minimum degree of an additional point is 3, while the maximum possible degree is l, e.g. it happens in a star graph. Therefore, by adding all B_j, we get w, the total number of additional points, i.e., $\sum_{j=3}^{l} \mathrm{B}_j = w$. Furthermore, in our tree surface graph, the vertices can be either rotational centers or additional points. Note that the sum of all degrees of vertices in a graph is equal to 2 times the number of its edges. Since in a tree graph with v vertices the number of edges is $v - 1$, we can state the following equation:

$$\sum_{i=1}^{l-1} i \cdot \mathrm{A}_i + \sum_{j=3}^{l} j \cdot \mathrm{B}_j = 2(l + w - 1) = n,$$

where n is the number of vertices (sides of the fundamental polygon to the tree graph of vertices $v = l + w$ and edges $v - 1 = l + w - 1$).

Therefore, all of possible tree graphs for G have to satisfy the solutions $\{A_i, B_j\}$, $i = 1 \dots l-1$, $j = 3 \dots l$ of the following "linear Diophantine equation system":

$$\sum_{i=1}^{l-1} i \cdot \mathrm{A}_i + \sum_{j=3}^{l} j \cdot \mathrm{B}_j = 2(l + w - 1) = n \tag{15}$$

$$\sum_{i=1}^{l-1} \mathrm{A}_i = l \tag{16}$$

$$\sum_{j=3}^{l} \mathrm{B}_j = w \tag{17}$$

$$\mathrm{A}_i, \mathrm{B}_j, \in \mathbb{N} \cup \{0\}, \text{ where}(0 \le w \le l-2) \tag{18}$$

Example as before: Let $G = [3, 3, 3, 3]$, i.e., $l = 4$. Possible additional points are $w = 0, 1, 2$. The corresponding linear Diophantine equation system is given by:

$$\mathrm{A}_1 + 2\mathrm{A}_2 + 3\mathrm{A}_3 + 3\mathrm{B}_3 + 4\mathrm{B}_4 = 2(4 + w - 1)$$
$$\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3 = 4, \ \mathrm{B}_3 + \mathrm{B}_4 = w, \text{ where } w = 0, 1, 2.$$

The complete 5 solutions of the system above and their corresponding tree surface graphs, see Figure 1, are presented in the Table 2 for $l = 4$ rotational centers with

Additional points	A_1	A_2	A_3	B_3	B_4	Tree surface graph
0	2	2	0	0	0	Type-1
0	3	0	1	0	0	Type-2
1	4	0	0	0	1	Type-3
1	3	1	0	1	0	Type-4
2	4	0	0	2	0	Type-5

Table 2: *The Diophantine equation system solution and its tree surface graph representations for* $G = [3, 3, 3, 3]$

maximum $l - 2$ additional points, (14). Consider $w = l - 2$ maximum additional points added; then the corresponding solution of (15)-(18), is $\mathrm{A}_1 = l$, $\mathrm{A}_i = 0$ for $i \neq 1$, and $\mathrm{B}_3 = l - 2$, $\mathrm{B}_j = 0$ for $j \neq 3$. The corresponding inscribed circle radius of each linear Diophantine solution (tree surface graph types) could be described in the next two subsections.

5.2. The constrained optimum problem in a single equation

Consider a tree surface graph and its fundamental domain of G. Let R_i be a rotational center with i adjacent edges ($i \in \{1, 2, 3, \dots, l-1\}$). The pair of scissors dissecting in this tree surface graph yields the fundamental domain, particularly the rotational center with i edges are dissected into i identical angles, i.e., $\frac{1}{i}\frac{2\pi}{3}$. Furthermore, the corresponding trigonometric relation formed by the right triangle in Figure 5 can be written as follows: $\cos\left(\frac{\alpha_i}{2}\right) = \cosh x \sin\left(\frac{\beta_i}{2}\right)$ it leads to $\cos\left(\frac{1}{i}\frac{\pi}{3}\right) = \cosh x \cdot \sin\left(\frac{\beta_i}{2}\right)$. Then $\beta_i = 2\sin^{-1}\left(\frac{\cos\left(\frac{1}{i}\frac{\pi}{3}\right)}{\cosh x}\right)$, for $i = 1, 2, \ldots, l-1$. Particularly, if $i = 1$, i.e., the rotational center appears as a "leaf" in the tree surface graph, and we have

$$\cosh x = \frac{1}{2\sin\left(\frac{\beta_1}{2}\right)}. \tag{19}$$

Remark: The conditions $\cosh x > 1$ in (19) impact the boundness of β_1, i.e., we can define the interval for β_1, that is, $\beta_1 \in (0, \frac{\pi}{3})$.
By substituting the expression $\cosh x$ (19) into β_i's we obtain:

