ISSN 1330–0016
CODEN FIZBE7

THE TRIANGLE WILSON LOOP IN $1+1$ DIMENSIONS

ANĐELKA ANDRAŠI

Rudjer Bošković Institute, 41001 Zagreb, P. O. B. 1016, Croatia

Dedicated to Professor Mladen Paić on the occasion of his 90$^{\mathbf{th}}$ birthday

Received 17 May 1995

UDC 539.12

PACS 11.10.Kk, 11.15.-q

We study the triangle Wilson loop in $2+\epsilon$ dimensions to order g^2 in the lightcone gauge with Mandelstam-Leibbrandt prescription. The complete result agrees with the calculation performed in the Feynman gauge. However, at intermediate stages the new 'ambiguous' terms of the form $\omega^{\frac{\epsilon}{2}-1}\epsilon^{-1}$ appear which are not controlled by any sort of Ward identity.

1. *Introduction*

The favourite among non-covariant gauges is the lightcone gauge with Mandelstam's prescription [1]. This gauge has been extensively used in QCD, supersymmetric Yang-Mills, string theories and gravity. It decreases the number of diagrams in a particular problem, however the individual diagrams are more complicated to evaluate than in the Feynman gauge. Mandelstam's lightcone gauge is defined by two lightlike vectors satisfying

$$n^2 = n^{*2} = 0, \quad n \cdot n^* = 2, \tag{1}$$

and the propagator

$$\frac{\delta_{ab}}{k^2+\mathrm{i}\eta}\{g_{\mu\nu}-\frac{k_\mu n_\nu+k_\nu n_\mu}{n\cdot k+\mathrm{i}\omega n^*\cdot k}\}. \tag{2}$$

However, this prescription shows unusual features. To higher orders in perturbation theory it leads to non-local counterterms [2] and does not satisfy the optical theorem [3]. The Wilson loop [4] is a gauge invariant operator and therefore a convenient laboratory for testing various prescriptions. Here we take the triangle Wilson loop with two sides in the direction of the two lightlike vectors and the base in spacelike direction, defined by

$$x^1_\mu = n^*_\mu L - n_\mu L t$$

$$x^2_\mu = v_\mu 2L(1-s)$$

$$x^3_\mu = n^*_\mu L t \tag{3}$$

where

$$n^*_\mu = (1,0,0,1)$$

$$n_\mu = (1,0,0,-1)$$

$$v_\mu = (0,0,0,1)$$

$$0 \le t \le 1$$

$$0 \le s \le 1. \tag{4}$$

2. *Self energy diagram*

While in the Feynman gauge there are four diagrams contributing to the complete result for the triangle Wilson loop to order g^2, in the lightcone gauge there are only two. These are the self-energy diagram on the spacelike Wilson line and the vertex graph where the gluon propagates between the base in the direction v_μ and the side in the direction n^*_μ. The self-energy diagram in the momentum space contributes

$$W_2 = \mathrm{i}g^2 C_R \int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{1}{k^2+\mathrm{i}\eta}\frac{k_-}{k_+ + \mathrm{i}\omega k_-}\frac{1}{k_3^2}(\cos 2Lk_3 - 1) \tag{5}$$

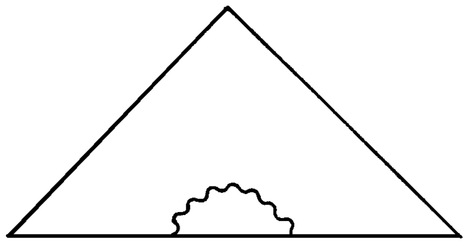

Fig. 1. The self energy diagram with the gluon exchanged on the spacelike Wilson line.

