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# Risk assessment in project management by a graph-theory-based group decision making method with comprehensive linguistic preference information

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**ABSTRACT**

Risk assessment is a vital part in project management. It is possible that experts may provide comprehensive linguistic preference information in distinct forms with respect to different aspects of the risk assessment problem in investment management. It is a challenge to model and deal with comprehensive linguistic preference assessments in multiple forms given by experts. In this regard, this paper defines the generalised probabilistic linguistic preference relation (GPLPR) to represent different forms of linguistic preference information in a unified structure. Then, a probability cutting method is proposed to simplify the representation of a GPLPR. Afterwards, a graph-theory-based method is developed to improve the consistency degree of a GPLPR. A group decision making method with GPLPRs is then proposed to carry on the risk assessment in project management. Discussions regarding the comparative analysis and managerial insights are given.

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## 1. Introduction

Risk assessment refers to the quantitative analysis of the impact on people's life and property caused by risk events. Risk assessment is an important task in determining project investment. Due to the lack of reliable historical data, risk assessments are usually based on the experience and knowledge of experts (Qiu et al., 2018). When giving assessments, it is easier for experts to give preference information based on pairwise comparisons of alternatives, compared with giving comprehensive assessments of each alternative directly. A preference relation is a matrix in which each element is a preference degree between two objects (Saaty, 1980). The preference relation is a good tool to reduce the difficulty in risk representation (Tang et al., 2018).

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Originally, the elements of a preference relation were represented in crisp numbers (Saaty, 1980). With the expertise of experts being enriched, experts could provide precise and complex preference information. In this regard, the preference information given by experts may be multiple values (Zhu et al., 2014) or intervals (Zhang & Pedrycz, 2019). For decision making problems associated with qualitative criteria, experts may prefer to give their preference information in single (Xu, 2004a), multiple (Wang & Xu, 2015; Zhu & Xu, 2014) or interval linguistic terms (Xu, 2004b). In addition, to address the different importance of linguistic terms, the probabilistic linguistic preference relation (PLPR) was presented (Zhang, et al., 2016). The PLPR assigns probabilities to single linguistic terms to represent the preference information; however, multiple or interval linguistic terms are ubiquitous in people's preference perceptions. When different forms of linguistic preferences such as single, multiple or interval linguistic representations occur simultaneously, as far as we know, there is no good method to represent such type of uncertain preference information. To solve this issue, this study introduces a generalised PLPR (GPLPR) to model single, multiple and interval linguistic preference assessments in a unified structure.

The consistency of a preference relation ensures that the pairwise assessments are logical. Failure to meet the requirement of consistency may lead to wrong decision results. The consistency of a preference relation was originally defined by transitivity, such as the additive transitivity (Herrera-Viedma et al., 2004). There were many studies about the consistency measurement and improving procedures for PLPRs (Gao et al., 2020; Zhang et al., 2016). For the GPLPR which is more complex than the PLPR, it is necessary to propose an effective method to deal with its consistency issue. Xu et al. (2013) introduced the graph theory (Boffey, 1982) to denote the consistency of preference relations intuitively. Wang and Xu (2015) pointed out that if the weights of a digraph are well defined, then the additive consistency can be explained intuitively by the graph theory. Based on the graph theory, scholars have defined the consistency indices for distinct preference relations, such as the fuzzy preference relation (Xu et al., 2013), extended hesitant fuzzy linguistic preference relation (Wang & Xu, 2015) and PLPR (Zhang et al., 2016). Inspired by the aforementioned work, this study develops a probability cutting method to simplify the consistency measurement process for GPLPRs and proposes a graph theory-based group decision making method with GPLPRs.

This study dedicates to conduct the following innovative work:

1. The GPLPR is proposed to facilitate experts to express assessments in risk assessment.
2. The probability cutting method is proposed to simplify the representation of a GPLPR.
3. The consistency index of a GPLPR based on the graph theory is introduced to check and improve the additive consistency of a GPLPR. Then, a graph theory-based group decision making method with GPLPRs is proposed. The method is further implemented in a case study regarding the risk assessment of investment projects to validate the efficiency of the proposed method.

This paper is organized as follows: Section 2 recalls the related work. Section 3 proposes a graph-theory-based group decision making method with generalised linguistic preference information. Section 4 applies the proposed method to deal with the risk assessment of investment projects, and the related discussions are given in Section 5. Section 6 ends the paper.

## 2. Related work

Before introducing our theoretical model, we provide a short review on the risk assessment in project management. Then, we introduce the PLPR and GPLTS to facilitate further presentation.

### 2.1. A short review for the risk assessment in project management

Investment risk refers to the uncertainty in future investment, which may lead to the loss of profit or even loss of principal. The idea of risk assessment before decision making was first put forward by Athenians (Bernstein, 1996). It was pointed out that there is a significant relationship between risk management methods and the success of projects (Acharyya, 2008; Voetsch et al., 2004). Risk assessment refers to the comprehensive analysis of the possibility of risk and the degree of loss, combined with other factors to comprehensively analyze investment projects. When deciding whether to invest or not, it is necessary to first carry out detailed and systematic risk assessments, and then make decisions according to such assessments. Risk assessment has attracted the attention of many scholars in the past decades (see Table 1 for details).

It can be seen from Table 1 that experts tend to give linguistic evaluations when evaluating project risks, which may be single term, a set of multiple terms or interval linguistic term. Probability has been introduced to express the preference information given by experts (Liang et al., 2021; Zhang et al., 2016), and sometimes the probability may be incomplete. In addition, to reduce the difficulty of project selection, preference relations have been introduced to establish evaluation procedures (Tang et al., 2018; Zeng et al., 2007; Zhang et al., 2016). However, there is still no model which can not only express linguistic preferences with multiple forms, but also allow incomplete information. How to establish a risk assessment process with incomplete linguistic preference assessments in multiple forms is an unsolved problem.

### 2.2. Probabilistic linguistic preference relation

Suppose that a discrete linguistic term set (LTS) is $S_0 = \{s_\alpha | \alpha = 1, 2, \dots, t\}$, where $t$ is a positive integer (Herrera et al., 1996). A symmetric set of discrete LTS $S = \{s_\alpha | \alpha = -\tau, \dots, 0, \dots, \tau\}$ was proposed to intuitively express the meanings of linguistic terms, satisfying: (1) $s_i > s_j$, if $i > j$; (2) $neg(s_\alpha) = s_{-\alpha}$, where $\tau$ is a positive integer (Xu, 2004a). A symmetric continuous LTS can be defined as $\bar{S} = \{s_\alpha | \alpha \in [-q, q]\} (q > \tau)$. To further deal with uncertainty, Xu (2004b) proposed an uncertain LTS as $\tilde{S} = \left\{\tilde{s} | \tilde{s} = [s_i, s_j], -\tau \le i \le j \le \tau\right\}$. For any three uncertain linguistic terms $\tilde{s} = [s_i, s_j]$, $\tilde{s}_1 = [s_{i1}, s_{j1}]$, $\tilde{s}_2 = [s_{i2}, s_{j2}] \in \tilde{S}$, and $\mu, \mu_1, \mu_2 \in [0, 1]$, the following operations were defined (Xu, 2004b): 1) $\mu\tilde{s} = \mu[s_i, s_j] = [s_{\mu i}, s_{\mu j}]$; 2)$(\mu_1 + \mu_2)s_1 = \mu_1 s_1 + \mu_2 s_1$.

**Table 1.** Researches on risk assessment of investment projects.

| References | Methods | Risk investment | Advantages | Disadvantages |
|---|---|---|---|---|
| Tyebjee and Bru (1984) | Factor analysis method | Venture capitalist investment evaluations | • The steps of venture capitalist investment risk activity are established. | • Lack theoretical basis. |
| Gustafsson and Salo (2005) | Goal programming model | Research and development risky projects | • The states of nature to capture exogenous uncertainties are used.<br>• The resources are modeled through dynamic state variables. | • Evaluation values can only be represented in numerical values.<br>• Evaluation values cannot be incomplete. |
| Zeng et al. (2007) | Analytic hierarchy process; Fuzzy reasoning technology | Construction risk assessment | • Judgment matrix reduces the difficulty of evaluation.<br>• The information was represented in single linguistic terms and their membership degrees. | • Evaluation values can only be represented by single linguistic term.<br>• Evaluation values cannot be incomplete. |
| Sun and Tan (2012) | Goal programming models | Risky investment project for power generation | • The models can maximize benefits and minimize risk value. | • Evaluation values can only be represented in numerical values.<br>• Evaluation values cannot be incomplete. |
| Aouni et al. (2013) | Goal programming model | Venture capitalist investment evaluations | • Four objectives, such as the investment return, the survival rate, the intellectual capital rate, and the investment risk, are considered simultaneously in the model. | • Evaluation values can only be represented in numerical values.<br>• Evaluation values cannot be incomplete. |
| Zhang et al. (2016) | PLPR | Risk assessment of investment projects | • Judgment matrix reduces the difficulty of evaluation.<br>• The information was represented in several single linguistic terms and their probabilities. | • Evaluation values cannot be represented by different forms of linguistic terms.<br>• Evaluation values cannot be incomplete. |
| Tang et al. (2018) | Hesitant fuzzy uncertain linguistic preference relations | Nature disaster risk assessments | • Judgment matrix reduces the difficulty of evaluation.<br>• The information was represented in single and several linguistic terms. | • Evaluation values cannot be represented by interval linguistic terms.<br>• Evaluation values cannot be incomplete. |
| Đurić et al. (2019) | Triangular or trapezoidal fuzzy information | Risk assessment in supply chains | • The information was represented in triangular or trapezoidal fuzzy numbers. | • Evaluation values can only be represented in numerical values. |

(*continued*)

**Table 1.** Continued.

| References | Methods | Risk investment | Advantages | Disadvantages |
|---|---|---|---|---|
| Lan et al. (20211) | Interval-valued bipolar uncertain linguistic information | Risk assessment of Chinese enterprises' overseas mergers and acquisitions | • The information was represented in interval linguistic terms. | • Evaluation values cannot be represented by different forms of linguistic terms. |
| Gou et al. (2021) | Linguistic preference ordering | Construction project investment risks | • The information was represented in multiple linguistic terms. | • Evaluation values cannot be represented by different forms of linguistic terms. |
| Liang et al. (2021) | Multi-granular linguistic distribution assessments | Renewable energy project risk assessment | • The information was represented in multiple linguistic terms and their probabilities. | • Evaluation values cannot be represented by different forms of linguistic terms. |
| Ilbahar et al. (2022) | Interval-valued Intuitionistic Fuzzy Analytic Hierarchy Process (IVIF AHP) method | Renewable energy investment risks | • A hierarchical model was constructed with seventeen risks under four main risk categories.<br>• The information was represented in interval numbers. | • Evaluation values can only be represented in numerical values. |

Source: The Authors.

For $n$ alternatives $X = \{x_1, x_2, ..., x_n\}$, let $S = \{s_\alpha | \alpha = -\tau, ..., 0, ..., \tau\}$ be an LTS. Then, an uncertain linguistic preference relation can be defined as (Xu, 2004b): $D = (\tilde{s}_{ij})_{n\times n} \subset X \times X$, where $\tilde{s}_{ij} = [s_{ij}^-, s_{ij}^+]$, $s_{-\tau} \le s_{ij}^- \le s_{ij}^+ \le s_\tau$, $s_{ij}^- \oplus s_{ji}^+ = s_{ij}^+ \oplus s_{ji}^- = s_0$, and $s_{ii}^- = s_{ii}^+ = s_0$ for all $i, j = 1, 2, ..., n$. Since different linguistic assessments may be with different preference intensities, Pang et al. (2016) introduced the concept of PLTS which uses probabilities to express the preference degrees of linguistic terms. To apply PLTSs to preference relations, Zhang et al. (2016) proposed the concept of PLPR as $D = (h_{ij})_{n\times n} \subset X \times X$, where $h_{ij}(p) = \{s_t(p_t) | s_t \in S, \quad p_t \ge 0, t = 1, 2, ..., T, \sum_{t=1}^{T} p_t \le 1\}$ with $p_t > 0$ and $\sum_{t=1}^{T} p_t \le 1$, and $s_t(p_t)$ is the $t$th linguistic term $s_t$ associated with the probability $p_t$, and $T$ is the number of different linguistic terms in $h_{ij}(p)$ arranged in ascending order. The PLPR is a good tool to model single linguistic terms and their probabilities. However, when experts give different forms of linguistic preference relations at the same time, such as single term, a set of multiple linguistic terms or interval linguistic term, the PLPR fails to model such preference relations. In this regard, this paper proposes a GPLPR to model preference relations with several forms, which will be described in Section 3 for details.