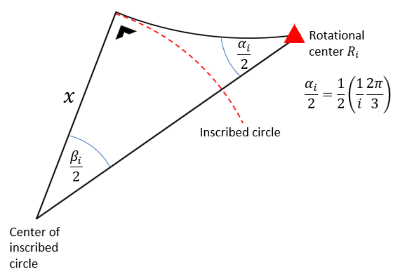

Figure 5: *Right triangle with rotational center. The larger the α_i, the smaller the β_i.*

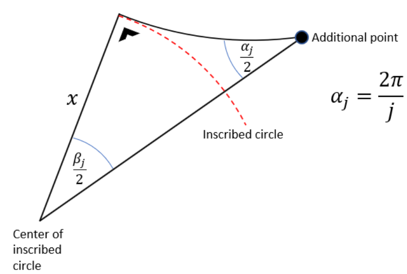

Figure 6: *Right triangle with additional point*

$$\beta_i = 2\sin^{-1}\left(2\cos\left(\frac{1}{i}\frac{\pi}{3}\right)\sin\left(\frac{\beta_1}{2}\right)\right) \tag{20}$$

Remark: The argument of $\sin^{-1}$ in (20) needs to be naturally on the interval $[-1, 1]$ (in this situation $[0, 1]$). This means that

$$0 \le 2\cos\left(\frac{1}{i}\frac{\pi}{3}\right)\sin\left(\frac{\beta_1}{2}\right) \le 1, \text{ for every } i = 1, \ldots, l-1,$$

and it means β_1 is bounded, i.e.,

$$0 \le \beta_1 \le 2\sin^{-1}\left(\frac{1}{2\cos\left(\frac{1}{i}\frac{\pi}{3}\right)}\right), \text{ for every } i = 1, \ldots, l-1. \tag{21}$$

It means β_1 is bounded by the least upper bound, i.e., $0 \le \beta_1 \le 2\sin^{-1}\left(\frac{1}{2\cos\left(\frac{1}{l-1}\frac{\pi}{3}\right)}\right)$, for fixed $l \ge 4$. A similar argumentation is applied to the right triangle with the additional point as vertex, see Figure 6. Unlike the rotational center case, in this case we have $\alpha_j = \frac{2\pi}{j}$. Then the trigonometric relationship in the triangle related to additional points is given by

$$\beta_j = 2\sin^{-1}\left(2\cos\left(\frac{\pi}{j}\right)\sin\left(\frac{\beta_1}{2}\right)\right), \text{ for } j = 3, 4, \ldots, l. \tag{22}$$

Again, since the argument of $\sin^{-1}$ should be on $[-1,1]$ (in our case $[0,1]$), by the analogous consideration as in (21), we have

$$0 \leq \beta_1 \leq 2\sin^{-1}\left(\frac{1}{2\cos\left(\frac{\pi}{j}\right)}\right), \text{ for every } j = 3, \ldots, l. \tag{23}$$

It means we have $0 \leq \beta_1 \leq 2\sin^{-1}\left(\frac{1}{2\cos\left(\frac{\pi}{l}\right)}\right)$, for fixed $l \geq 4$. The sum of all central angles of the inscribed circle, i.e., β_i's and β_j's should be equal to 2π, one complete rotation. That is, the following conditions should be fulfilled for every $\{\mathrm{A}_i; \mathrm{B}_j\}$ solutions of (15)-(18):

$$\sum_{i=1}^{l-1} i\mathrm{A}_i\beta_i + \sum_{j=3}^{l} j\mathrm{B}_j\beta_j = 2\pi \tag{24}$$