where

$$k_- = n^* \cdot k = k_0 - k_3,$$

$$k_+ = n \cdot k = k_0 + k_3. \tag{6}$$

Closeing the contour in the upper k_0 half plane, we meet two poles

$$k_0 = -k + \mathrm{i}\eta$$

$$k_0 = -k_3 + 2\mathrm{i}\omega k_3 \theta(k_3), \tag{7}$$

which give

$$W_2 = \pi g^2 C_R (2\pi)^{-n} \int \mathrm{d}k_3 \mathrm{d}^\epsilon K \frac{1}{k} \frac{k + k_3}{k - k_3 + \mathrm{i}\omega(k + k_3)} \frac{1}{k_3^2} (\cos 2Lk_3 - 1)$$

$$-4\pi g^2 C_R (2\pi)^{-n} \int_0^\infty \mathrm{d}k_3 \int \mathrm{d}^\epsilon K \frac{1}{K^2 + 4\mathrm{i}\omega k_3^2 - \mathrm{i}\eta} \frac{1}{k_3} (\cos 2Lk_3 - 1). \tag{8}$$

Naively, one would let $\omega \to 0$ before $\epsilon = 2 - n$, in the integrand. Then, the latter integral in Eq.(8) is a tadpole in the perpendicular momentum K, and vanishes according to the postulates of dimensional regularization. However, after the introduction of polar coordinates

$$k_3 = k \cos\theta = kx,$$

$$\mathrm{d}^{1+\epsilon} k = k^\epsilon \mathrm{d}k (1 - x^2)^{\frac{\epsilon}{2} - 1} \times \int \mathrm{d}\phi \tag{9}$$

where

$$\int \mathrm{d}\phi = \frac{2\pi^{\frac{\epsilon}{2}}}{\Gamma(\frac{\epsilon}{2})} \tag{10}$$

is the integral over the remaining angles in $1+\epsilon$ dim. space, the first integral leads to an integral which is not defined for any ϵ (we change the variable of integration $x^2 = y$). Therefore we have to keep ω to the end of the calculation.

$$W_2 = 2\pi i\Gamma(\epsilon-2)\cos\frac{\epsilon\pi}{2}(2L)^{2-\epsilon}\frac{2\pi^{\frac{\epsilon}{2}}}{\Gamma(\frac{\epsilon}{2})}g^2C_R(2\pi)^{-n}$$

$$\times\{\int_0^1 \mathrm{d}y(1-y)^{\frac{\epsilon}{2}-1}y^{\frac{1-\epsilon}{2}}\{1-y(1-4\mathrm{i}\omega)\}^{-1}+\frac{1}{2}\int_0^1 \mathrm{d}y(1-y)^{\frac{\epsilon}{2}-1}y^{-\frac{\epsilon}{2}-\frac{1}{2}}\}$$

$$-4\pi\mathrm{i}\pi^{\frac{\epsilon}{2}}\Gamma(1-\frac{\epsilon}{2})\Gamma(\epsilon-2)\omega^{\frac{\epsilon}{2}-1}L^{2-\epsilon}e^{-\frac{\mathrm{i}\pi\epsilon}{4}}\cos\frac{\epsilon\pi}{2}g^2C_R(2\pi)^{-n} \qquad (11)$$

Had we neglected ω in Eq.(11), the first integral in the curly brackett would be defined in the region

$$2 < \epsilon < 3 \qquad (12)$$

and the second in the region

$$0 < \epsilon < 1. \qquad (13)$$

We choose to evaluate the integrals in the strip Eq.(13) in which the Feynman gauge integrals are defined. Then the only unknown integral in Eq.(11) is

$$I = \int_0^1 \mathrm{d}y(1-y)^{\frac{\epsilon}{2}-1}y^{\frac{1-\epsilon}{2}}\{1-y(1-4\mathrm{i}\omega)\}^{-1} = B(\frac{3-\epsilon}{2},\frac{\epsilon}{2})F(1,\frac{3-\epsilon}{2};\frac{3}{2};1-4\mathrm{i}\omega). \qquad (14)$$

Expanding the hypergeometric function in powers of ω and neglecting the terms of order $\omega^{\frac{\epsilon}{2}}$ and higher which vanish in the limit $\omega\to 0$ before $\epsilon\to 0$, it becomes

$$I = B(\frac{3-\epsilon}{2},\frac{\epsilon}{2})\{\frac{\Gamma(\frac{3}{2})\Gamma(\frac{\epsilon}{2}-1)}{\Gamma(\frac{1}{2})\Gamma(\frac{\epsilon}{2})}+(4\omega e^{\frac{\mathrm{i}\pi}{2}})^{\frac{\epsilon}{2}-1}\frac{\Gamma(\frac{3}{2})\Gamma(1-\frac{\epsilon}{2})}{\Gamma(\frac{3-\epsilon}{2})}\}. \qquad (15)$$