### 2.3. Generalised probabilistic linguistic term set and its normalization process

In PLTSs, each linguistic term is associated with a probability, but in many cases, experts may also give a set of multiple linguistic terms or interval linguistic terms. To model the probabilistic linguistic assessments with multiple forms of linguistic expressions, Fang et al. (2021) introduced the GPLTS as:

$$\begin{aligned} GL(p) &= \{GL^q(p^q) | q = 1, 2, \ldots, \#GL(p)\} \\ &= \{\{s_{\alpha_k}{}^{q_1}\}(p^{q_1}), [s_{t_1}{}^{q_2}, s_{t_2}{}^{q_2}](p^{q_2}) | s_{\alpha_k}{}^{q_1}, s_{t_1}{}^{q_2}, s_{t_2}{}^{q_2} \in S, k = 1, 2, \ldots, K, \ t_1 \le t_2, q_1 = 1, 2, \ldots, Q_1, \\ &\quad q_2 = 1, 2, \ldots, Q_2, \textstyle\sum_{q_1=1}^{Q_1} p^{q_1} + \sum_{q_2=1}^{Q_2} p^{q_2} \le 1\} \end{aligned} \tag{1}$$

where $S = \{s_\alpha | \alpha = -\tau, ..., 0, ..., \tau\}$ is a discrete LTS, $K$ is the number of linguistic terms in $\{s_{\alpha_k}{}^{q_1}\}$, and $\#GL(p)$ is the number of different linguistic expressions in $GL(p)$, satisfying $\#GL(p) = Q_1 + Q_2$.

**Note 1.** The motivation of introducing the GPLTS is not to propose a complicated representation model for decision making, but to model the precise linguistic assessments given by experts directly and comprehensively. Due to the continuous changes of objects and the limited cognition of human beings, only key states of change can be perceived by experts. Thus, the differentiation of $S$ and $\bar{S}$ can be achieved, where $S$ is used to express experts' assessments and $\bar{S}$ is applied to describe the state of objects.

**Note 2.** In a GPLTS, there may be incomplete probability which is not assigned to any linguistic term in the GPLTS. In this case, we need to normalize the original GPLTS. In the original study of GPLTSs (Fang, et al., 2021), there was no clear explanation about why it is necessary and possible to normalize GPLTSs. Thus, we further explain the normalization process in detail in Appendix A.

## 3. A graph-theory-based group decision making method with generalised linguistic preference information

Risk assessment in project management is a complex decision-making process. Since preference relations do not require to identify criteria, researchers (Gou et al., 2021; Tang et al., 2018; Zeng et al., 2007; Zhang et al., 2016) have introduced different forms of preference relations for risk assessment. Preference relations for risk assessment involve several linguistic representations such as single linguistic term (Liang et al., 2021; Zeng et al., 2007), a set of multiple linguistic terms (Tang et al., 2018), or interval linguistic term (Lan et al., 2021), but existing methods cannot deal with these representations at the same time. In this regard, this paper proposes a GPLPR to model different linguistic representations at the same time, a probability cutting method to simplify the representation of a GPLPR, and a graph theory-based method to check and improve the additive consistency of GPLPR. Then, a graph theory-based group decision making method with GPLPRs is developed, which can be demonstrated intuitively as Figure 1. As can be seen, the contributions of this paper, marked with red words, mainly include modelling several linguistic representations of preference relation, simplifying the representations, checking and improving the additive consistency, and calculating the utility values of alternatives. In the following, the way to model several linguistic representations of preference relation is described in Section 3.1, and the way to simplify the representations is discussed in Section 3.2. Section 3.3 addresses the process to check and improve the additive consistency. Section 3.4 shows the process to calculate the utility values and generate the ranking of alternatives.

### 3.1. Establish the generalised probabilistic linguistic preference relation

For a practical risk assessment problem with $n$ alternatives $X = \{x_1, x_2, \ldots, x_n\}$, experts are invited to give the preference degree between each pair of alternatives based on the LTSs $S = \{s_\alpha | \alpha = -\tau, \ldots, 0, \ldots, \tau\}$ and $\bar{S} = \{s_\alpha | \alpha \in [-\tau, \tau]\}$. The experts may use different forms of linguistic expressions, such as single linguistic terms, multiple linguistic terms or interval linguistic terms. To represent multiple linguistic forms comprehensively, we define the GPLPR as follows:

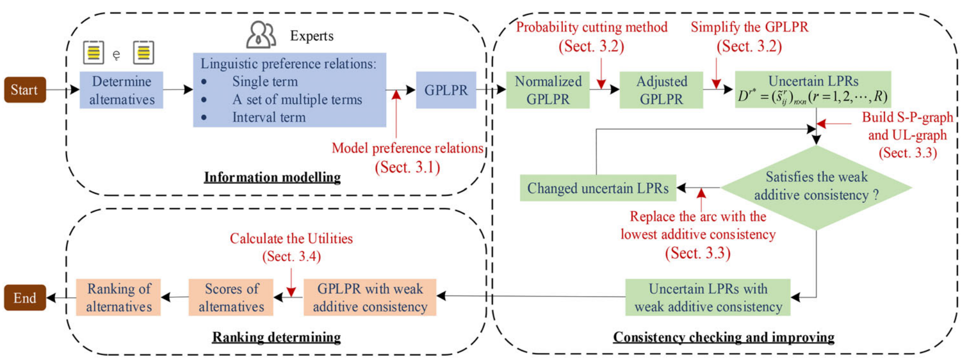



**Figure 1.** Framework of the graph theory-based group decision making method with GPLPRs for risk assessment.
Source: The Authors.

**Definition 1.** A GPLPR on $X = \{x_1, x_2, ..., x_n\}$ is denoted as $D = (GL_{ij}(p))_{n\times n} \subset X \times X$ for all $i, j = 1, 2, ..., n$, with

$$\begin{aligned} GL_{ij}(p) &= \{GL_{ij}^q(p^q)|q = 1, 2, \ldots, \#GL_{ij}(p)\} \\ &= \{\{s_{\alpha_k, ij}{}^{q_1}\}(p_{ij}^{q_1}), [s_{t_1, ij}{}^{q_2}, s_{t_2, ij}{}^{q_2}](p_{ij}^{q_2})|s_{\alpha_k, ij}{}^{q_1}, s_{t_1, ij}{}^{q_2}, s_{t_2, ij}{}^{q_2} \in S, k = 1, 2, \ldots, K,\ t_1 \le t_2, \\ &\qquad q_1 = 1, 2, \ldots, Q_1, q_2 = 1, 2, \ldots, Q_2, \textstyle\sum_{q_1=1}^{Q_1} p_{ij}^{q_1} + \sum_{q_2=1}^{Q_2} p_{ij}^{q_2} \le 1\} \end{aligned} \tag{2}$$

where $K$ is the number of linguistic terms in $\{s_{\alpha_k, ij}{}^{q_1}\}$, and $\#GL_{ij}(p)$ is the number of different linguistic expressions in $GL_{ij}(p)$, satisfying $\#GL_{ij}(p) = Q_1 + Q_2$. As a GPLPR, $D = (GL_{ij}(p))_{n\times n} \subset X \times X$ satisfies: $p_{ij}^{q_1} = p_{ji}^{q_1}$, $p_{ij}^{q_2} = p_{ji}^{q_2}$, $GL_{ii}(p) = \{s_0(1)\}$, $\#GL_{ij}(p) = \#GL_{ji}(p)$, $s_{\alpha_k, ij}{}^{q_1} = neg(s_{\alpha_k, ij}{}^{q_1})$, and $[s_{t_1, ij}{}^{q_2}, s_{t_2, ij}{}^{q_2}] = neg([s_{t_1, ji}{}^{q_2}, s_{t_2, ji}{}^{q_2}])$.

The GPLPR $D = (GL_{ij}(p))_{n\times n}$ on $S$ satisfies: if $i<j$, then $[s_{t_1, ij}{}^{q_2}, s_{t_2, ij}{}^{q_2}]$ with $s_{-\tau} \le s_{t_1, ij}{}^{q_2} \le s_{t_2, ij}{}^{q_2} \le s_\tau$; if $i>j$, then $[s_{t_1, ij}{}^{q_2}, s_{t_2, ij}{}^{q_2}] = [neg(s_{t_1, ij}{}^{q_2}), neg(s_{t_2, ij}{}^{q_2})]$ with $s_{-\tau} \le s_{t_2, ij}{}^{q_2} \le s_{t_1, ij}{}^{q_2} \le s_\tau$. A GPLPR can be simplified as $D = (s_{ij}(p))_{n\times n} \subset X \times X$ where $s_{ij}(p) = \{s_{ij}^{(k)}(p_{ij}^{(k)})|k = 1, 2, ..., K_{ij}\}$ for all $i, j = 1, 2, ..., n$, satisfying $p_{ij}^{(k)} = p_{ji}^{(k)}$ and $s_{ij}^{(k)} = neg(s_{ji}^{(k)})$. Especially, if all $s_{ij}$ (for $i, j = 1, 2, ..., n$) are single linguistic terms, then the GPLPR degenerates into a PLPR; if all $s_{ij}$ (for $i, j = 1, 2, ..., n$) are a set of multiple linguistic terms, then the GPLPR degenerates into a hesitant fuzzy linguistic preference relation (Zhu & Xu, 2014) or extended hesitant fuzzy linguistic preference relation (Wang & Xu, 2015); if all $s_{ij}$ (for $i, j = 1, 2, ..., n$) are interval linguistic terms, then the GPLPR degenerates into an uncertain linguistic preference relation. The introduction of the GPLPR makes it possible to model different linguistic expressions in one model.

### 3.2. Simplify the GPLPR: a probability cutting method

To ensure that preference information is logical and non-random, it is essential to check the consistency of the preference relation. Due to the probability information assigned to different forms of linguistic information, the consistency processing for PLPRs and GPLPRs is much difficult. To simplify the consistency checking process of preference relations, the $\alpha$-cut method (Liao et al., 2019) was introduced. However, since the threshold $\alpha$ may be different values, the information gained by the $\alpha$-cut method may be misleading. In this subsection, motivated by the idea to measure the consistency by the distance between PLTSs (Fang et al., 2021; Wu & Liao, 2019; Xu et al., 2013), we propose a probability cutting method to address the consistency of GPLPRs.

The main idea of the probability cutting method is to adjust the probability distribution of the GPLPR to the same structure, and then divide the GPLPR into several preference relations with linguistic information. The consistency of a GPLPR can be achieved by dealing with the consistency of all the obtained preference relations with linguistic information. For a normalized GPLPR, we define its consistency as follows:

**Definition 2.** Given a normalized GPLPR, $\bar{D} = (\bar{s}_{ij}(p))_{n\times n}$ is said to satisfy the additive consistency if $\bar{s}_{ij}(p) = \bar{s}_{ie}(p) \oplus \bar{s}_{ej}(p)$, for all $i, e, j = 1, 2, ..., n, i \ne j$. If

$\bar{D} = (\bar{s}_{ij}(p))_{n\times n}$ satisfies the additive consistency, then $D = (s_{ij}(p))_{n\times n}$ satisfies the additive consistency.

To get an adjusted GPLPR whose elements have the same expression, an adjustment method is proposed.

**Definition 3.** Given a normalized GPLPR $\bar{D} = (\bar{s}_{ij}(p))_{n\times n}$, where $\bar{s}_{ij}(p) = \{s_{ij}^{(k)}(p_{ij}^{(k)})|k = 1,2,\ldots,K_{ij}\}$ $(i,j = 1,2,\ldots,n)$ satisfying $E(s_{ij}^{(k)})<E(s_{ij}^{(k+1)})$. Let $p* = \{p^r|r = 1,2,\ldots,R\}$ be the rearranged probability set for all $p_{ij}^{(k)}(k = 1,2,\ldots,K_{ij}$, $i,j = 1,2,\ldots,n)$, and $R$ be the number of probabilities in $p*$. The adjusted GPLPR can be denoted as $D* = (s_{ij}^*(p*))_{n\times n}$, where $s_{ij}^*(p*) = \{s_{ij}^r(p^r)|r = 1,2,\ldots,R\}$.

The adjustments only change the form of the original preference relation, but do not change its substance. Therefore, the matrixes as $D* = (s_{ij}^*(p*))_{n\times n}$ and $\bar{D} = (\bar{s}_{ij}(p))_{n\times n}$ represent the same assessments, satisfying $E(s_{ij}^*(p*)) = E(\bar{s}_{ij}(p))$. Based on $D* = (s_{ij}^*(p*))_{n\times n}$, we can reduce the dimension of $D*$ by $p* = \{p^r|r = 1,2,\ldots,R\}$, and obtain $R$ uncertain linguistic preference relations as $D^{r*} = (s_{ij}^r)_{n\times n}(r = 1,2,\ldots,R)$. Example 1 is given to show the process of reducing the dimension of GPLPR.

