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SIGNIFICANCE OF THE SPINORIAL BASIS IN RELATIVISTIC QUANTUM MECHANICS

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The problems related to the choice of the spinorial basis in the $(j,0)\oplus(0,j)$ representation space are discussed. This choice is shown to have a profound significance in the relativistic quantum theory and for physical applications. From the methodological viewpoint, this fact is related to the important dynamical role which the space-time symmetries play for all kinds of interactions.

1. Introduction

We have become accustomed to thinking of the particle world from a viewpoint of the principle of gauge invariance. Profound significance of this principle seems to be clear to everybody and it deserves to be in the place that it occupies now. Remarkable experimental confirmations of both quantum electrodynamics [1] and its non-Abelian extensions (Weinberg-Salam-Glashow model, quantum chromodynamics), Refs. 2 and 3, proved the applicability of this principle. As is known, the principle has primarily been deduced from the interaction of charged particles with the electromagnetic potential. For long, it has

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been recognized that for other kinds of particles (namely, for truly neutral particles which are supposed to be described by self/anti-self charge conjugate states), a change of phase leads to the destruction of self/anti-self conjugacy [4]. It is in this field of modern science (neutrino physics, gluon contributions in QCD etc.) that we have now most consistent indications for new physics. Without any intention to cast a shade on the great achievements of the theories based on the use of 4-vector potentials, I am going to consider the subject from a slightly different point of view. I will discuss here the constructs based only on the use of the $(j,0)\oplus(0,j)$ Lorentz group representations for a description of the particle world and of the interactions. I hope that the presented thoughts may be useful for a deeper understanding of the surprising symmetries of the Dirac equation and of the unexpected rich structure of the $(j,0)\oplus(0,j)$ representation space[2].

The spinorial basis in the standard representation of the Dirac equation[3],

$$u^{(1)}(\overset{\circ}{p}^{\mu}) = \sqrt{m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad u^{(2)}(\overset{\circ}{p}^{\mu}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\0\end{pmatrix},$$
$$v^{(1)}(\overset{\circ}{p}^{\mu}) = \sqrt{m}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad v^{(2)}(\overset{\circ}{p}^{\mu}) = \sqrt{m}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \tag{1}$$

is well understood and acceptable for the description of a Dirac particle. However, let ask ourselves, what forced us to choose the real basis for complex spinors?.. Let me attack the problem of the choice of a spinorial basis from the most general position.

I am going to consider theories based on the following four postulates:

- For an arbitrary j, the right- $(j,0)$ and the left-handed $(0,j)$ spinors transform in the following ways (according to the Wigner's ideas [11,12]):

$$\phi_R(p^\mu) = \Lambda_R(p^\mu \leftarrow \overset{\circ}{p}^{\mu})\,\phi_R(\overset{\circ}{p}^{\mu}) = \exp(+\vec{J}\cdot\vec{\varphi})\,\phi_R(\overset{\circ}{p}^{\mu}), \tag{2a}$$

$$\phi_L(p^\mu) = \Lambda_L(p^\mu \leftarrow \overset{\circ}{p}^{\mu})\,\phi_L(\overset{\circ}{p}^{\mu}) = \exp(-\vec{J}\cdot\vec{\varphi})\,\phi_L(\overset{\circ}{p}^{\mu}). \tag{2b}$$

  $\Lambda_{R,L}$ are the matrices for Lorentz boosts, $\vec{J}$ are the spin matrices for spin j and $\vec{\varphi}$ are the parameters of the a given boost. We restrict ourselves to the case of bradyons, defined by (e.g., Refs. 9 and 6):

$$\cosh(\varphi) = \gamma = \frac{1}{\sqrt{1-v^2}} = \frac{E}{m}, \quad \sinh(\varphi) = v\gamma = \frac{|\vec{p}|}{m}, \quad \hat{\varphi} = \vec{n} = \frac{\vec{p}}{|\vec{p}|}. \tag{3}$$

[2] Of course, spin-1/2 fermions, which transform according to the $(1/2,0)\oplus(0,1/2)$ representation of the Lorentz group, could be considered in such a framework as particular cases. The discussion of recent achievements [5,6] in the Weinberg $2(2j+1)$ component theory [7] can be found in Ref. 8.

[3] I use here and below the notation of Refs. 9, 6 and 10. For the 4-momentum of a particle in the rest one uses $\overset{\circ}{p}^{\mu}$.