Then two terms in Eq.(11) each contain the factor $\omega^{\frac{\epsilon}{2}-1}\epsilon^{-1}$ which has no limit as $\omega\to 0$. However these poles cancel in the sum leaving a finite contribution to the self energy.

$$W_2 = 2\pi\mathrm{i}L^2\cdot g^2C_R(2\pi)^{-2}. \qquad (16)$$

The same diagram in the Feynman gauge contains single divergences.

3. The vertex graph

We devide the integrand for the vertex graph into three parts,

$$W_1 = W_1^a + W_1^b + W_1^c \tag{17}$$

where

$$W_1^a = -2\mathrm{i}g^2 C_R \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{1}{k^2 + \mathrm{i}\eta} \frac{1}{k_3} \frac{1}{k_+ + \mathrm{i}\omega k_-} (e^{\mathrm{i}k_+ L} - 1) \tag{18}$$

$$W_1^b = 2\mathrm{i}g^2 C_R \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{1}{k^2 + \mathrm{i}\eta} \frac{1}{k_3} \frac{1}{k_+ + \mathrm{i}\omega k_-} (e^{\mathrm{i}k_- L} - 1) \tag{19}$$

$$W_1^c = 2\mathrm{i}g^2 C_R \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{1}{k^2 + \mathrm{i}\eta} \frac{1}{k_+ + \mathrm{i}\omega k_-} \frac{1}{k_3} (e^{2\mathrm{i}k_3 L} - 1). \tag{20}$$

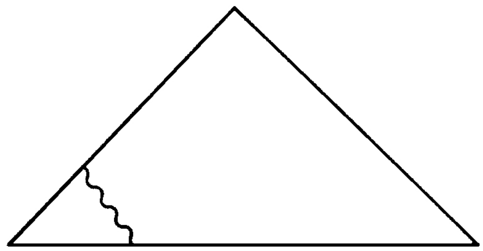

Fig. 2. The vertex graph W_1. The gluon propagates between the Mandelstam lightlike vector n^* and the spacelike Wilson line.

Again we close the contour in the upper k_0 half plane. Using

$$\int_0^\infty \mathrm{d}k k^{\epsilon-3} \sin^2 Lkx = -\Gamma(\epsilon-2) \cos\frac{(2-\epsilon)\pi}{2} \frac{1}{2} (2Lx)^{2-\epsilon} \tag{21}$$

we obtain

$$W_1^a = -2\pi\mathrm{i}\Gamma(\epsilon-2) L^{2-\epsilon} e^{\frac{\mathrm{i}\pi}{2}(2-\epsilon)} \frac{2\pi^{\frac{\epsilon}{2}}}{\Gamma(\frac{\epsilon}{2})} g^2 C_R (2\pi)^{-n} \times E. \tag{22}$$

The remaining integral we devide into two parts

$$E = \int_0^1 \mathrm{d}x (1-x^2)^{\frac{\epsilon}{2}-1} \frac{1}{x} [(1-x)^{1-\epsilon} - (1+x)^{1-\epsilon}]$$

$$= \int_0^1 \mathrm{d}x(1+x)^{\frac{\epsilon}{2}-1}(1-x)^{\frac{\epsilon}{2}}\frac{1}{x}[(1-x)^{1-\epsilon}-(1+x)^{1-\epsilon}]$$

$$+ \int_0^1 \mathrm{d}x(1+x)^{\frac{\epsilon}{2}-1}(1-x)^{\frac{\epsilon}{2}}\frac{1}{1-x}[(1-x)^{1-\epsilon}-(1+x)^{1-\epsilon}]. \tag{23}$$

The factor multiplying E in Eq. (22) is finite for $\epsilon \to 0$ and therefore we can take ϵ -expansion of the f irst part in Eq. (23).