**Example 1.** For a GPLPR on $\bar{S}_2$ in Example 1 as $D_1$, it is easy to obtain $p* = \{0.3, 0.3, 0.4\}$.

$$D_1 = \begin{bmatrix} \{s_0(1)\} & \{[s_1,s_2](0.3),[s_2,s_3](0.7)\} & \{[s_{-1},s_0](0.6),[s_0,s_1](0.4)\} \\ \{[s_{-1},s_{-2}](0.3),[s_{-2},s_{-3}](0.7)\} & \{s_0(1)\} & \{[s_0,s_1](0.3),[s_1,s_2](0.7)\} \\ \{[s_1,s_0](0.6),[s_0,s_{-1}](0.4)\} & \{[s_0,s_{-1}](0.3),[s_{-1},s_{-2}](0.7)\} & \{s_0(1)\} \end{bmatrix}$$

Next, we have

$$D_1^* = \begin{bmatrix} \{s_0(0.3),s_0(0.3),s_0(0.4)\} & \{[s_1,s_2](0.3),[s_2,s_3](0.3),[s_2,s_3](0.4)\} & \{[s_{-1},s_0](0.3),[s_{-1},s_0](0.3),[s_0,s_1](0.4)\} \\ \{[s_{-1},s_{-2}](0.3),[s_{-2},s_{-3}](0.3),[s_{-2},s_{-3}](0.4)\} & \{s_0(0.3),s_0(0.3),s_0(0.4)\} & \{[s_0,s_1](0.3),[s_1,s_2](0.3),[s_1,s_2](0.4)\} \\ \{[s_1,s_0](0.3),[s_1,s_0](0.3),[s_0,s_{-1}](0.4)\} & \{[s_0,s_{-1}](0.3),[s_{-1},s_{-2}](0.3),[s_{-1},s_{-2}](0.4)\} & \{s_0(0.3),s_0(0.3),s_0(0.4)\} \end{bmatrix}$$

Then, three ULPRs are obtained by the dimension reduction of $D_1{}^*$ according to $p* = (0.3, 0.3, 0.4)^T$:

$$D_1^{2*} = \begin{bmatrix} s_0 & [s_2,s_3] & [s_{-1},s_0] \\ [s_{-2},s_{-3}] & s_0 & [s_1,s_2] \\ [s_1,s_0] & [s_{-1},s_{-2}] & s_0 \end{bmatrix}, D_1^{1*} = \begin{bmatrix} s_0 & [s_1,s_2] & [s_{-1},s_0] \\ [s_{-1},s_{-2}] & s_0 & [s_0,s_1] \\ [s_1,s_0] & [s_0,s_{-1}] & s_0 \end{bmatrix}, D_1^{3*} = \begin{bmatrix} s_0 & [s_2,s_3] & [s_0,s_1] \\ [s_{-2},s_{-3}] & s_0 & [s_1,s_2] \\ [s_0,s_{-1}] & [s_{-1},s_{-2}] & s_0 \end{bmatrix}$$

**Theorem 1.** *If all $D^{r*} = (s_{ij}^r)_{n\times n}$ (for $r = 1,2,\ldots,R$) satisfy the additive consistency, then $D* = (s_{ij}^*(p*))_{n\times n}$ satisfies the additive consistency. In other words, if $s_{ie}^r(p)\oplus s_{ej}^r(p) = s_{ij}^r(p)$ for all $r = 1,2,\ldots,R$, and $j = 1,2,\ldots,n, i \neq j$, then $s_{ie}^*(p*)\oplus s_{ej}^*(p*) = s_{ij}^*(p*)$ for all $r = 1,2,\ldots,R$, and $j = 1,2,\ldots,n, i \neq j$.*

The proof of Theorems 1 is shown in Appendix B. According to Theorem 1, we can transform a GPLPR into several uncertain linguistic preference relations by the probability cutting method, which greatly reduces the difficulty of the consistency processing and does not change the meaning of original information.

### 3.3. Deal with the consistency of GPLPRs based on the graph theory

Graph theory is an important method to solve decision-making problems. In the research of preference relationship persistence, Wang and Xu (2015) and Zhang et al. (2016) applied the preference relation graph (P-graph) and symmetric preference relation graph (S-P-graph) to deal with the consistency of the extended hesitant fuzzy LPR and PLPR respectively. In this regard, it may be a good attempt to use the graph theory to intuitively deal with the consistency of GPLPRs.

For a GPLPR $D = (GL_{ij}(p))_{n\times n} \subset X \times X$, we normalize it by the method presented in Section 3.1 to get $\bar{D} = (\bar{G}L_{ij}(p))_{n\times n} \subset X \times X$, where $\bar{G}L_{ij}(p) = \{GL_{ij}^q(p_{ij}^q)|q = 1, 2, ..., \#\bar{G}L_{ij}(p)\} = \{[U_{ij}^q, V_{ij}^q](p_{ij}^q)|q = 1, 2, ..., \#\bar{G}L_{ij}(p)\}$. Then, by Section 3.2, an adjusted GPLPR of $D$ as $D* = (GL_{ij}^*(p))_{n\times n}$ where $GL_{ij}^*(p) = \{[U_{ij}^r, V_{ij}^r](p_{ij}^r)|r = 1, 2, ..., R\}$, and its dimension reduction as $p* = \{p^r|r = 1, 2, ..., R\}$ can be obtained, respectively. After that, $R$ uncertain linguistic preference relations as $D^{r*} = (\tilde{s}_{ij}^r)_{n\times n}$ with $\tilde{s}_{ij}^r = [U_{ij}^r, V_{ij}^r]$ $(r = 1, 2, ..., R)$ can be generated.

To calculate conveniently and embody the symmetry of a preference relation, the uncertain linguistic preference relation is introduced as $D = (\tilde{s}_{ij})_{n\times n} \subset X \times X$, satisfying if $i<j$, then $\tilde{s}_{ij} = [s_{ij}^-, s_{ij}^+]$, $s_{-\tau} \le s_{ij}^- \le s_{ij}^+ \le s_\tau$; if $i = j$, then $\tilde{s}_{ii} = [s_{ii}^-, s_{ii}^+] = [s_0, s_0]$; if $i>j$, then $\tilde{s}_{ij} = [s_{ij}^-, s_{ij}^+] = [s_{-ji}^-, s_{-ji}^+]$. In the right upper triangle of $D = (\tilde{s}_{ij})_{n\times n}$, the preference relationship satisfies $s_{-\tau} \le s_{ij}^- \le s_{ij}^+ \le s_\tau$, and in the left lower triangle, the preference relationship $s_{-\tau} \le s_{ij}^+ \le s_{ij}^- \le s_\tau$. $D$ is called an additively consistent uncertain linguistic preference relation if $\tilde{s}_{ie} \oplus \tilde{s}_{ej}$ for any $i, e, j = 1, 2, ..., n, i \ne j$.

To deal with consistency, the weighted S-P-graph of $D$ is defined as $G_{S-UL}(V, A)$, where $V = \{v_1, v_2, ..., v_n\}$ is the set of vertices and $A = \{(v_i, v_j)|i \ne j, i, j = 1, 2, ..., n)\}$ is the set of arcs. $(v_i, v_j)$ represents a directed line segment from $v_i$ to $v_j$ with the weight $w(v_i, v_j) = [r_{ij}^-, r_{ij}^+]$, where $r_{ij}^-$ and $r_{ij}^+$ represent the subscripts of $s_{ij}^-$ and $s_{ij}^+$, respectively. Example 2 is given to show the S-P-graph of $D$.

**Example 2.** Given an uncertain linguistic preference relation on $S$ as $D_2$, then, its S-P-graph is shown as Figure 2.

As can be seen from Figure 2, there are multiple paths from $v_i$ to $v_j$. For a consistent uncertain linguistic preference relation, the average length of the path $len(v_i, (v_i, v_j), v_j)$ should be equal to the average length of the path

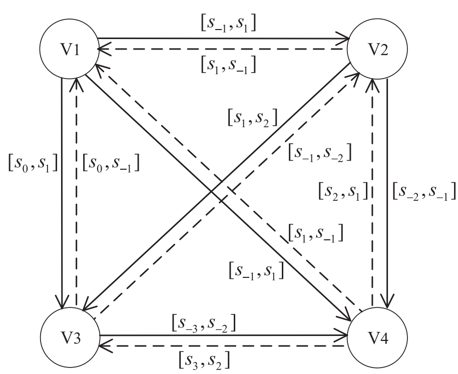



**Figure 2.** The S-P-graph of $D_2$.
Source: The Authors.

$len(v_i, (v_i, v_e), v_e, (v_e, v_j), v_j)$, where $i, e, j = 1, 2, ..., n, i \neq j$. Then, we define the additively consistent uncertain linguistic preference relation as:

$$D_2 = \begin{bmatrix} [s_0, s_0] & [s_{-1}, s_1] & [s_0, s_1] & [s_{-1}, s_1] \\ [s_1, s_{-1}] & [s_0, s_0] & [s_1, s_2] & [s_{-2}, s_{-1}] \\ [s_0, s_{-1}] & [s_{-1}, s_{-2}] & [s_0, s_0] & [s_{-3}, s_{-2}] \\ [s_1, s_{-1}] & [s_2, s_1] & [s_3, s_2] & [s_0, s_0] \end{bmatrix}$$

**Definition 4.** Let $D = (\tilde{s}_{ij})_{n\times n}$ be an uncertain linguistic preference relation. $D$ is an additively consistent uncertain linguistic preference relation if $\tilde{s}_{ij} = \tilde{s}_{ie} \oplus \tilde{s}_{ej}$ for any $i, e, j = 1, 2, ..., n, i \neq j$.

**Theorem 2.** *The uncertain linguistic preference relation $\bar{D}$ is additively consistent if*

$$\bar{s}_{ij} = \begin{cases} \frac{1}{n}\left[\oplus_{e=1}^{n}(\tilde{s}_{ie}\oplus\tilde{s}_{ej})\right], & i,j = 1,2,\ldots,n, i\neq j \\ [s_0, s_0], & \text{otherwise} \end{cases} \tag{3}$$

The proof of Theorems 2 is shown in Appendix C. Based on the additively consistent uncertain linguistic preference relation, according to the similarity between $\tilde{s}_{ij}$ in $D = (\tilde{s}_{ij})_{n\times n}$ and $\bar{s}_{ij}$ in $\bar{D} = (\bar{s}_{ij})_{n\times n}$, the consistency index of $\tilde{s}_{ij}$ can be calculated by Eq. (4):

$$CI_{ij} = 1 - \frac{|r_{ij}^- - \bar{r}_{ij}^-| + |r_{ij}^+ - \bar{r}_{ij}^+|}{4\tau} \tag{4}$$

where $r_{ij}^-$ and $r_{ij}^+$ are the subscripts of $\tilde{s}_{ij}^-$ and $\tilde{s}_{ij}^+$, and $\bar{r}_{ij}^-$ and $\bar{r}_{ij}^+$ are the subscripts of $\bar{s}_{ij}^-$ and $\bar{s}_{ij}^+$, respectively. The higher the consistency index is, the more logical the preference relation given by the expert is. Especially, if $r_{ij}^- = r_{ij}^+ = r_{ij}$ and $\bar{r}_{ij}^- = \bar{r}_{ij}^+ = \bar{r}_{ij}$, then,

$$CI_{ij} = 1 - \frac{|r_{ij} - \bar{r}_{i,}|}{2\tau}, \text{ for } i,j = 1,2,\ldots,n, i\neq j$$

According to the obtained $CI_{ij}$ of $\tilde{s}_{ij}$, we can get the consistency level of $D = (\tilde{s}_{ij})_{n\times n}$ by

$$CI(D) = \frac{1}{n^2 - n}\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=1, j\neq i}^{n} CI_{ij} \tag{5}$$

The strict additive consistency is a strong condition for evaluating preference relations. Due to the complexity of decision-making problems, such conditions are usually difficult to meet. Therefore, we introduce the weak consistency of uncertain linguistic preference relations.

**Definition 5.** Let $D = (\tilde{s}_{ij})_{n\times n}$ be an uncertain linguistic preference relation. If when $s_{ik}^- \geq s_\alpha$ and $s_{kj}^- \geq s_\alpha$, for $i, j, k \in \{1, 2, ..., n\}$ □ $i \neq j \neq k$, there is $s_{ij}^+ \geq s_\alpha$, then $D$ is said to satisfy the weak consistency.

To intuitively represent the weak consistency of uncertain linguistic preference relations, an uncertain linguistic preference graph (UL-graph) is proposed, which is a weighted digraph $G_{UL}(V, A)$. $V = \{v_1, v_2, ..., v_n\}$ is the set of vertices and $A = \{(v_i, v_j)|i<j, i, j = 1, 2, ..., n)\}$ is the set of arcs, where $(v_i, v_j)$ is a directed arc from $v_i$ to $v_j$. The weight of $(v_i, v_j)$ is $w(v_i, v_j) = [r_{ij}^-, r_{ij}^+]$, where $r_{ij}^-$, $r_{ij}^+$ are the subscripts of $s_{ij}^-$ and $s_{ij}^+$, respectively. In the UL-graph of $D$, if $s_{ik}^- \geq s_\alpha$, then there is an arc $(v_i, v_k)$; if $s_{kj}^- \geq s_\alpha$, then there is $(v_k, v_j)$; while if $s_{ij}^+ \leq s_\alpha$, then there is $(v_i, v_j)$, which leads to a circular triad $(v_i, (v_i, v_k), v_k, (v_k, v_j), v_j, (v_j, v_i), v_i)$. If there is no circular triad in the UL-graph, then $D$ is said to satisfy the weak consistency.