- ϕ_L and ϕ_R are the eigenspinors of the helicity operator $(\vec{J}\cdot\vec{n})$:

$$(\vec{J}\cdot\vec{n})\,\phi_{R,L} \,=\, h\,\phi_{R,L} \tag{4}$$

  ($h = -j, -j+1, \ldots j$ is the helicity quantum number).

- The relativistic dispersion relation $E^2 - \vec{p}^{\,2} = m^2$ holds for free particles.

- Physical results do not depend on rotations of the spatial coordinate axes (in other words: the 3-space is uniform).

Since the spin-1/2 particles are most important objects in physical applications, and the Maxwell's spin-1 equations can be written in the similar 4-component form (see e.g., Ref. 13), we concentrate on the analysis of the $(1/2,0)\oplus(0,1/2)$ representation space. For the sake of a compact description the 2-spinors (left- or right-handed) are denoted as ξ. From the condition (see the second item):

$$\frac{1}{2}\,(\vec{\sigma}\cdot\vec{n})\,\xi \,=\, \pm\,\frac{1}{2}\,\xi, \tag{5}$$

and using the expressions for $\vec{n}$ in the spherical coordinates:

$$n_x = \sin\theta\cos\phi, \tag{6a}$$
$$n_y = \sin\theta\sin\phi, \tag{6b}$$
$$n_z = \cos\theta, \tag{6c}$$

we find that the Pauli spinor ξ for the eigenvalue $h = 1/2$ of the helicity operator can be parametrized as

$$\xi_{+\,1/2} \,=\, \begin{pmatrix} \xi_1 \\ \tan(\theta/2)\,e^{i\phi}\,\xi_1 \end{pmatrix} \quad \text{or} \quad \xi_{+\,1/2} \,=\, \begin{pmatrix} \cot(\theta/2)\,e^{-i\phi}\,\xi_2 \\ \xi_2 \end{pmatrix} \tag{7}$$

in terms of the azimuthal angle θ and the polar angle ϕ associated with the vector $\vec{p}\to 0$ [14,10], and for the $h = -1/2$ eigenvalue as

$$\xi_{-\,1/2} \,=\, \begin{pmatrix} \xi_1 \\ -\cot(\theta/2)\,e^{i\phi}\,\xi_1 \end{pmatrix} \quad \text{or} \quad \xi_{-\,1/2} \,=\, \begin{pmatrix} -\tan(\theta/2)\,e^{-i\phi}\,\xi_2 \\ \xi_2 \end{pmatrix}. \tag{8}$$

From the normalization condition $\xi^\dagger_{\pm 1/2}\xi_{\pm 1/2} = N^2$ (N^2 is the normalization factor) we have that the form of spinors can be chosen[4]

$$\xi_{+1/2} = N\,e^{i\,\vartheta_1^+}\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix} \quad \text{or} \quad \xi_{+1/2} = N e^{i\vartheta_2^+}\begin{pmatrix} \cos(\theta/2)e^{-i\phi} \\ \sin(\theta/2) \end{pmatrix}, \tag{9a}$$

[4] The second parametrization differs from the first one by an overall phase factor, what does not have influence on the physical results.

$$\xi_{-1/2} = N e^{i\vartheta_1^-} \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2)e^{i\phi} \end{pmatrix} \quad \text{or} \quad \xi_{-1/2} = N e^{i\vartheta_2^-} \begin{pmatrix} -\sin(\theta/2)e^{-i\phi} \\ \cos(\theta/2) \end{pmatrix}. \qquad (9b)$$

This parametrization is the same as in Eqs. (22a,22b) of Ref. 10b and with the formulas of Ref. 14 (p. 87), within the overall phase factors $\vartheta^{\pm}$. Let me note the interesting identities:

$$\xi_{+1/2}(\overset{\circ}{p}{}'^{\mu}) = e^{i(\vartheta^+ - \vartheta^-)}\xi_{-1/2}(\overset{\circ}{p}{}^{\mu}), \quad \xi_{-1/2}(\overset{\circ}{p}{}'^{\mu}) = e^{i(\vartheta^- - \vartheta^+)}\xi_{+1/2}(\overset{\circ}{p}{}^{\mu}), \qquad (10)$$

where $\overset{\circ}{p}{}'^{\mu}$ is the parity conjugated 4-momentum ($\theta' = \pi - \theta$, $\phi' = \phi + \pi$).