By substituting β's from (20) and (22) we have a nice relation as follows:

$$\sum_{i=1}^{l-1} i\mathrm{A}_i 2\sin^{-1}\left(2\cos\left(\frac{1}{i}\frac{\pi}{3}\right)\sin\left(\frac{\beta_1}{2}\right)\right) + \sum_{j=3}^{l} j\mathrm{B}_j 2\sin^{-1}\left(2\cos\left(\frac{\pi}{j}\right)\sin\left(\frac{\beta_1}{2}\right)\right) = 2\pi.$$

Note that based on (21) and (23), β_1 is defined on

$$0 \leq \beta_1 \leq 2\sin^{-1}\left(\frac{1}{2\cos\left(\frac{1}{l-1}\frac{\pi}{3}\right)}\right).$$

In this last equation, we need to find β_1 only to determine the corresponding inradius x in each Diophantine solution. For convenience, we write β_1 as β, and the upper bound $2\sin^{-1}\left(\frac{1}{2\cos\left(\frac{1}{l-1}\frac{\pi}{3}\right)}\right) =: K_l$, for fixed $l \geq 4$. Finally, we formulate our problem concretely as follows:

Lemma 3. *In each tree surface graphs of $G = [3, 3, 3, \ldots, 3]$ of $l \geq 4$ rotational centers of order 3 there is a Diophantine system* (15)-(18)*, its solution* $\{\mathrm{A}_i; \mathrm{B}_j\}$ *$i = 1, \ldots l-1$, $j = 3, \ldots l$; and the radius of inscribed circle x is obtained by*

$$\cosh x = \frac{1}{2\sin\left(\frac{\beta}{2}\right)},$$

where β is the root of equation

$$\sum_{i=1}^{l-1} i\mathrm{A}_i 2\sin^{-1}\left(2\cos\left(\frac{1}{i}\frac{\pi}{3}\right)\sin\left(\frac{\beta}{2}\right)\right) + \sum_{j=3}^{l} j\mathrm{B}_j 2\sin^{-1}\left(2\cos\left(\frac{\pi}{j}\right)\sin\left(\frac{\beta}{2}\right)\right) - 2\pi = 0, \tag{25}$$

in the interval $[0, K_l]$*, where* $K_l = 2\sin^{-1}\left(\frac{1}{2\cos\left(\frac{1}{l-1}\frac{\pi}{3}\right)}\right)$*. One could observe that the smaller root* β *is obtained, the larger inradius* x *is determined.* □

5.3. Proof of Theorem 3

By observations in Section 3, we have seen that the theorem holds in $G = [3, 3, 3, 3]$, $l = 4$. Hence, it is sufficient to prove the remaining cases, i.e., $l \geq 5$.
Firstly, we denote the previous function h in (25) in Lemma 3 as follows: For every fixed solution $\{\mathrm{A}_i, \mathrm{B}_j\}, i = 1 \ldots l-1, j = 3, \ldots l$ of Diophantine system (15)-(18), we define a function, extended at the endpoint of its interval

$$h : [0, K_l] \longrightarrow \mathbb{R}, \text{ where } K_l = 2\sin^{-1}\left(\frac{1}{2\cos\left(\frac{1}{l-1}\frac{\pi}{3}\right)}\right),$$

and h is defined by

$$\begin{aligned} h(\beta) = & \sum_{i=1}^{l-1} i\mathrm{A}_i 2\sin^{-1}\left(2\cos\left(\frac{1}{i}\frac{\pi}{3}\right)\sin\left(\frac{\beta}{2}\right)\right) \\ & + \sum_{j=3}^{l} j\mathrm{B}_j 2\sin^{-1}\left(2\cos\left(\frac{\pi}{j}\right)\sin\left(\frac{\beta}{2}\right)\right) - 2\pi. \end{aligned}$$

Observe that $K_l = 2\sin^{-1}\left(\frac{1}{2\cos\left(\frac{1}{l-1}\frac{\pi}{3}\right)}\right) < 2\sin^{-1}\left(\frac{1}{2\cos\left(\frac{\pi}{3}\right)}\right) = \pi$. Hence, $[0, K_l] \subset [0, \pi]$, in particular, $\frac{\beta}{2} \in [0, \frac{K_l}{2}] \subset [0, \frac{\pi}{2}]$. Note that h is a strictly increasing function in $[0, K_l]$ since h appears as a linear combination of composition terms of $\sin^{-1}$ and $\sin$, with $\sin\left(\frac{\beta}{2}\right)$ increasing on $[0, K_l] \subset [0, \frac{\pi}{2}]$, and also $\sin^{-1}$ increasing on $[0, 1]$. As Lemma 3 stated, once we solve $h(\beta) = 0$ for β, then the inradius x can be simply computed. In this setting, we want to minimize the root β. Remark: The existence of the root β in $[0, K_l]$ is guaranteed by the continuity of h. In fact, $h(0) = -2\pi < 0$, and also:

$$\begin{aligned} h(K_l) = & \sum_{i=1}^{l-1} i\mathrm{A}_i 2\sin^{-1}\left(2\cos\left(\frac{1}{i}\frac{\pi}{3}\right)\sin\left(\frac{K_l}{2}\right)\right) \\ & + \sum_{j=3}^{l} j\mathrm{B}_j 2\sin^{-1}\left(2\cos\left(\frac{\pi}{j}\right)\sin\left(\frac{K_l}{2}\right)\right) - 2\pi \\ \geq & \sum_{i=1}^{l-1} i\mathrm{A}_i 2\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) + \sum_{j=3}^{l} j\mathrm{B}_j 2\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) - 2\pi \\ = & \frac{\pi}{3}\left(\sum_{i=1}^{l-1} i\mathrm{A}_i + \sum_{j=3}^{l} j\mathrm{B}_j\right) - 2\pi = \frac{\pi}{3}\left(2(l-1+w)\right) - 2\pi \geq 0. \end{aligned}$$

Then by the intermediate value theorem, we may validate this claim. Moreover, since h is strictly increasing in $[0, K_l]$, the root is unique in that interval. Our claim is that the maximum inscribed circle radius r is realized by the solution to the Diophantine system whose number of additional points is the maximum $w = l - 2$. In this situation, the Diophantine system has exactly one unique solution, i.e., $\mathrm{A}_1 = l$, $A_i = 0$ for $i = 2, \ldots, l-1$ and $B_3 = l - 2$, $B_j = 0$ for $j = 4, \ldots, l$. The corresponding function h is $h_{l-2}(\beta) = (l + 3(l-2)) \cdot 2\sin^{-1}(\sin(\frac{\beta}{2})) - 2\pi = (4l - 6)\beta - 2\pi$. Clearly, the root of $h_{l-2}(\beta) = 0$ is $\beta_{l-2} = \frac{2\pi}{4l-6}$ that gives the corresponding radius $r_{l-2} = \cosh^{-1}\left(\frac{1}{2\sin(\frac{\beta_{l-2}}{2})}\right) = \cosh^{-1}\left(\frac{1}{2\sin(\frac{\pi}{4l-6})}\right)$, as expressed in the theorem.

Suppose indirectly that there exists a solution to the Diophantine system with fewer additional points w, $0 \leq w < l - 2$, say $\{\mathrm{A}_i^*, \mathrm{B}_j^*\}$, and the corresponding equation $h^*(\beta) = 0$, such that it has a root β^* whose resulting radius r^* is greater than r_{l-2}, $r^* > r_{l-2}$. It is equivalent to $\beta^* < \beta_{l-2}$. Since h^* is strictly increasing, $0 = h^*(\beta^*) < h^*(\beta_{l-2})$, i.e., $h^*(\beta_{l-2}) > 0$. Meanwhile, we have $h_{l-2}(\beta_{l-2}) = 0$ already. It would lead to the following inequality:

$$h^*(\beta_{l-2}) > 0,$$

or explicitly

$$\sum_{i=1}^{l-1} i\mathrm{A}_i^* 2\sin^{-1}\left(2\cos\left(\frac{1}{i}\frac{\pi}{3}\right)\sin\left(\frac{\beta_{l-2}}{2}\right)\right) + \sum_{j=3}^{l} j\mathrm{B}_j^* 2\sin^{-1}\left(2\cos\left(\frac{\pi}{j}\right)\sin\left(\frac{\beta_{l-2}}{2}\right)\right) - 2\pi > 0, \tag{26}$$

as an indirect assumption.