For a weakly consistent $D = (\tilde{s}_{ij})_{n\times n}$, it conforms to logical relations and can be used to solve decision-making problems. If $D = (\tilde{s}_{ij})_{n\times n}$ does not satisfy the weak consistency, there is a need to find circular triads whose arcs are denoted by $A_{TRI} = \{(v_i^m, v_j^m)|m$ is the number of arcs in $A_{TRI}\}$. In $A_{TRI}$, we need to find the arc with the worst consistency, i.e., $(v_i^m, v_j^m)$ with $\min_m\{CI_{ij}^m\}$, and replace its weight with the average weight of the other two arcs in its circular triad. Then, we check circular triad again and process it until there is no circular triad in the UL-graph. Finally, the $D = (\tilde{s}_{ij})_{n\times n}$ satisfying weak consistency is obtained.

The weak consistency of uncertain linguistic preference relations ensures that the assessments satisfy the logic and avoids the influence on the accuracy of the decision result due to the misjudgements. For ease of application, we summarize the procedure in Algorithm I and give an example as Example 3.

**Algorithm I**: Weak consistency checking and improving for uncertain linguistic preference relations

*Step 1.* Let $p = 0$ and $D^{(p)} = D = (\tilde{s}_{ij})_{n\times n}$ be an uncertain linguistic preference relation.

*Step 2.* Build the S-P-graph and UL-graph of $D^{(p)}$.

*Step 3.* If $D^{(p)}$ satisfies the weak consistency, then go to Step 7; else go to Step 4.

*Step 4.* Find out all the circular triads to form $A_{TRI}$. Next, we calculate the consistency index $CI_{ij}$ of $(v_i^m, v_j^m)$ by Eq. (4) and select the arc with the lowest additive consistency level and go to Step 5.

*Step 5.* Replace the weight of the arc $(v_i^m, v_j^m)$ found in Step 4 with $\bigoplus_{k=1, k\neq i, j}^n (\tilde{s}_{ik} \oplus \tilde{s}_{kj})/(n-2)$, and change the corresponding uncertain LTSs in $D^{(p)}$.

*Step 6.* Let $D^{(p+1)} = D^{(p)}, p = p + 1$. Then, go to Step 3.

*Step 7.* Let $\tilde{D} = D^{(p)}$. End.

**Example 3.** Given an uncertain linguistic preference relation on $\bar{S}_2$ in Example 1 as $D_3$.

$$D_3 = \begin{bmatrix} [s_0, s_0] & [s_{-1}, s_1] & [s_0, s_1] & [s_{-1}, s_1] \\ [s_1, s_{-1}] & [s_0, s_0] & [s_1, s_2] & [s_{-2}, s_{-1}] \\ [s_0, s_{-1}] & [s_{-1}, s_{-2}] & [s_0, s_0] & [s_0, s_1] \\ [s_1, s_{-1}] & [s_2, s_1] & [s_0, s_{-1}] & [s_0, s_0] \end{bmatrix} \tilde{D}_3 = D_3^{(1)} = \begin{bmatrix} [s_0, s_0] & [s_{-1}, s_1] & [s_0, s_1] & [s_{-1}, s_1] \\ [s_1, s_{-1}] & [s_0, s_0] & [s_1, s_2] & [s_{-0.5}, s_{2.5}] \\ [s_0, s_{-1}] & [s_{-1}, s_{-2}] & [s_0, s_0] & [s_0, s_1] \\ [s_1, s_{-1}] & [s_{0.5}, s_{-2.5}] & [s_0, s_{-1}] & [s_0, s_0] \end{bmatrix}$$

$$(CI_{ij})_{n\times n} = \begin{bmatrix} - & 0.8958 & 0.9583 & 0.9167 \\ 0.8958 & - & 0.9167 & 1 \\ 0.9583 & 0.9167 & - & 0.9167 \\ 0.9167 & 1 & 0.9167 & - \end{bmatrix}$$

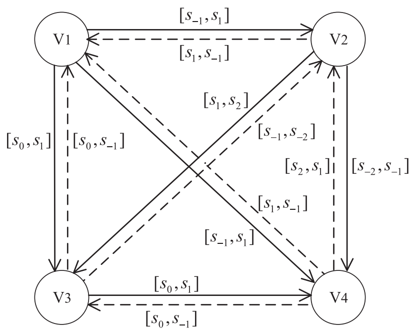



**Figure 3.** The S-UL-graph of $D_3^{(0)}$.
Source: The Authors.

*Step 1*. Let $p = 0$ and $D_3^{(p)} = D_3$.
*Step 2*. Build the S-UL-graph and UL-graph of $D_2^{(0)}$ as Figures 3 and 4.
*Step 3*. There is a circular triad $(v_2, (v_2, v_3), v_3, (v_3, v_4), v_4, (v_4, v_2), v_2)$ in $D_3^{(0)}$, then$A_{TRI} = \{(v_2, v_3), (v_3, v_4), (v_4, v_2)\}$.
*Step 4*. By Eq. (4), we can obtain $CI_{23} = 0.8125$, $CI_{34} = 0.8125$, $CI_{42} = 0.7917$. Thus, $(v_4, v_2)$ is the arc with the lowest consistency index $CI_{42} = 0.7917$.
*Step 5*. Since $w(v_4, v_2) = \frac{\oplus_{k=1, k\neq, j}^{4}(\tilde{s}_{4k}\oplus\tilde{s}_{k2})}{4-2} = [s_{0.5}, s_{-2.5}]$, then we have $\tilde{s}_{42} = [s_{0.5}, s_{-2.5}]$ and $\tilde{s}_{24} = [s_{-0.5}, s_{2.5}]$.
*Step 6*. Let $D_3^{(1)} = D_3^{(0)}, p = 1$; go to Step 3, and there is no circular triad in $D_3^{(1)}$. Then, we have $\tilde{D}_3$. For $\tilde{D}_3$ satisfying the weak consistency, its consistency level can be calculated by Eqs. (4) and (5) as $(CI_{ij})_{n\times n}$. Then, we can get $CI(\tilde{D}_3) = 0.9340$. According to Algorithm I, the uncertain linguistic preference relation $\tilde{D}^{r*} = (\tilde{s}_{ij}^{r})_{n\times n}$ meets the consistency requirements, then, and the GPLPR $\tilde{D}$ meets the consistency requirements.

### 3.4. Generate the ranking of alternatives

In group decision making process, if a GPLPR satisfies the acceptable additive consistency, the preference information needs to be aggregated to obtain an overall assessment matrix and then generate the ranking the alternatives. In this subsection, the classical arithmetic average operator is introduced to illustrate the information fusion of GPLPR.

**Definition 6.** For $n$ normalized GPLTSs $\bar{G}L_i(p) = \{[U_i^q, V_i^q](p_i^q)|q = 1, 2, ..., \#\bar{G}L_i(p)\}(i = 1, 2, ..., n)$, the arithmetic average operator of them can be defined as:

$$\begin{aligned} GPLA(GL_1(p), GL_2(p), \ldots, GL_n(p)) &= \frac{1}{n}(GL_1(p)\oplus GL_2(p)\oplus\cdots\oplus GL_n(p)) \\ &= \bigcup_{\substack{[U_1^q, V_1^q](p_1^q)\in GL_1(p),\\ [U_2^q, V_2^q](p_2^q)\in GL_2(p),\ \ldots,\\ [U_n^q, V_n^q](p_n^q)\in GL_n(p)}} \left\{\frac{1}{n}p_1^q\left[U_1^q, V_1^q\right]\oplus\frac{1}{n}p_2^q\left[U_2^q, V_2^q\right]\oplus\cdots\oplus\frac{1}{n}p_n^q\left[U_n^q, V_n^q\right]\right\} \end{aligned} \tag{6}$$

The preference information of alternatives can be aggregated by this operator, and obtain the corresponding comprehensive assessment. Then, according to the score

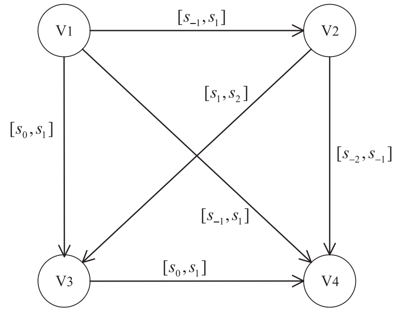



**Figure 4.** The UL-graph of $D_3^{(0)}$.
Source: The Authors.

function and deviation function of the comprehensive assessment, the alternatives can be ranked.

**Definition 7.** Let $\bar{G}L(p) = \{[U^q, V^q](p^q) | q = 1, 2, \ldots, \#\bar{G}L(p)\}$ be a normalized GPLTS. Then, 1) the score function of $\bar{G}L(p)$ is $E(\bar{G}L(p)) = s_{\bar{\alpha}}$, where $\bar{\alpha} = \sum_{q=1}^{\#\bar{G}L(p)} [\frac{r_U^q + r_V^q}{2} \cdot p^q]$ with $r_U^q$ and $r_V^q$ being the subscribes of $U^q$ and $V^q$, respectively. 2) the deviation function of $\bar{G}L(p)$ is $\sigma(\bar{G}L(p)) = s_\sigma$, where $\sigma = \sqrt{\sum_{q=1}^{\#\bar{G}L(p)} [\left(\frac{r_U^q + r_V^q}{2} - \bar{\sigma}\right) \cdot p^q]^2}$.

For any two normalized GPLTSs $\bar{G}L_1(p)$ and $\bar{G}L_2(p)$ :1) $\bar{G}L_1(p) > \bar{G}L_2(p)$, if $E(\bar{G}L_1(p)) > E(\bar{G}L_2(p))$; 2) $\bar{G}L_1(p) < \bar{G}L_2(p)$, if $E(\bar{G}L_1(p)) < E(\bar{G}L_2(p))$; 3) if $E(\bar{G}L_1(p)) = E(\bar{G}L_2(p))$, then: a) $\bar{G}L_1(p) > \bar{G}L_2(p)$, if $\sigma(\bar{G}L_1(p)) > \sigma(\bar{G}L_2(p))$; b) $\bar{G}L_1(p) \sim \bar{G}L_2(p)$, if $\sigma(\bar{G}L_1(p)) = \sigma(\bar{G}L_2(p))$; c) $\bar{G}L_1(p) < \bar{G}L_2(p)$, if $\sigma(\bar{G}L_1(p)) < \sigma(\bar{G}L_2(p))$.

Based on the above analysis of GPLPR, Algorithm II for group decision making with GPLPR is developed.

**Algorithm II:** A graph theory-based method for group decision making with GPLPRs

*Step 1.* Determine the GPLPR $D = (GL_{ij}(p))_{n \times n} \subset X \times X$.

*Step 2.* Get the normalized GPLPR $\bar{D} = (\bar{G}L_{ij}(p))_{n \times n}$ of $D$.

*Step 3.* Adjust $\bar{D}$ to $D* = (GL_{ij}^*(p))_{n \times n}$.

*Step 4.* Reduce the dimension of $D*$ to obtain $R$ matrices in the uncertain linguistic preference relation $D^{r*} = (\tilde{s}_{ij}^r)_{n \times n}$ $(r = 1, 2, \ldots, R)$. In each probability $p^r$ $(r = 1, 2, \ldots, R)$, we can apply Algorithm I to deal with the additive consistency of $D^{r*} = (\tilde{s}_{ij}^r)_{n \times n}$. Then, we can get $\tilde{D}^{r*} = (\tilde{s}_{ij}^r)_{n \times n}$ which satisfies the weakly additive consistency, and then calculate the corresponding consistency index $CI^{r*}$.

*Step 5.* According to $D^{r*} = (\tilde{s}_{ij}^r)_{n \times n}$ in $p^r$ $(r = 1, 2, \ldots, R)$, we can transform the preference relation from a uncertain linguistic preference relation to a GPLPR to get $\tilde{D} = (\tilde{s}_{ij})_{n \times n}$ which satisfies the weakly additive consistency, and the consistency level is $CI(\tilde{D}) = \sum_{r=1}^{R} CI^{r*}$.

*Step 6.* Aggregate the preference information in $\tilde{D}$ by Eq. (6), and then rank the alternatives. End the algorithm.

## 4. Case study: risk assessment of investment projects

This section solves a case study about risk assessment of investment projects, and then conducts comparative analyses to illustrate the advantages of the proposed method.