If we know the spin matrices for an arbitrary j, one can find similar parametrizations for spinors of higher dimension by resolving the set of equations of the $(2j+1)$- order for each value of the helicity[5]. One has a certain freedom in the choice of the spinorial basis in the $(1/2,0)$ (or $(0,1/2)$) space since, according to the fourth postulate, physical results do not depend on rotations of the spatial axes and, furthermore, one has arbitrary phase factors $e^{i\vartheta^{\pm}}$. Therefore, the commonly-used choice ($\vec{p}\,||Oz$)

$$\xi_{+1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad (11)$$

is only a convenience.

In the Dirac equation, one has two kinds of spinors, ϕ_R and ϕ_L. In Refs. 9, 6 and 10, the relation between them in the rest frame,

$$\phi_R(\overset{\circ}{p}{}^{\mu}) = \pm\phi_L(\overset{\circ}{p}{}^{\mu}), \qquad (12)$$

has been named the Ryder-Burgard relation (see also Ref. 15). It was shown (see footnote #1 in Ref. 10b) that the relation (12) can be used to derive the Dirac equation, the equation that describes eigenstates of the charge operator. Moreover, if we accept this form of the relation for $(1,0)\oplus(0,1)$ bispinors, one can construct a version of the Foldy-Nigam-Bargmann-Wightman-Wigner (FNBWW) type quantum field theory [16,12,6]. The remarkable feature of this *Dirac*-like modification of the Weinberg theory [7] is the fact that the boson and its antiboson have opposite relative intrinsic parities (like the Dirac fermions).

However, a more general form of Eq. (12) could be used. Let us assume that $\phi_R(\overset{\circ}{p}{}^{\mu})$ and $\phi_L(\overset{\circ}{p}{}^{\mu})$ are connected by the complex matrix $\mathcal{A}$, an arbitrary linear transformation, namely, $\phi_R(\overset{\circ}{p}{}^{\mu}) = \mathcal{A}\phi_L(\overset{\circ}{p}{}^{\mu})$. The unit matrix and the three Pauli σ-matrices form a complete set. Therefore, the matrix corresponding of the linear transformation $\mathcal{A}$ can be expanded as

$$\begin{aligned} \phi_R^{\pm}(\overset{\circ}{p}{}^{\mu}) &= \mathcal{A}\phi_L^{\pm}(\overset{\circ}{p}{}^{\mu}) = \left[\mathbf{1}\, c_1^0 + \vec{\sigma}\cdot\vec{c}_1\right]\phi_L^{\pm}(\overset{\circ}{p}{}^{\mu}) = \\ &= \left[c_1^0 \pm (|\Re e\,\vec{c}_1| + i|\Im m\,\vec{c}_1|)\right]\phi_L^{\pm}(\overset{\circ}{p}{}^{\mu}) = e^{i\alpha_\pm}\,\phi_L^{\pm}(\overset{\circ}{p}{}^{\mu}), \end{aligned} \qquad (13)$$

[5]See, e.g., the formulas (23a-c) in Ref. 10b and below.

where c_i are complex coefficients [6]. Above we have assumed that ϕ_L and ϕ_R are the eigenspinors of the helicity operator and have chosen the parametrization of the coefficients $\left[c_1^0 \pm (|\Re e\,\vec{c}_1| + \mathrm{i}|\Im m\,\vec{c}_1|)\right] \equiv \mathrm{e}^{\mathrm{i}\alpha_\pm}$. The modulus of the bracketed quantity (the determinant of the $\mathcal{A}$ matrix) should be equal to the one, because of the condition of the invariance of the normalization of spinors. Equation (12) corresponds to the particular choices of $\alpha_\pm = 0, \pm\pi$. By using the generalized Ryder-Burgard relation, and the fact that

$$\left[\Lambda_{L,R}(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\right]^{-1} = \left[\Lambda_{R,L}(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\right]^\dagger, \tag{14}$$

we obtain the "generalized" Dirac equation[7]:

$$\begin{aligned}\phi_R^\pm(p^\mu) &= \Lambda_R(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\,\phi_R^\pm(\overset{\circ}{p}{}^\mu) = \mathrm{e}^{\mathrm{i}\alpha_\pm}\,\Lambda_R(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\,\phi_L^\pm(\overset{\circ}{p}{}^\mu)\\ &= \mathrm{e}^{\mathrm{i}\alpha_\pm}\,\Lambda_R(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\,\Lambda_L^{-1}(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\,\phi_L^\pm(p^\mu),\end{aligned} \tag{15a}$$