$$E = -2\ln 2 + \frac{1}{1-\frac{\epsilon}{2}}F(1,1-\frac{\epsilon}{2};2-\frac{\epsilon}{2};-1) - \frac{2}{\epsilon}F(1,\frac{\epsilon}{2};1+\frac{\epsilon}{2};-1). \tag{24}$$

Here to order ϵ we have

$$F(1,1,2,-1) = \ln 2 \tag{25}$$

$$\frac{2}{\epsilon}F(1,\frac{\epsilon}{2};1+\frac{\epsilon}{2};-1) = \frac{2}{\epsilon}\times 2^{-\frac{\epsilon}{2}}F(\frac{\epsilon}{2},\frac{\epsilon}{2};1+\frac{\epsilon}{2};\frac{1}{2}) = \frac{2}{\epsilon} - \ln 2. \tag{26}$$

The exact result for W_1^a contains single poles $\frac{1}{\epsilon}$

$$W_1^a = -2\pi\mathrm{i}\Gamma(\epsilon-2)L^{2-\epsilon}e^{\frac{\mathrm{i}\pi}{2}(2-\epsilon)}\frac{2\pi^{\frac{\epsilon}{2}}}{\Gamma(\frac{\epsilon}{2})}(-\frac{2}{\epsilon})g^2C_R(2\pi)^{-n}. \tag{27}$$

The integrands W_1^b and W_1^c show the same features as the self energy diagram. At intermediate stages there are ambiguous terms, which however cancel.

$$W_1^b = W_1^a \tag{28}$$

$$W_1^c = -2\pi\mathrm{i}\Gamma(\epsilon-2)\cos\frac{\epsilon\pi}{2}(2L)^{2-\epsilon}\frac{\Gamma(\frac{\epsilon}{2}-1)\Gamma(\frac{3-\epsilon}{2})}{\Gamma(\frac{1}{2})}\times\frac{2\pi^{\frac{\epsilon}{2}}}{\Gamma(\frac{\epsilon}{2})}g^2C_R(2\pi)^{-n}. \tag{29}$$

The complete vertex graph contributes

$$W_1 = 2\pi\mathrm{i}L^2(\mathrm{i}\pi-1)g^2C_R(2\pi)^{-2}. \tag{30}$$

The total result for the triangle Wilson loop in 2 dim. space is

$$W = W_1 + W_2 = -\frac{1}{2}L^2g^2C_R. \qquad (31).$$

4. The Feynman gauge

Here we only list the final results for the individual graphs using Feynman gauge.

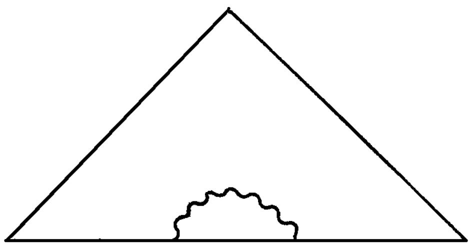

Fig. 3. The self energy graph on the spacelike Wilson line in the Feynman gauge.

Self energy

$$W_4 = -4\pi \mathrm{i} g^2 C_R \pi^{\frac{\epsilon}{2}} \Gamma(\frac{\epsilon}{2})(2-\epsilon)^{-1}(1-\epsilon)^{-1}(2\pi)^{-n} L^{2-\epsilon} \tag{32}$$

It contains single poles.

Vertex I

$$W_1 = -4\pi \mathrm{i} g^2 C_R \pi^{\frac{\epsilon}{2}} \Gamma(\frac{\epsilon}{2}) L^{2-\epsilon} \frac{1}{2-\epsilon} \{ e^{\frac{\mathrm{i}\pi}{2}(2-\epsilon)} B(\frac{2-\epsilon}{2}, 3-\frac{\epsilon}{2}) - \frac{1}{2-\epsilon} \} \tag{33}$$

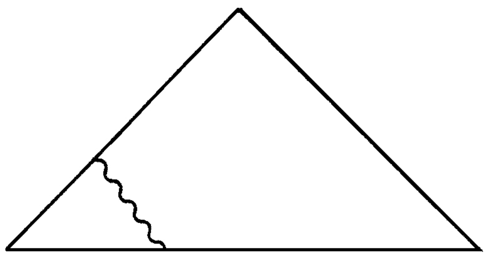

Fig. 4. Vertex I. The gluon propagates between the spacelike Wilson line and the lightlike vector n^*.