### 4.1. Case description

Suppose that there is a company, whose main business projects are to establish industry, material supply and marketing industry, import and export business, and real estate development. For the sustainable operation and development, the company needs to choose new projects for investment. Whether for social responsibility or high profits, the purpose of investment is for the development of the company. However, as long as new projects are invested, the risks are inevitable. Now, the company plans to invest in one of four projects $x_i$ $(i = 1, 2, 3, 4)$. For these projects, there is no significant difference in profit and other benefits. Thus, decision makers mainly consider the risk situation to choose the optimal investment project. The basic information of these four projects is described in Table 2.

Next, the graph theory-based group decision making method with GPLPR is applied to deal with this risk assessment of investment projects.

For four investment projects $x_i$ $(i = 1, 2, 3, 4)$, ten experts are invited to compare their risk levels in pairs based on the LTS $S = \{s_{-3} = \text{very low}, s_{-2} = \text{low}, s_{-1} = \text{a little low}, s_0 = \text{fair}, s_1 = \text{a little high}, s_2 = \text{high}, s_3 = \text{very high}\}$. The evaluations given by experts are shown as Table A.1. Then, such evaluations can be expressed by GPLPR as Table 3.

### 4.2. Solve the case study by the proposed method

It can be seen from Table 3 that there are incomplete evaluations. To model the incomplete evaluations, the envelope of GPLTS is introduced. In this regard, the normalized GPLPR is obtained as Table A.2. Based on this, the rearranged probability set $p* = (0.3, 0.3, 0.4)^T$ can be obtained by cutting the probability of $\bar{D}$. Then, the adjusted GPLPR can be generated as Table 4.

Then, three ULPRs are obtained by dimension reduction of $\bar{D}*$ according to $p* = (0.3, 0.3, 0.4)^T$, shown as Table 5.

**Table 2.** The basic information of projects.

| | Project type | Target groups |
|---|---|---|
| $x_1$ | Construction of smart community | The technology lovers, the elderly and people with mobility disabilities at the middle and high-class group of society |
| $x_2$ | Construction of green community | The environmental protection enthusiasts and the elderly at the middle and high-class group of society |
| $x_3$ | Construction of sanatorium community | The elderly and the sick at the middle and high-class of society |
| $x_4$ | Construction of single apartments | The people who are single young or tend to live in small rooms in the middle-class of society |

Source: The Authors.

**Table 3.** Decision matrix for risk assessment in GPLPR.

| $D$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $x_1$ | $\{s_0(1)\}$ | $\{[s_1,s_2](0.6),\{s_2,s_3\}(0.4)\}$ | $\{\{s_0,s_1\}(0.3),[s_1,s_2](0.7)\}$ | $\{[s_{-1},s_0](0.3),[s_1,s_2](0.3)\}$ |
| $x_2$ | $\{[s_{-1},s_{-2}](0.6),\{s_{-2},s_{-3}\}(0.4)\}$ | $\{s_0(1)\}$ | $\{[s_{-1},s_0](0.3),\{s_{-1},s_1\}(0.3)\}$ | $\{[s_0,s_1](0.3),[s_1,s_2](0.3),\ \{s_2,s_3\}(0.4)\}$ |
| $x_3$ | $\{\{s_0,s_{-1}\}(0.3),[s_{-1},s_{-2}](0.7)\}$ | $\{[s_1,s_0](0.3),\{s_1,s_{-1}\}(0.3)\}$ | $\{s_0(1)\}$ | $\{\{s_1,s_2\}(0.3),[s_2,s_3](0.7)\}$ |
| $x_4$ | $\{[s_1,s_0](0.3),[s_{-1},s_{-2}](0.3)\}$ | $\{[s_0,s_{-1}](0.3),[s_{-1},s_{-2}](0.3),\ \{s_{-2},s_{-3}\}(0.4)\}$ | $\{\{s_{-1},s_{-2}\}(0.3),[s_{-2},s_{-3}](0.7)\}$ | $\{s_0(1)\}$ |

Source: The Authors.

**Table 4.** Decision matrix for risk assessment in adjusted GPLPR.

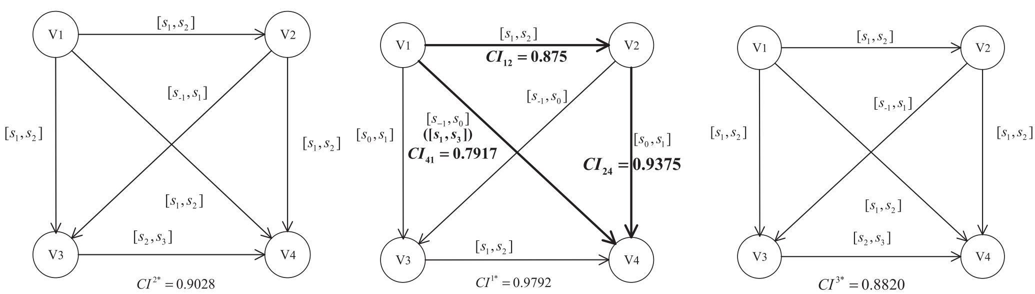
| $\bar{D}^*$ | $x_1$ | $x_2$ |
|---|---|---|
| $x_1$ | $\{[s_0,s_0](0.3),[s_0,s_0](0.3),[s_0,s_0](0.4)\}$ | $\{[s_1,s_2](0.3),[s_1,s_2](0.3),[s_2,s_3](0.4)\}$ |
| $x_2$ | $\{[s_{-1},s_{-2}](0.3),[s_{-1},s_{-2}](0.3),[s_{-2},s_{-3}](0.4)\}$ | $\{[s_0,s_0](0.3),[s_0,s_0](0.3),[s_0,s_0](0.4)\}$ |
| $x_3$ | $\{[s_0,s_{-1}](0.3),[s_{-1},s_{-2}](0.3),[s_{-1},s_{-2}](0.4)\}$ | $\{[s_1,s_0](0.3),[s_1,s_{-1}](0.3),[s_0,s_{-1}](0.4)\}$ |
| $x_4$ | $\{[s_1,s_0](0.3),[s_{-1},s_{-2}](0.3),[s_1,s_{-2}](0.4)\}$ | $\{[s_0,s_{-1}](0.3),[s_{-1},s_{-2}](0.3),[s_{-2},s_{-3}](0.4)\}$ |
| | $x_3$ | $x_4$ |
| $x_1$ | $\{[s_0,s_1](0.3),[s_1,s_2](0.3),[s_1,s_2](0.4)\}$ | $\{[s_{-1},s_0](0.3),[s_1,s_2](0.3),[s_{-1},s_2](0.4)\}$ |
| $x_2$ | $\{[s_{-1},s_0](0.3),[s_{-1},s_1](0.3),[s_0,s_1](0.4)\}$ | $\{[s_0,s_1](0.3),[s_1,s_2](0.3),[s_2,s_3](0.4)\}$ |
| $x_3$ | $\{[s_0,s_0](0.3),[s_0,s_0](0.3),[s_0,s_0](0.4)\}$ | $\{[s_1,s_2](0.3),[s_2,s_3](0.3),[s_2,s_3](0.4)\}$ |
| $x_4$ | $\{[s_{-1},s_{-2}](0.3),[s_{-2},s_{-3}](0.3),[s_{-2},s_{-3}](0.4)\}$ | $\{[s_0,s_0](0.3),[s_0,s_0](0.3),[s_0,s_0](0.4)\}$ |

Source: The Authors.

**Table 5.** Three ULPRs for risk assessment.

| | $p^1=0.3(\bar{D}^{1*})$ | | | | $p^2=0.3(\bar{D}^{2*})$ | | | | $p^3=0.4(\bar{D}^{3*})$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
| $x_1$ | $[s_0,s_0]$ | $[s_1,s_2]$ | $[s_0,s_1]$ | $[s_{-1},s_0]$ | $[s_0,s_0]$ | $[s_1,s_2]$ | $[s_1,s_2]$ | $[s_1,s_2]$ | $[s_0,s_0]$ | $[s_2,s_3]$ | $[s_1,s_2]$ | $[s_1,s_3]$ |
| $x_2$ | $[s_{-1},s_{-2}]$ | $[s_0,s_0]$ | $[s_{-1},s_0]$ | $[s_0,s_1]$ | $[s_{-1},s_{-2}]$ | $[s_0,s_0]$ | $[s_{-1},s_1]$ | $[s_1,s_2]$ | $[s_{-2},s_{-3}]$ | $[s_0,s_0]$ | $[s_0,s_1]$ | $[s_2,s_3]$ |
| $x_3$ | $[s_0,s_{-1}]$ | $[s_1,s_0]$ | $[s_0,s_0]$ | $[s_1,s_2]$ | $[s_{-1},s_{-2}]$ | $[s_1,s_{-1}]$ | $[s_0,s_0]$ | $[s_2,s_3]$ | $[s_{-1},s_{-2}]$ | $[s_0,s_{-1}]$ | $[s_0,s_0]$ | $[s_2,s_3]$ |
| $x_4$ | $[s_1,s_0]$ | $[s_0,s_{-1}]$ | $[s_{-1},s_{-2}]$ | $[s_0,s_0]$ | $[s_{-1},s_{-2}]$ | $[s_{-1},s_{-2}]$ | $[s_{-2},s_{-3}]$ | $[s_0,s_0]$ | $[s_{-1},s_{-3}]$ | $[s_{-2},s_{-3}]$ | $[s_{-2},s_{-3}]$ | $[s_0,s_0]$ |

Source: The Authors.




(a) The UL-graph of $\bar{D}^{1*}$ (b) The UL-graph of $\bar{D}^{2*}$ (c) The UL-graph of $\bar{D}^{3*}$

**Figure 5.** Three UL-graphs by the rearranged probability set $p*=(0.3,0.3,0.4)^T$.
Note: The bold line is the circular triad. The interval terms in the bracket are the evaluation after consistency correction.
Source: The Authors.

Based on the three ULPRs, we can obtain three UL-graphs as Figure 5 respectively. After the consistency checking, we can get the GPLPR $\tilde{D}*$ with the weakly additive consistency as Table 6, whose consistency level $CI(\tilde{D}*)=0.9174$.

By Eq. (6), we can get the scores of $x_i(i=1,2,3,4)$ as: $E(x_1)=s_{1.2375}$, $E(x_2)=s_{-0.0625}$, $E(x_3)=s_{0.2375}$, and $E(x_4)=s_{-1.4125}$. Then, $x_1>x_3>x_2>x_4$ can be obtained.

The results show that $x_4$ (construction of single apartments) is the investment project with the least risk. Thus, $x_4$ is the best choice for the company to invest. Compared with the construction of smart community ($x_1$), green community ($x_2$) and sanatorium community ($x_3$), the risk of investment in constructing single apartments ($x_4$) is the lowest. The reason for this may be that the number of single people buying an apartment is increasing. Due to the Chinese traditional yearning for home, the social people usually have a pursuit of apartments. Because they do not pursue

**Table 6.** The GPLPR with the weakly additive consistency.

| $\tilde{D}*$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $x_1$ | $\{s_0(1)\}$ | $\{[s_1, s_2](0.6), [s_2, s_3](0.4)\}$ | $\{[s_0, s_1](0.3), [s_1, s_2](0.7)\}$ | $\{[s_1, s_2](0.3), [s_1, s_3](0.7)\}$ |
| $x_2$ | $\{[s_{-1}, s_{-2}](0.6), [s_{-2}, s_{-3}](0.4)\}$ | $\{s_0(1)\}$ | $\{[s_{-1}, s_0](0.3), [s_{-1}, s_1](0.3), [s_0, s_1](0.4)\}$ | $\{[s_0, s_1](0.3), [s_1, s_2](0.3), [s_2, s_3](0.4)\}$ |
| $x_3$ | $\{[s_0, s_{-1}](0.3), [s_{-1}, s_{-2}](0.7)\}$ | $\{[s_1, s_0](0.3), [s_1, s_{-1}](0.3), [s_0, s_{-1}](0.4)\}$ | $\{s_0(1)\}$ | $\{[s_1, s_2](0.3), [s_2, s_3](0.7)\}$ |
| $x_4$ | $\{[s_{-1}, s_{-2}](0.3), [s_{-1}, s_{-3}](0.7)\}$ | $\{[s_0, s_{-1}](0.3), [s_{-1}, s_{-2}](0.3), [s_{-2}, s_{-3}](0.4)\}$ | $\{[s_{-1}, s_{-2}](0.3), [s_{-2}, s_{-3}](0.7)\}$ | $\{s_0(1)\}$ |

Source: The Authors.

large space or have limited funds, single person tends to choose single apartment with small space. In addition, the investment risk of constructing green community and sanatorium community is similar, and the risk of the former is slightly lower than that of the later. The risk of constructing smart community is the highest. The possible reason is that the construction of smart community needs a lot of money, and the domestic complete smart home is not perfect and popular. Based on this, the target population may have concerns about the function and quality of this community.

## 5. Discussions

To further understand the probability cutting method of GPLPRs, we have some discussions and analyses on it.