$$\begin{aligned}\phi_L^\pm(p^\mu) &= \Lambda_L(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\,\phi_L^\pm(\overset{\circ}{p}{}^\mu) = \mathrm{e}^{-\mathrm{i}\alpha_\pm}\,\Lambda_L(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\,\phi_R^\pm(\overset{\circ}{p}{}^\mu)\\ &= \mathrm{e}^{-\mathrm{i}\alpha_\pm}\,\Lambda_L(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\,\Lambda_R^{-1}(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\,\phi_R^\pm(p^\mu).\end{aligned} \tag{15b}$$

Using definitions of the Lorentz boost (2,3) one can rewrite Eqs. (15a,b) in the matrix form (provided that $m\neq 0$):

$$\begin{pmatrix} -m\mathrm{e}^{-\mathrm{i}\alpha_\pm} & p_0 + (\vec{\sigma}\cdot\vec{p}) \\ p_0 - (\vec{\sigma}\cdot\vec{p}) & -m\mathrm{e}^{i\alpha_\pm} \end{pmatrix} \begin{pmatrix} \phi_R(p^\mu) \\ \phi_L(p^\mu) \end{pmatrix} = 0, \tag{16}$$

or

$$(\hat{p} - m\mathcal{T})\,\Psi(p^\mu) = 0, \tag{17}$$

with

$$\mathcal{T} = \begin{pmatrix} \mathrm{e}^{-i\alpha_\pm} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\alpha_\pm} \end{pmatrix}. \tag{18}$$

Note the particular cases:

$$\alpha_\pm = 0, 2\pi \quad : \quad (\hat{p} - m)\Psi = 0, \tag{19a}$$
$$\alpha_\pm = \pm\,\pi \quad : \quad (\hat{p} + m)\Psi = 0, \tag{19b}$$
$$\alpha_\pm = +\pi/2 \quad : \quad (\hat{p} + \mathrm{i}m\gamma_5)\Psi = 0, \tag{19c}$$
$$\alpha_\pm = -\pi/2 \quad : \quad (\hat{p} - \mathrm{i}m\gamma_5)\Psi = 0. \tag{19d}$$

[6] The signs $\pm$ correspond to the helicity of the spinors.

[7] A reminder: the Lorentz boost matrices are Hermitian for any finite-dimensional representation of the group.

The equations (19a and b) are the well-known Dirac equations for positive- and negative-energy bispinors in the momentum space. The equations of the type (19c and d) had also been discussed in the old literature, e.g.,in Ref. 17. They have been named as the Dirac equations for 4-spinors of the second kind [18,19]. Their possible relevance for the description of neutrino has been mentioned in the cited papers. Let me still note that this idea has been proposed before the appearance of the two-component model of Landau, Lee, Salam and Yang.

Since the spinors are, in general, complex quantities, it is possible to set up the Ryder-Burgard relation in the following form: $\phi_R(\overset{\circ}{p}^\mu) = \mathcal{B}\phi_L^*(\overset{\circ}{p}^\mu)$. The operation of complex conjugation is not a linear operator. Therefore, these two forms of the relation are not equivalent. It is more convenient to expand $\mathcal{B}$ in the basis of another complete set[8]: σ_2 and $\sigma_i\sigma_2$. The norm should again be conserved: $\phi^\dagger_{R,L}\,\phi_{R,L} = \phi^T_{R,L}\,\phi^*_{R,L} = N^2$. By using the procedure analogous to the above, we obtain another form of the Ryder-Burgard relation:

$$\begin{aligned}\phi_R^\pm(\overset{\circ}{p}^\mu) &= \mathcal{B}[\phi_L^\pm(\overset{\circ}{p}^\mu)]^* = \left[c_2^0\sigma_2 + (\vec{\sigma}\cdot\vec{c}_2)\sigma_2\right][\phi_L^\pm(\overset{\circ}{p}^\mu)]^* \\ &= \left[\mathrm{i}\,c_2^0\,\Theta_{[1/2]} \mp \mathrm{i}\,(|\Re e\,\vec{c}_2| + \mathrm{i}\,|\Im m\,\vec{c}_2|)\,\Theta_{[1/2]}\right][\phi_L^\pm(\overset{\circ}{p}^\mu)]^* = \mathrm{i}\,e^{\mathrm{i}\beta_\mp}\,\Theta_{[1/2]}[\phi_L^\pm(\overset{\circ}{p}^\mu)]^*\end{aligned} \tag{20}$$