Substitute $\beta_{l-2} = \frac{2\pi}{4l-6}$ and apply the Jensen-type inequalities from the Appendix:

$$\sin^{-1}\left[2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{3i}\right)\right] < \frac{3}{\pi}\sin^{-1}\left[2\sin\left(\frac{\pi}{4l-6}\right)\right]\left(\frac{\pi}{2} - \frac{\pi}{3i}\right),$$

and

$$\sin^{-1}\left[2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{j}\right)\right] < \frac{3}{\pi}\sin^{-1}\left[2\sin\left(\frac{\pi}{4l-6}\right)\right]\left(\frac{\pi}{2} - \frac{\pi}{j}\right),$$

for $i = 1, \ldots, l-1$ and $j = 3, \ldots, l$. Therefore, in (26) the sums will be much simpler, we can refer to equations (15)-(18) in the Diophantine system for $(\mathrm{A}_i^*, \mathrm{B}_j^*)$ and we obtain the following:

$$\frac{3}{\pi}\sin^{-1}\left[2\sin\left(\frac{\pi}{4l-6}\right)\right]\left\{\frac{\pi}{2}\left[2(l+w-1)\right] - \frac{\pi}{3}l - \pi w\right\} - 2\pi > 0,$$

i.e.,

$$\sin^{-1}\left[2\sin\left(\frac{\pi}{4l-6}\right)\right](2l-3) > 2\pi, \text{ then } 2\sin\left(\frac{\pi}{4l-6}\right) > \sin\left(4 \cdot \frac{\pi}{4l-6}\right).$$

Since $l \geq 5$, thus $4l-6 \geq 14$, we get a contradiction in interval $(0, \frac{\pi}{14}]$ by a simple analysis of the sine function. $\square$

5.3.1. Remark about the area of fundamental domain $\mathcal{F}_G$

We have just found the optimum radius of the inscribed circle of G. This optimal radius provides the optimum density of the circle in its fundamental domain $\mathcal{F}_G$ immediately. Since the following observation shows that the area of $\mathcal{F}_G$ is constant for every Diophantine solution. The area of the fundamental domain $\mathcal{F}_G$ is proportional to the angle defect $\triangle$. In fact, $\mathcal{F}_G$ can be dissected into a number of right triangles, as illustrated in Figure 4, where the area of each right triangle could be simply computed through its defect angle, see figures 5, 6. Let $\triangle_i$ be the defect angle of the right triangle about rotational center R_i, (Figure 5). Similarly, let $\triangle_j$ be the defect angle of the right triangle about the rotational point (Figure 6). The dissection of $\mathcal{F}_G$ gives the result

$$\text{Area } \mathcal{F}_G = \sum_{i=1}^{l-1} i \text{ A}_i \; 2 \; \triangle_i + \sum_{j=3}^{l} j \text{ B}_j \; 2 \; \triangle_j,$$

where $\{\text{A}_i; \text{B}_j\}$ is a Diophantine solution in (15)-(18).
Now, by considering figures 5-6, the defect angles are exactly $\triangle_i = \pi - \frac{\pi}{2} - \frac{\alpha_i}{2} - \frac{\beta_i}{2}$ and $\triangle_j = \pi - \frac{\pi}{2} - \frac{\alpha_j}{2} - \frac{\beta_j}{2}$. Furthermore, by Diophantine conditions in (15)-(18) together with the central angle condition in (24), we can conclude

$$\begin{aligned}\text{Area } \mathcal{F}_G &= \sum_{i=1}^{l-1} i \text{ A}_i \; 2 \; \triangle_i + \sum_{j=3}^{l} j \text{ B}_j \; 2 \; \triangle_j \\ &= 2\sum_{i=1}^{l-1} i \text{ A}_i \; \left(\frac{\pi}{2} - \frac{\alpha_i}{2} - \frac{\beta_i}{2}\right) + 2\sum_{j=3}^{l} j \text{ B}_j \; \left(\frac{\pi}{2} - \frac{\alpha_j}{2} - \frac{\beta_j}{2}\right) \\ &= \pi(2(l+w-1)) - \frac{2}{3}\pi l - 2\pi w - 2\pi = \left(\frac{4}{3}l - 4\right)\pi. \; \square\end{aligned}$$

6. Appendix

Lemma 4. *The upper bounds of* $\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{3i}\right)\right)$ *and* $\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{j}\right)\right)$ *are given by*

$$\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{3i}\right)\right) < \frac{3}{\pi}\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\right)\left(\frac{\pi}{2} - \frac{\pi}{3i}\right),$$

and

$$\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{j}\right)\right) < \frac{3}{\pi}\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\right)\left(\frac{\pi}{2} - \frac{\pi}{j}\right),$$

for all $i = 1 \dots l-1$, $j = 3, \dots, l$ *and* $l \geq 5$.