### 5.1. Necessity of the proposed additive consistency of GPLPR

If the consistency of $\bar{D}$ is not processed, the original GPLPR is aggregated directly with Eq. (4), and the following results can be obtained:$E(x_1) = s_{1.05}$, $E(x_2) = s_{-0.0625}$, $E(x_3) = s_{0.2375}$, and $E(x_4) = s_{-1.225}$. Then, we have $x_1 > x_3 > x_2 > x_4$. Thus, $x_4$ with the lowest risk is still our best choice.

It can be seen that the processing of consistency has an impact on the score values of the original preference relations, and may affect the final decision-making results. In this case, since the consistency of the original GPLPR is high enough, the results satisfying the consistency may not change much compared with the original results. However, while in the decision-making problem of ranking a large number of alternatives, the influence of the consistency may be obvious. When experts compare a large number of alternatives in pairs, it is likely to cause the problem of logical inconsistency, so it is necessary to deal with the consistency, so as to get a logical preference relationship.

### 5.2. Comparative analysis

To illustrate the effectiveness of the proposed consistency checking method, we compare it with three methods, including the subscript calculation method (Zhang et al., 2016), the improved subscript calculation method (Liang et al., 2020) and the linear programming method (Xie et al., 2019). The characteristics of the four consistency checking methods are listed in Table 7.

By applying the three methods to deal with the same case study in Section 5.1, we can get the comparative analysis of the results as shown in Table 8. The detailed steps for the three methods are shown in Appendix E.

Based on the comparative analysis, we can find that the proposed method in this paper has the following four advantages:

1. Feasible in complex probabilistic linguistic information. These four methods can measure and improve the additive consistency of a PLPR, but only the proposed

**Table 7.** Comparisons on the characteristics of four consistency checking methods.

| Comparative item | Form of evaluations | Basic method | Representation of consistency index | Consistency improvement method | Disadvantages |
|---|---|---|---|---|---|
| The proposed method | GPLPR | Probabilistic cutting | Subscript deviation | Replace the maximum difference element. | – |
| Zhang et al. (2016) | PLPR | Subscript calculation | Distance | Change all elements in a direction close to the acceptable consistency matrix. | • Virtual LTSs are difficult to understand.<br>• The calculation is complex.<br>• All elements need to be changed to improve consistency. |
| Liang et al. (2020) | PLPR | Subscript calculation | Distance | All elements are adjusted by nonlinear programming with the goal of satisfying acceptable consistency. | • Virtual LTSs are difficult to understand.<br>• All elements need to be changed to improve consistency. |
| Xie et al. (2019) | PLPR | Linear programming | Differences in expectations | Replace all elements with the ideal expectable values. | • Taking the minimum difference as the objective function, the result may not be objective enough.<br>• All elements need to be changed to improve consistency. |

Source: The Authors.

**Table 8.** Comparative analysis of the results obtained by four consistency checking methods.

| Comparative item | Acceptable consistency | Iterations | Ranking result |
|---|---|---|---|
| The proposed method | ✗ | 1 | $x_1 > x_3 > x_2 > x_4$ |
| Zhang et al. (2016) | ✓ | – | $x_1 > x_3 > x_2 > x_4$ |
| Liang et al. (2020) | ✓ | – | $x_1 > x_3 > x_2 > x_4$ |
| Xie et al. (2019) | ✗ | 1 | $x_1 > x_2 > x_3 > x_4$ |

Source: The Authors.

method can further deal with the complex additive consistency of UPLPR and GPLPR.

2. Easy to understand. In the calculation process, the methods proposed by Zhang et al. (2016) and Liang et al. (2020) both used the subscript calculation of linguistic terms to transform the original preference information into virtual linguistic terms, which is convenient to the calculation but difficult to explain the meaning. In addition, the method proposed by Xie et al. (2019) transformed the original linguistic information into numerical information through semantic functions, which is also difficult to intuitively display the meaning of the transformed information. In the process of calculation, the proposed method always keeps preference information in the set linguistic category, so the meaning can be easily understood.
3. Keeping the original information. For the preference relation with unacceptable additive consistency, the other three methods need to change all the elements of the original information, while the proposed method only needs to change some key elements which have the highest deviation from other linguistic terms in the original preference information.
4. Simple calculation. Compared with the methods proposed by Zhang et al. (2016) and Liang et al. (2020) which need to get acceptable consistent preference relations based on virtual linguistic terms, and the method proposed by Xie et al. (2019) which needs to set the deviation thresholds between alternatives, the proposed method is simple and does not need to set any parameter.

### 5.3. Managerial implications

For practicing managers, this study has some implications. First, it proposes a GPLPR for the risk assessment of investment projects. This model allows experts to give pairwise comparison information of projects, which reduces the difficulty of giving assessments. At the same time, due to the diversity of linguistic expressions allowed by a GPLPR, it can directly model the original linguistic assessments given by experts, which makes the original information not lost in the process of modelling. For managers, the effective use of information can provide reliable evidence for decision making, which is conducive to the development of enterprises.

Second, the proposed probability cutting method can improve the efficiency of decision making. It is important for managers to ensure that the obtained information is logical and non-random. Based on this, it is a vital part to measure and improve the consistency of preference relations. But for the complex GPLPR, handling the consistency is not an easy task. In this regard, the probability cutting method can reduce the difficulty of consistency processing and improve the efficiency of the whole decision-making process.

Overall, an effective method for risk assessments has been developed. Managers can use the proposed method to assess their projects on risk and select the best investment project. Based on the implementation of accurate assessments, managers can have a clear understanding of the actual situation of each project, which plays a great role in the selection of projects. Choosing an appropriate investment project may affect the development of the enterprise in the future. Therefore, it is important for managers to select an appropriate investment project by using the correct decision-making method.

## 6. Conclusion

This paper proposed a graph theory-based group decision making method with GPLPRs to deal with the risk assessment problem of project investment. It could be noted that the existing probabilistic preference models mainly include linguistic preference relations, PLPRs and uncertain linguistic preference relations, which do not contain various forms of linguistic preference. In addition, for incomplete probabilities, these models usually assigned the remaining probability proportionally to the existing linguistic expressions. This way might ignore the uncertainty caused by incomplete probability, and produce unreasonable results. Therefore, in this paper, we first proposed the GPLPR, which can cover several linguistic probabilistic preference expressions, and then introduced a method to handle the incomplete probability. To simplify the measuring the consistency of GPLPRs, a probability cutting method was proposed. It was proved that the consistency of the GPLPR can be achieved by processing the consistency of the cut linguistic preference relations separately. Besides, based on the graph theory, we proposed a method to measure and improve the consistency of GPLPRs. Finally, a graph theory-based group decision making method with GPLPRs was developed and verified by carry on the risk assessment of project investment.

In the future, the probability cutting method should be applied to deal with probabilistic preference relations with hesitant fuzzy linguistic information, extended hesitant fuzzy linguistic information, intuitive fuzzy linguistic information or other linguistic expressions in the existing methods. In addition, while aggregating the consistency of preference relations, it may be interesting to combine different aggregation operators to support the proposed model. Besides, it will also be interesting to apply the proposed method to deal with more practical decision-making problems.

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## Appendix

### A. The normalization of GPLTSs

The normalization of GPLTSs can be divided into two steps: 1) the processing of incomplete probability, 2) the transformation from subjective assessments to objective ratings.

1) The processing of incomplete probability

In a GPLTS, if $\sum_{q_1=1}^{Q_1} p^{q_1} + \sum_{q_2=1}^{Q_2} p^{q_2} = 1$, the probability is regarded as complete; if $\sum_{q_1=1}^{Q_1} p^{q_1} + \sum_{q_2=1}^{Q_2} p^{q_2} < 1$, the probability is incomplete, which is called as probability uncertainty. There are two ways to assign the remaining probability $p^{env} = 1-(\sum_{q_1=1}^{Q_1} p^{q_1} + \sum_{q_2=1}^{Q_2} p^{q_2})$: one is to assign it to the original linguistic terms equally (Pang et al., 2016); the other is to assign it to the whole LTS $S$ (Fang et al., 2021; Yang & Singh,1994; Yang & Xu, 2002). The former directly converts the uncertainty caused by incomplete probability into deterministic information, while the latter deals with the uncertainty based on the ignorance of given assessments. To deal with the probabilistic uncertainty reasonably, we assign the remaining probability to the envelope of the assessment, which cannot only contain the essential probability uncertainty, but also make full use of the given assessment. After dealing with the probability uncertainty, a GPLTS can be depicted as: $GL(p) = \{\{s_{\alpha_k}{}^{q_1}\}(p^{q_1}), [s_{t_1}{}^{q_2}, s_{t_2}{}^{q_2}](p^{q_2}), [GL^-(p), GL^+(p)](p^{env})\}$ where $[GL^-(p), GL^+(p)]$ is the envelope of the GPLTS with $GL^-(p) = \min\{\min_{q_1}\{s_{\alpha_k}{}^{q_1}\}, \min_{q_2}\{s_{t_1}{}^{q_2}\}\}$ and $GL^+(p) = \max\{\max_{q_1}\{s_{\alpha_k}{}^{q_1}\}, \max_{q_2}\{s_{t_2}{}^{q_2}\}\}$.

2) The transformation from subjective assessments to objective ratings

In practice, due to the limited experience and expertise, experts are hard to distinguish the continuous changes regarding the states of an objects. For example, an expert believes that the state of an object may be $s_1$ (good) or $s_2$ (very good), *i.e.*, the subjective assessment can be represented as $\{s_1, s_2\}$. But in fact, the state of the object cannot jump, that is to say, the objective performance should be $[s_1, s_2]$. In this sense, it is necessary to transform subjective assessments into objective ratings, namely, we can translate $\{s_{\alpha_k}{}^{q_1}\}$ in a GPLTS $GL(p)$ into an interval $[\min_{q_1}\{s_{\alpha_k}{}^{q_1}\}, \max_{q_1}\{s_{\alpha_k}{}^{q_1}\}]$. After dealing with the uncertainty in the GPLTS $GL(p)$ and transforming subjective assessments into objective ratings, all linguistic assessments in $GL(p)$ can be expressed in interval forms. Then, we can get the following normalized GPLTS (Fang, et al., 2021):$\bar{G}L(p) = \{GL^q(p^q)|q = 1, 2, \ldots, \#\bar{G}L(p)\} = \{[U^q, V^q](p^q)| \quad q = 1, 2, \ldots, \#\bar{G}L(p)\}$ where $U, V \in \bar{S}$, $U \le V$, and $\sum_{q=1}^{\#\bar{G}L(p)} p^q = 1$. If $p^{env} = 0$, then $\#\bar{G}L(p) = \#GL(p)$; if $p^{env}>0$, then $\#\bar{G}L(p) = \#GL(p) + 1$.

The following example is given to facilitate the normalization process of GPLTSs. Let $S_2 = \{s_{-3} = \text{extremely bad}, s_{-2} = \text{very bad}, s_{-1} = \text{bad}, s_0 = \text{fair}, s_1 = \text{good}, s_2 = \text{very good}, s_3 = \text{extremely good}\}$ be a discrete LTS. The corresponding continuous LTS is $\bar{S}_2 = \{s_\alpha | \alpha \in [-3, 3]\}$. Ten experts are invited to give evaluations on a production. Three experts think the production is 'fair or good', five experts think it is ranging from 'good' to 'very good', and two experts do not express their opinions. In this regard, a GPLTS can be formed as $GL_1(p) = \{\{s_0, s_1\}(0.3), [s_1, s_2](0.5)\}$, and its normalized GPLTS is $\bar{G}L_1(p) = \{[s_0, s_1](0.3), [s_1, s_2](0.5), [s_0, s_2](0.2)\}$.