and, hence, the inverse one,

$$\phi_L^\pm(\overset{\circ}{p}^\mu) = -\mathrm{i}\,e^{\mathrm{i}\beta_\mp}\,\Theta_{[1/2]}[\phi_R^\pm(\overset{\circ}{p}^\mu)]^*. \tag{21}$$

We used above that σ_2 matrix is related to the Wigner operator by $\Theta_{[1/2]} = -\mathrm{i}\sigma_2$, and the property of the Wigner operator for any spin $\Theta_{[j]}\vec{J}\Theta_{[j]}^{-1} = -\vec{J}^*$. So, if $\phi_{L,R}$ is the eigenstate of the helicity operator, then $\Theta_{[j]}\phi^*_{L,R}$ is the eigenstate with the opposite helicity quantum number:

$$(\vec{J}\cdot\vec{n})\,\Theta_{[j]}\left[\phi^h_{L,R}(p^\mu)\right]^* = -\,h\,\Theta_{[j]}\left[\phi^h_{L,R}\right]^*. \tag{22}$$

Therefore, from Eqs. (20,21), we have

$$\phi_R^\pm(p^\mu) = +\,\mathrm{i}\,e^{\mathrm{i}\beta_\mp}\,\Lambda_R(p^\mu \leftarrow \overset{\circ}{p}^\mu)\,\Theta_{[1/2]}\,[\Lambda_L^{-1}(p^\mu \leftarrow \overset{\circ}{p}^\mu)]^*\,[\phi_L^\pm(p^\mu)]^*, \tag{23a}$$
$$\phi_L^\pm(p^\mu) = -\,\mathrm{i}\,e^{\mathrm{i}\beta_\mp}\,\Lambda_L(p^\mu \leftarrow \overset{\circ}{p}^\mu)\,\Theta_{[1/2]}\,[\Lambda_R^{-1}(p^\mu \leftarrow \overset{\circ}{p}^\mu)]^*\,[\phi_R^\pm(p^\mu)]^*. \tag{23b}$$

Using the mentioned property of the Wigner operator, we transform Eqs. (23a,b) to

$$\phi_R^\pm(p^\mu) = +\,\mathrm{i}e^{\mathrm{i}\beta_\mp}\Theta_{[1/2]}[\phi_L^\pm(p^\mu)]^*, \tag{24a}$$
$$\phi_L^\pm(p^\mu) = -\,\mathrm{i}e^{\mathrm{i}\beta_\mp}\Theta_{[1/2]}[\phi_R^\pm(p^\mu)]^*. \tag{24b}$$

[8] It is known that after a multiplication by a non-singular matrix, the property of completeness of a set remains valid.

In the matrix form one has

$$\begin{pmatrix}\phi_R(p^\mu)\\ \phi_L(p^\mu)\end{pmatrix} = e^{i\beta_\mp}\begin{pmatrix}0 & i\Theta_{[1/2]}\\ -i\Theta_{[1/2]} & 0\end{pmatrix}\begin{pmatrix}\phi_R^*(p^\mu)\\ \phi_L^*(p^\mu)\end{pmatrix} = S^c_{[1/2]}\begin{pmatrix}\phi_R(p^\mu)\\ \phi_L(p^\mu)\end{pmatrix}, \qquad (25)$$

where $S^c_{[1/2]}$ is the operator of charge conjugation in the $(1/2,0)\oplus(0,1/2)$ representation space, see, e.g., Ref. 20. In fact, we obtain the conditions of the self/anti-self charge conjugacy:

$$\Psi(p^\mu) = \pm\Psi^c(p^\mu). \qquad (26)$$

Thus, depending on the relations between the left- and right-handed spinors (in fact, depending on the choice of the spinorial basis), we obtain physical excitations of a different physical nature. In the first version of the Ryder-Burgard relation, we have the Dirac equations; in the framework of the second version, neutral fermions[9].