Proof. We will provide the proof for the first inequality, then the second inequality can be proven in a similar way. Consider

$$2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{3i}\right) < 2\sin\left(\frac{\pi}{4l-6}\right) < \cos\left(\frac{\pi}{3i}\right),\ \text{for all } i=1,\dots l-1,\ l\geq 5.$$

Since $\sin^{-1}$ is increasing in $(0,1]$, then we have

$$\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{3i}\right)\right) < \sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\right) < \sin^{-1}\left(\cos\left(\frac{\pi}{3i}\right)\right).$$

Since $\sin^{-1}$ is concave up, then the slope of its secant line through the origin $(0,0)$ and $(x,\sin^{-1}(x))$ is increasing, that is, $\frac{\sin^{-1}(x_1)}{x_1} < \frac{\sin^{-1}(x_2)}{x_2}$, if $x_1 < x_2$. Therefore,

$$\frac{\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{3i}\right)\right)}{2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{3i}\right)} < \frac{\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\right)}{2\sin\left(\frac{\pi}{4l-6}\right)} < \frac{\sin^{-1}\left(\cos\left(\frac{\pi}{3i}\right)\right)}{\cos\left(\frac{\pi}{3i}\right)}.$$

Now, we multiply all (positive) sides in the inequality by (positive) $2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{3i}\right)$ to have

$$\begin{aligned}\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{3i}\right)\right) &< \sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\right)\cdot\cos\left(\frac{\pi}{3i}\right)\\ &< 2\sin\left(\frac{\pi}{4l-6}\right)\cdot\sin^{-1}\left(\cos\left(\frac{\pi}{3i}\right)\right).\end{aligned}$$

Hence, we have

$$\frac{\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{3i}\right)\right)}{2\sin\left(\frac{\pi}{4l-6}\right)\cdot\sin^{-1}\left(\cos\left(\frac{\pi}{3i}\right)\right)} < \frac{\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\right)\cdot\cos\left(\frac{\pi}{3i}\right)}{2\sin\left(\frac{\pi}{4l-6}\right)\cdot\sin^{-1}\left(\cos\left(\frac{\pi}{3i}\right)\right)}.$$

Note that $\cos\left(\frac{\pi}{3i}\right)\geq\cos\left(\frac{\pi}{3}\right)$ for all $i=1\dots l-1$. Since $\sin^{-1}$ is concave up, then we have $\frac{\sin^{-1}\left(\cos\left(\frac{\pi}{3i}\right)\right)}{\cos\left(\frac{\pi}{3i}\right)}\geq\frac{\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)}{\cos\left(\frac{\pi}{3}\right)}=\frac{\pi}{3}$. Therefore, $\frac{\cos\left(\frac{\pi}{3i}\right)}{\sin^{-1}\left(\cos\left(\frac{\pi}{3i}\right)\right)}\leq\frac{3}{\pi}$. Then we have

$$\frac{\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{3i}\right)\right)}{2\sin\left(\frac{\pi}{4l-6}\right)\cdot\sin^{-1}\left(\cos\left(\frac{\pi}{3i}\right)\right)} < \frac{3}{\pi}\frac{\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\right)}{2\sin\left(\frac{\pi}{4l-6}\right)}.$$

By simplifying

$$\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{3i}\right)\right) < \frac{3}{\pi}\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\right)\sin^{-1}\left(\cos\left(\frac{\pi}{3i}\right)\right)$$

since $\sin^{-1}\left(\cos\left(\frac{\pi}{3i}\right)\right)=\left(\frac{\pi}{2}-\frac{\pi}{3i}\right)$, then

$$\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\cos\left(\frac{\pi}{3i}\right)\right) < \frac{3}{\pi}\sin^{-1}\left(2\sin\left(\frac{\pi}{4l-6}\right)\right)\left(\frac{\pi}{2}-\frac{\pi}{3i}\right),$$

as we claimed. $\square$

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