## B. *The proof of Theorem 1*

**Proof:** For any $i,j=1,2,\ldots,n, i\neq j$, and all $D^{r*}=(s_{ij}^r)_{n\times n}$ (for $r=1,2,\ldots,R$) satisfying the additive consistency, we have

$$\begin{aligned} E(s_{ie}^*(p*)\oplus s_{ej}^*(p*)) &= E(s_{ie}^*(p*))\oplus E(s_{ej}^*(p*)) \\ &= E(\textstyle\sum_{r=1}^R p^r s_{ie}^r)\oplus E(\sum_{r=1}^R p^r s_{ej}^r) = \sum_{r=1}^R p^r E(s_{ie}^r\oplus s_{ej}^r) \\ &= \textstyle\sum_{r=1}^R p^r E(s_{ij}^r) = E(\sum_{r=1}^R p^r s_{ij}^r) = E(s_{ij}^*(p*)) \end{aligned}$$

## C. *The proof of Theorem 2*

***Proof:*** For any $i,j=1,2,\ldots,n, i\neq j$, $E(\bar{s}_{ij})=E(\frac{1}{n}[\oplus_{e=1}^n(\tilde{s}_{ie}\oplus\tilde{s}_{ej})])=\frac{1}{n}E(\oplus_{e=1}^n(\tilde{s}_{ie}\oplus\tilde{s}_{ej}))$. We have

$$\begin{aligned} E(\bar{s}_{ie}\oplus\bar{s}_{ej}) &= E(\frac{1}{n}(\oplus_{q=1}^n(\tilde{s}_{iq}\oplus\tilde{s}_{qe}))\oplus\frac{1}{n}(\oplus_{q=1}^n(\tilde{s}_{eq}\oplus\tilde{s}_{qj}))) \\ &= \frac{1}{n}E((\oplus_{q=1}^n(\tilde{s}_{iq}\oplus\tilde{s}_{qe}\oplus\tilde{s}_{eq}\oplus\tilde{s}_{qj}))) \\ &= \frac{1}{n}E((\oplus_{q=1}^n(\tilde{s}_{eq}\oplus\tilde{s}_{qe}))\oplus(\oplus_{q=1}^n(\tilde{s}_{iq}\oplus\tilde{s}_{qj}))) \\ &= \frac{1}{n}(E(\oplus_{q=1}^n(\tilde{s}_{eq}\oplus\tilde{s}_{qe}))\oplus E(\oplus_{q=1}^n(\tilde{s}_{iq}\oplus\tilde{s}_{qj}))) \\ &= s_0\oplus\frac{1}{n}E(\oplus_{q=1}^n(\tilde{s}_{iq}\oplus\tilde{s}_{qj})) = E(\bar{s}_{ij}) \end{aligned}$$

where $\bar{s}_{ij}=\bar{s}_{ie}\oplus\bar{s}_{ej}$. Thus, $\bar{D}=(\bar{s}_{ij})_{n\times n}$ is an additively consistent uncertain linguistic preference relation.

## D. *Tables*

**Table A.1.** Decision matrix for risk assessment.

| | $x_1$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $x_1$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ |
| $x_2$ | $[s_{-1},s_{-2}]$ | $\{s_{-2},s_{-3}\}$ | $[s_{-1},s_{-2}]$ | $[s_{-1},s_{-2}]$ | $[s_{-1},s_{-2}]$ | $\{s_{-2},s_{-3}\}$ | $[s_{-1},s_{-2}]$ | $[s_{-1},s_{-2}]$ | $\{s_{-2},s_{-3}\}$ | $\{s_{-2},s_{-3}\}$ |
| $x_3$ | $[s_{-1},s_{-2}]$ | $\{s_0,s_{-1}\}$ | $[s_{-1},s_{-2}]$ | $[s_{-1},s_{-2}]$ | $[s_{-1},s_{-2}]$ | $[s_{-1},s_{-2}]$ | $\{s_0,s_{-1}\}$ | $\{s_0,s_{-1}\}$ | $[s_{-1},s_{-2}]$ | $[s_{-1},s_{-2}]$ |
| $x_4$ | – | – | $[s_1,s_0]$ | $[s_1,s_0]$ | $[s_{-1},s_{-2}]$ | $[s_1,s_0]$ | – | $[s_{-1},s_{-2}]$ | $[s_{-1},s_{-2}]$ | – |
| | $x_2$ | | | | | | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $x_1$ | $[s_1,s_2]$ | $\{s_2,s_3\}$ | $[s_1,s_2]$ | $\{s_2,s_3\}$ | $[s_1,s_2]$ | $[s_1,s_2]$ | $[s_1,s_2]$ | $\{s_2,s_3\}$ | $[s_1,s_2]$ | $\{s_2,s_3\}$ |
| $x_2$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ |
| $x_3$ | $[s_1,s_0]$ | – | $[s_1,s_0]$ | $[s_1,s_0]$ | – | $\{s_1,s_{-1}\}$ | – | $\{s_1,s_{-1}\}$ | $\{s_1,s_{-1}\}$ | – |
| $x_4$ | $[s_0,s_{-1}]$ | $[s_0,s_{-1}]$ | $[s_{-1},s_{-2}]$ | $\{s_{-2},s_{-3}\}$ | $\{s_{-2},s_{-3}\}$ | $[s_0,s_{-1}]$ | $\{s_{-2},s_{-3}\}$ | $[s_{-1},s_{-2}]$ | $[s_{-1},s_{-2}]$ | $\{s_{-2},s_{-3}\}$ |
| | $x_3$ | | | | | | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $x_1$ | $\{s_0,s_1\}$ | $[s_1,s_2]$ | $[s_1,s_2]$ | $[s_1,s_2]$ | $[s_1,s_2]$ | $[s_1,s_2]$ | $[s_1,s_2]$ | $\{s_0,s_1\}$ | $\{s_0,s_1\}$ | $[s_1,s_2]$ |
| $x_2$ | $[s_{-1},s_0]$ | – | $[s_{-1},s_0]$ | – | $\{s_{-1},s_1\}$ | – | $\{s_{-1},s_1\}$ | $\{s_{-1},s_1\}$ | – | $[s_{-1},s_0]$ |
| $x_3$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ |
| $x_4$ | $[s_{-2},s_{-3}]$ | $\{s_{-1},s_{-2}\}$ | $[s_{-2},s_{-3}]$ | $[s_{-2},s_{-3}]$ | $\{s_{-1},s_{-2}\}$ | $[s_{-2},s_{-3}]$ | $[s_{-2},s_{-3}]$ | $[s_{-2},s_{-3}]$ | $\{s_{-1},s_{-2}\}$ | $[s_{-2},s_{-3}]$ |
| | $x_4$ | | | | | | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $x_1$ | $[s_{-1},s_0]$ | $[s_1,s_2]$ | – | $[s_1,s_2]$ | – | $[s_1,s_2]$ | – | $[s_{-1},s_0]$ | $[s_{-1},s_0]$ | – |
| $x_2$ | $[s_1,s_2]$ | $\{s_2,s_3\}$ | $\{s_2,s_3\}$ | $[s_0,s_1]$ | $[s_1,s_2]$ | $[s_1,s_2]$ | $[s_0,s_1]$ | $[s_0,s_1]$ | $\{s_2,s_3\}$ | $\{s_2,s_3\}$ |
| $x_3$ | $\{s_1,s_2\}$ | $[s_2,s_3]$ | $[s_2,s_3]$ | $[s_2,s_3]$ | $[s_2,s_3]$ | $[s_2,s_3]$ | $[s_2,s_3]$ | $[s_2,s_3]$ | $\{s_1,s_2\}$ | $\{s_1,s_2\}$ |
| $x_4$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ | $s_0$ |

**Table A.2.** Decision matrix for risk assessment in normalized GPLPR.

| $\bar{D}$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $x_1$ | $\{s_0(1)\}$ | $\{[s_1,s_2](0.6),\{s_2,s_3\}(0.4)\}$ | $\{\{s_0,s_1\}(0.3),[s_1,s_2](0.7)\}$ | $\{[s_{-1},s_0](0.3),[s_1,s_2](0.3),\ [s_{-1},s_2](0.4)\}$ |
| $x_2$ | $\{[s_{-1},s_{-2}](0.6),\{s_{-2},s_{-3}\}(0.4)\}$ | $\{s_0(1)\}$ | $\{[s_{-1},s_0](0.3),\{s_{-1},s_1\}(0.3),\ [s_0,s_1](0.4)\}$ | $\{[s_0,s_1](0.3),[s_1,s_2](0.3),\ \{s_2,s_3\}(0.4)\}$ |
| $x_3$ | $\{\{s_0,s_{-1}\}(0.3),[s_{-1},s_{-2}](0.7)\}$ | $\{[s_1,s_0](0.3),\{s_1,s_{-1}\}(0.3),\ [s_0,s_{-1}](0.4)\}$ | $\{s_0(1)\}$ | $\{\{s_1,s_2\}(0.3),[s_2,s_3](0.7)\}$ |
| $x_4$ | $\{[s_1,s_0](0.3),[s_{-1},s_{-2}](0.3),\ [s_1,s_{-2}](0.4)\}$ | $\{[s_0,s_{-1}](0.3),[s_{-1},s_{-2}](0.3),\ \{s_{-2},s_{-3}\}(0.4)\}$ | $\{\{s_{-1},s_{-2}\}(0.3),[s_{-2},s_{-3}](0.7)\}$ | $\{s_0(1)\}$ |

## E. *The calculation process of the consistency checking method*

1) The calculation process of the consistency checking method based on subscript calculation proposed by Zhang et al. (2016).

To deal with the additive consistency of PLPRs, Zhang et al. (2016) proposed a subscript calculation method based on the graph theory to directly measure and improve the additive consistency of PLPR. Next, we will use this method to solve the same case study in Section 5.1. The detailed steps are shown as follows.

*Step 1*. Establish the judgement matrix with PLPR. In the method of Zhang et al. (2016), the assessments given by experts should be PLPRs and the subscript of linguistic terms should be interval. Since the subscript calculation method is a consistency checking tool for PLPR, it is necessary to transform the original GPLPR to PLPR for the convenience of comparison. Based on this, to facilitate the comparative analysis of the two methods, there is a need to transform the GPLPR $D = (GL_{ij}(p))_{4\times4}$ to a PLPR $D' = (L_{ij}(p))_{4\times4}$ by Eq.(7).

$$L_{ij}(p) = \begin{cases} \{\max\limits_{\alpha_k}\{s_{\alpha_k,ij}{}^{q_1}\}(p_{ij}^{q_1}), s_{t_2,ij}{}^{q_2}(p_{ij}^{q_2})\} & , \quad i<j \\ s_0(1) & , \quad i=j \\ \{neg(\max\limits_{\alpha_k}\{s_{\alpha_k,ij}{}^{q_1}\})(p_{ij}^{q_1}), neg(s_{t_2,ij}{}^{q_2})(p_{ij}^{q_2})\} & , \quad i<j \end{cases} \tag{7}$$

Note: If $i<j$, we take the upper bound of each linguistic representation in $G_{ij}(p)$ to form $L_{ij}(p)$, and for $i<j$, the $L_{ij}(p) = neg(L_{ji}(p))$.

$$D' = \begin{bmatrix} \{s_0(1)\} & \{s_2(0.6), s_3(0.4)\} & \{s_1(0.3), s_2(0.7)\} & \{s_0(0.3), s_2(0.3)\} \\ \{s_{-2}(0.6), s_{-3}(0.4)\} & \{s_0(1)\} & \{s_0(0.3), s_1(0.3)\} & \{s_1(0.3), s_2(0.3), s_3(0.4)\} \\ \{s_{-1}(0.3), s_{-2}(0.7)\} & \{s_0(0.3), s_{-1}(0.3)\} & \{s_0(1)\} & \{s_2(0.3), s_3(0.7)\} \\ \{s_0(0.3), s_{-2}(0.3)\} & \{s_{-1}(0.3), s_{-2}(0.3), s_{-3}(0.4)\} & \{s_{-2}(0.3), s_{-3}(0.7)\} & \{s_0(1)\} \end{bmatrix}$$

*Step 2*. Normalize the PLPR. The remaining probability in each PLTS is assigned to the existing linguistic terms in proportion, and the linguistic terms with zero probability are added so that all elements in the judgment matrix have the same number of linguistic terms. By this way, the normalized PLPR is generated as:

$$\bar{D}' = \begin{bmatrix} \{s_0(1)\} & \{s_2(0), s_2(0.6), s_3(0.4)\} & \{s_1(0), s_1(0.3), s_2(0.7)\} & \{s_0(0), s_0(0.5), s_2(0.5)\} \\ \{s_{-2}(0), s_{-2}(0.6), s_{-3}(0.4)\} & \{s_0(1)\} & \{s_0(0), s_0(0.5), s_1(0.5)\} & \{s_1(0.3), s_2(0.3), s_3(0.4)\} \\ \{s_{-1}(0), s_{-1}(0.3), s_{-2}(0.7)\} & \{s_0(0), s_0(0.5), s_{-1}(0.5)\} & \{s_0(1)\} & \{s_2(0), s_2(0.3), s_3(0.7)\} \\ \{s_0(0), s_0(0.5), s_{-2}(0.5)\} & \{s_{-1}(0.3), s_{-2}(0.3), s_{-3}(0.4)\} & \{s_{-2}(0), s_{-2}(0.3), s_{-3}(0.7)\} & \{s_0(1)\} \end{bmatrix}$$