Vertex II

$$W_2 = W_1 \tag{34}$$

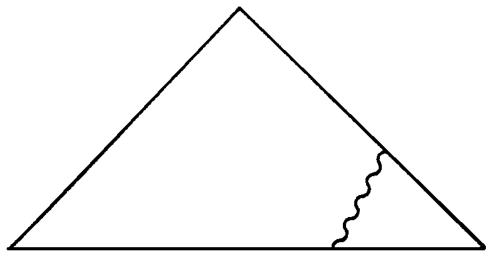

Fig. 5. Vertex II. The gluon is exchanged between the spacelike Wilson line and the line in the direction of the lightlike vector n.

Vertex III

$$W_3 = -8\pi \mathrm{i} g^2 C_R \pi^{\frac{\epsilon}{2}} \Gamma(\frac{\epsilon}{2}) L^{2-\epsilon} (2-\epsilon)^{-2} (2\pi)^{-n}. \tag{35}$$

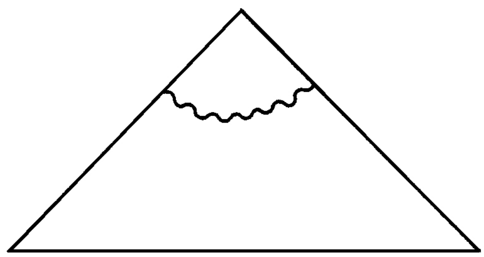

Fig. 6. Vertex III. The gluon is exchanged between the two lightlike Wilson lines.

The complete exact result in the Feynman gauge is

$$W = W_1 + W_2 + W_3 + W_4 \tag{36}$$

$$W = -4\pi \mathrm{i} g^2 C_R \pi^{\frac{\epsilon}{2}} \Gamma(\frac{\epsilon}{2}) L^{2-\epsilon} \{ \frac{4}{(2-\epsilon)^2} e^{\frac{\mathrm{i}\pi}{2}(2-\epsilon)} \frac{\Gamma^2(2-\frac{\epsilon}{2})}{\Gamma(3-\epsilon)} + \frac{1}{(2-\epsilon)(3-\epsilon)} \} \tag{37}$$

Each diagram in the Feynman gauge contains single divergences, but single poles cancel in the sum leaving a finite contribution which coincides with Eq. (31).

5. Discussion

We have evaluated the triangle Wilson loop using the M-L lightcone gauge and the Feynman gauge in 2 dim. space. The results do agree. However at intermediate stages the M-L prescription leads to ill defined integrals giving ambiguous poles $\omega^{\frac{\epsilon}{2}-1}\epsilon^{-1}$. These poles cancel to order g^2 in perturbation theory, but we do not know their behaviour to higher orders. Individual diagrams have different degrees of divergence depending on the gauge.

References

1) S. Mandelstam, Nucl. Phys. **B213** (1983) 149;

2) G. Leibbrandt, Phys. Rev. **D29** (1984) 1699;

3) A. Andraši and J. C. Taylor, Nucl. Phys. **B310** (1988) 222;

4) K. G. Wilson, Phys. Rev. **D10** (1974) 2445.

TROKUTASTA WILSONOVA PETLJA U $1+1$ DIMENZIJI

ANĐELKA ANDRAŠI

Institut Ruđer Bošković, 41001 Zagreb, P. O. B. 1016, Hrvatska

UDK 539.12

PACS 11.10.Kk, 11.15.-q

Proučavana je trokutasta Wilsonova petlja u $2+\epsilon$ dimenziji do reda g^2 u uvjetu svjetlosnog konusa i Mandelstam-Leibbrandtovoj preskripciji. Konačni rezultat slaže se s računom provedenim u Feynmanovom baždarnom uvjetu. Međutim u međukoracima pojavljuju se novi "proizvoljni" članovi oblika $\omega^{\frac{\epsilon}{2}-1}\epsilon^{-1}$ koji ne podliježu kontroli Wardovih identiteta.