*Step 3*. Generate the additively consistent PLPR as $\bar{D}_C{}'$ by

$$\bar{L}_{ij}^C(p) = \begin{cases} \frac{1}{n}(\oplus_{e=1}^n(\bar{L}_{ie}^C(p) \oplus \bar{L}_{ej}^C(p))) & , i,j = 1,2,\ldots,n, i \neq j \\ \{s_0(1)\} & , otherwise \end{cases}$$

$$\bar{D}_C{}' = \begin{bmatrix} \{s_0(1)\} \\ \{s_0(0.0252), s_{-0.25}(0.0633), s_{-0.5}(0.1138), s_{-0.75}(0.1683), s_{-1}(0.1876), s_{-1.25}(0.1783), s_{-1.5}(0.1362), s_{-1.75}(0.0817), s_{-2}(0.0372), s_{-2.25}(0.0084)\} \\ \{s_{-0.25}(0), s_{-0.25}(0), s_{-0.25}(0.00945), s_{-0.5}(0.0639), s_{-0.75}(0.16635), s_{-1}(0.2353), s_{-1.25}(0.24055), s_{-1.5}(0.1861), s_{-1.75}(0.08365), s_{-2}(0.0147)\} \\ \{s_{-1.5}(0.00405), s_{-1.75}(0.02565), s_{-2}(0.06975), s_{-2.25}(0.12945), s_{-2.5}(0.18825), s_{-2.75}(0.20155), s_{-3}(0.18345), s_{-3.25}(0.11735), s_{-3.5}(0.0609), s_{-3.75}(0.0196)\} \end{bmatrix}$$

$$\begin{matrix} \{s_0(0.0252), s_{0.25}(0.0633), s_{0.5}(0.1138), s_{0.75}(0.1683), s_1(0.1876), s_{1.25}(0.1783), s_{1.5}(0.1362), s_{1.75}(0.0817), s_2(0.0372), s_{2.25}(0.0084)\} \\ \{s_0(1)\} \\ \{s_1(0), s_1(0), s_1(0.0063), s_{0.75}(0.04575), s_{0.5}(0.14025), s_{0.25}(0.2461), s_0(0.2756), s_{-0.25}(0.19555), s_{-0.5}(0.07785), s_{-0.75}(0.0126)\} \\ \{s_{-0.25}(0.0027), s_{-0.5}(0.01845), s_{-0.75}(0.0585), s_{-1}(0.12255), s_{-1.25}(0.1869), s_{-1.5}(0.21475), s_{-1.75}(0.1915), s_{-2}(0.12745), s_{-2.25}(0.0604), s_{-2.5}(0.0168)\} \end{matrix}$$

$$\begin{matrix} \{s_{0.25}(0), s_{0.25}(0), s_{0.25}(0.00945), s_{0.5}(0.0639), s_{0.75}(0.16635), s_1(0.2353), s_{1.25}(0.24055), s_{1.5}(0.1861), s_{1.75}(0.08365), s_2(0.0147)\} \\ \{s_{-1}(0), s_{-1}(0), s_{-1}(0.0063), s_{-0.75}(0.04575), s_{-0.5}(0.14025), s_{-0.25}(0.2461), s_0(0.2756), s_{0.25}(0.19555), s_{0.5}(0.07785), s_{0.75}(0.0126)\} \\ \{s_0(1)\} \\ \{s_{-0.5}(0), s_{-0.5}(0.004725), s_{-0.75}(0.033525), s_{-1}(0.096375), s_{-1.25}(0.162075), s_{-1.5}(0.202375), s_{-1.75}(0.200275), s_{-2}(0.162225), s_{-2.25}(0.104125), s_{-2.5}(0.0343)\} \end{matrix}$$

$$\begin{matrix} \{s_{1.5}(0.00405), s_{1.75}(0.02565), s_2(0.06975), s_{2.25}(0.12945), s_{2.5}(0.18825), s_{2.75}(0.20155), s_3(0.18345), s_{3.25}(0.11735), s_{3.5}(0.0609), s_{3.75}(0.0196)\} \\ \{s_{0.25}(0.0027), s_{0.5}(0.01845), s_{0.75}(0.0585), s_1(0.12255), s_{1.25}(0.1869), s_{1.5}(0.21475), s_{1.75}(0.1915), s_2(0.12745), s_{2.25}(0.0604), s_{2.5}(0.0168)\} \\ \{s_{0.5}(0), s_{0.5}(0.004725), s_{0.75}(0.033525), s_1(0.096375), s_{1.25}(0.162075), s_{1.5}(0.202375), s_{1.75}(0.200275), s_2(0.162225), s_{2.25}(0.104125), s_{2.5}(0.0343)\} \\ \{s_0(1)\} \end{matrix} \Bigg]$$

*Step 4.* Measure the consistency index of $D'$ by Eq. (8) as $CI(D') = 0.0113$. Zhang et al. (2016) used the deviation degree between $\bar{D}'$ and $\bar{D}_C{}'$ to measure the consistency index of $D'$. Next, it is necessary to compare the consistency index $CI(D')$ with the maximum acceptable deviation degree $\bar{CI}(D')$. If $CI(D') \leq \bar{CI}(D')$, then the consistency level of $D'$ is acceptable; otherwise, the consistency level of $D'$ is unacceptable. According to reference (Dong et al., 2008), for the judgment matrix with $n = 4$, the maximum acceptable deviation degree is $\bar{CI}(D') = 0.173$. Due to $CI(D') = 0.0113 < \bar{CI}(D') = 0.173$, $D'$ is of acceptable consistency obviously.

$$CI(D') = d(\bar{D}', \bar{D}_C{}') = \frac{1}{2\tau + 1}\sqrt{\frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n}\left(\sum_{r=1}^{R}(\bar{p}'_{ij}r \cdot \bar{p}_{C_{ij}}'r \cdot (\alpha'_{ij}r - \alpha_{C_{ij}}'r))\right)^2} \tag{8}$$

*Step 5.* Rank alternatives. Experts have no preference on the four projects, so the weight vector is $w = (1/4, 1/4, 1/4, 1/4)^T$. By the probabilistic linguistic weighted averaging operator (Zhang et al., 2016), the comprehensive preference values of alternatives can be obtained as: $PV_1 = \{s_0, s_{0.375}, s_{0.9}\}$, $PV_2 = \{s_{-0.225}, s_{-0.15}, s_{0.425}\}$, $PV_3 = \{s_{-0.35}, s_{0.075}, s_{0.4}\}$, $PV_4 = \{s_{-1.075}, s_{-0.3}, s_{-0.075}\}$.

Then, the scores of the comprehensive preference values of $x_i (i = 1, 2, 3, 4)$ are generated as $E(PV_1) = s_{0.425}$, $E(PV_2) = s_{0.0167}$, $E(PV_3) = s_{0.0417}$, $E(PV_4) = s_{-0.4833}$. Therefore, the order of alternative projects is obtained as $x_1 > x_3 > x_2 > x_4$.

2) The proposed calculation process of the consistency checking based on subscript calculation proposed by Liang et al. (2020).

Liang et al. (2020) pointed out that there is a deficiency in the original formula for calculating consistency index based on subscript calculation, that is, when the subscript values of linguistic terms are the same, the consistency index is zero whatever their respective probability values are, which ignores the role of probability values in consistency measurement. Based on this, Liang et al. (2020) proposed a consistency measurement method, which takes the subscript value and probability value of linguistic terms into account.

In the original subscript calculation method, the calculation formula of consistency index $CI(D')$ is replaced by

$$CI(D') = d(\bar{D}', \bar{D}_C{}') = \frac{1}{2\tau + 1}\sqrt{\frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n}\left(\frac{1}{R}\sum_{r=1}^{R}(\bar{p}'_{ij}r \cdot \alpha'_{ij}r - \bar{p}_{C_{ij}}'r \cdot \alpha_{C_{ij}}'r)^2\right)} \tag{9}$$

Due to $CI(D') = 0.0681 < \bar{CI}(D') = 0.173$, $D'$ satisfies acceptable consistency (Saaty, 1980).

In addition, other steps are consistent with the subscript calculation method. In this regard, the order of alternative projects is obtained as $x_1 > x_3 > x_2 > x_4$.

3) The calculation process of the consistency checking based on linear programming proposed by Xie et al. (2019).

Based on the normalized PLPR $\bar{D}' = (A_{ij}(p))_{n\times n}$ and $f(s_\alpha) = (I(s_\alpha) + \tau)/2\tau$ where $I(s_\alpha) = \alpha(\alpha \in \{-\tau, \ldots, 0, \ldots, \tau\})$, the transformed additive PLPR (TAP) can be generated as

$$TAP = (f(A_{ij})(p))_{n\times n}$$
$$= \begin{bmatrix} 0.5 & \{\frac{5}{6}(0),\frac{5}{6}(0.6),1(0.4)\} & \{\frac{2}{3}(0),\frac{2}{3}(0.3),\frac{5}{6}(0.7)\} & \{0.5(0),0.5(0.5),\frac{5}{6}(0.5)\} \\ \{\frac{1}{6}(0),\frac{1}{6}(0.6),0(0.4)\} & 0.5 & \{0.5(0),0.5(0.5),\frac{2}{3}(0.5)\} & \{\frac{2}{3}(0.3),\frac{5}{6}(0.3),1(0.4)\} \\ \{\frac{1}{3}(0),\frac{1}{3}(0.3),\frac{1}{6}(0.7)\} & \{0.5(0),0.5(0.5),\frac{1}{3}(0.5)\} & 0.5 & \{\frac{5}{6}(0),\frac{5}{6}(0.3),1(0.7)\} \\ \{0.5(0),0.5(0.5),\frac{1}{6}(0.5)\} & \{\frac{1}{3}(0.3),\frac{1}{6}(0.3),0(0.4)\} & \{\frac{1}{6}(0),\frac{1}{6}(0.3),0(0.7)\} & 0.5 \end{bmatrix}$$

Then, the expected transformed additive PLPR (ETAP) can be expressed by $ETAP = (ETA_{ij})_{n\times n}$, where $ETA_{ij} = \sum_{k=1}^{K_{ij}(p)} f(A_{ij}^{(k)})p_{ij}^{(k)}$ and $K_{ij}(p)$ is the number of linguistic terms in $A_{ij}$.

$$ETAP = (ETA_{ij})_{n\times n} = \begin{bmatrix} 0.5 & \frac{9}{10} & \frac{47}{60} & \frac{2}{3} \\ \frac{1}{10} & 0.5 & \frac{7}{12} & \frac{17}{20} \\ \frac{13}{60} & \frac{5}{12} & 0.5 & \frac{19}{20} \\ \frac{1}{3} & \frac{3}{20} & \frac{1}{20} & 0.5 \end{bmatrix}$$

By Eqs. (10) and (11), the priority weight vector can be generated as $v_1 = 0.4944$, $v_2 = 0.2944$, $v_3 = 0.2112$, and $v_4 = 0$, and $CI = 0.1371 > 0.01$ which is unacceptable. In this regard, the $ETAP$ can be modified as:

$$ETAP = METAP = (META_{ij})_{n\times n} = \begin{bmatrix} 0.5 & 0.7 & 0.7833 & 0.9945 \\ 0.3 & 0.5 & 0.5833 & 0.7944 \\ 0.2167 & 0.4167 & 0.5 & 0.7111 \\ 0.0055 & 0.2056 & 0.2889 & 0.5 \end{bmatrix}$$

After that, a linear programming is constructed to determine the priority weight vector of alternatives, by minimizing the deviation between the practical ETAP and the ideal ETAP. The linear programming model is shown as:

$$\begin{aligned} &\min \textstyle\sum_{i=1}^{n-1}\sum_{j=2,j>i}^{n}(d_{ij}^{+}+d_{ij}^{-}) \\ s.t.\quad & ETA_{ij} - \zeta(v_i - v_j) - 0.5 - d_{ij}^{+} + d_{ij}^{-} = 0, \\ & v_i \in [0,1],\ \textstyle\sum_{i=1}^{n} v_i = 1, \\ & d_{ij}^{+}, d_{ij}^{-} \geq 0, \\ & i,j = 1,2,\ldots,n, i<j \end{aligned} \tag{10}$$

where $d_{ij}^{+}$ and $d_{ij}^{-}$ are deviation variables, $v_i$ is the priority weight of $x_i$, and $\zeta$ reflects the deviation between two alternatives. Here, we have $\zeta = 1$ by experience (Xie et al., 2019). By this way, the priority weight vector can be generated as $v_1 = 0.4944$, $v_2 = 0.2944$, $v_3 = 0.2112$, and $v_4 = 0$.

Then, the consistency index $CI$ can be generated by:

$$CI = (2/n(n-1))\sum_{i=1}^{n-1}\sum_{j=2,j>i}^{n}\left|ETA_{ij} - \zeta(v_i - v_j) - 0.5\right| \tag{11}$$

where $\eta$ is the given threshold and $\eta = 0.01$. If $CI \leq \eta$, then the consistency level of PLPR is acceptable. By Eq. (11), $CI = 0.1371 > 0.01$ which is unacceptable. In this regard, the $ETAP$ should be modified into $METAP = (META_{ij})_{n\times n}$, where $META_{ij} = ETA_{ij} - d_{ij}^{+} + d_{ij}^{-}$. Let $ETAP = METAP$, then by Eqs. (10) and (11), we can obtain the priority weight vector as $v_1 = 0.494425$, $v_2 = 0.294425$, $v_3 = 0.211125$, and $v_4 = 0.000025$, with $CI = 0.00002 < 0.01$. Since $v_1 > v_2 > v_3 > v_4$, the ranking of alternatives can be obtained as $x_1 > x_2 > x_3 > x_4$.