The most general form of the Ryder-Burgard relation is[10]

$$\phi_R(\overset{\circ}{p}{}^\mu) = \mathcal{A}\phi_L(\overset{\circ}{p}{}^\mu) + \mathcal{B}\phi_L^*(\overset{\circ}{p}{}^\mu), \qquad (27)$$

which results in

$$\phi_R(\overset{\circ}{p}{}^\mu) = A\,e^{i\alpha_\pm}\,\phi_L(\overset{\circ}{p}{}^\mu) + i\,B\,e^{i\beta_\mp}\,\Theta_{[1/2]}\,\phi_L^*(\overset{\circ}{p}{}^\mu) \qquad (28a)$$

$$\phi_L(\overset{\circ}{p}{}^\mu) = A\,e^{-i\alpha_\pm}\,\phi_R(\overset{\circ}{p}{}^\mu) - i\,B\,e^{i\beta_\mp}\,\Theta_{[1/2]}\,\phi_R^*(\overset{\circ}{p}{}^\mu). \qquad (28b)$$

The equation, that could be considered as a mathematical generalization of the Dirac equation, is then

$$\begin{pmatrix}\mathbf{-1} & Ae^{i\alpha_\pm}\Lambda_R\Lambda_L^{-1} + iBe^{i\beta_\mp}\Theta_{[1/2]}\mathcal{K}\\ Ae^{-i\alpha_\pm}\Lambda_L\Lambda_R^{-1} - iBe^{i\beta_\mp}\Theta_{[1/2]}\mathcal{K} & \mathbf{-1}\end{pmatrix}\begin{pmatrix}\phi_R(p^\mu)\\ \phi_L(p^\mu)\end{pmatrix} = 0, \qquad (29)$$

where $A^2+B^2=1$ and $\mathcal{K}$ is the operation of complex conjugation. In a symbolic form, it is rewritten as

$$\left[A\frac{\hat{p}}{m} + B\,\mathcal{T}\,S^c_{[1/2]} - \mathcal{T}\right]\Psi(p^\mu) = 0. \qquad (30)$$

Using the computer algebra system MATEMATICA 2.2, it is easy to check that the equation satisfies the correct relativistic dispersion relation (see the third item of the set of postulates). What physical meaning could be attached to this equation?

Consider the problem of the choice of the spinorial basis in the $j=1$ case. In the present consideration, I use the Weinberg $2(2j+1)$–component formalism [7]. It is easy

[9] In fact, the second definition of the Ryder-Burgard relation leads to the Majorana-McLennan-Case spinors [21,22]. Recent discussions of this construct can be found in Refs. 23, 10 and 24.

[10] We deal above with the spinors of the same helicities. Without much effort, the reader can reveal what happens if one joins the spinors of different helicities, $\phi_R^\pm = \mathcal{A}\phi_L^\mp$ or $\phi_R^\pm = \mathcal{B}[\phi_L^\mp]^*$.

to show, by using the same procedure, that $j=1$ spinors ξ can be parametrized, e.g., in the following form

$$\xi_{+1} = N\,\mathrm{e}^{i\vartheta_+}\begin{pmatrix}\frac{1}{2}(1+\cos\theta)\mathrm{e}^{-i\phi}\\ \sqrt{\frac{1}{2}}\sin\theta\\ \frac{1}{2}(1-\cos\theta)\mathrm{e}^{+i\phi}\end{pmatrix}, \tag{31a}$$

$$\xi_{-1} = N\,\mathrm{e}^{i\vartheta_-}\begin{pmatrix}-\frac{1}{2}(1-\cos\theta)\mathrm{e}^{-i\phi}\\ \sqrt{\frac{1}{2}}\sin\theta\\ -\frac{1}{2}(1+\cos\theta)\mathrm{e}^{+i\phi}\end{pmatrix}, \tag{31b}$$

$$\xi_{0} = N\,\mathrm{e}^{i\vartheta_0}\begin{pmatrix}-\sqrt{\frac{1}{2}}\sin\theta\,\mathrm{e}^{-i\phi}\\ \cos\theta\\ \sqrt{\frac{1}{2}}\sin\theta\,\mathrm{e}^{+i\phi}\end{pmatrix}, \tag{31c}$$

provided they are eigenspinors of the helicity operator. In the isotropic-basis representation, the $j=1$ spin operators are expressed [25][11] in the following way:

$$J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \quad J_2 = \frac{i}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}, \quad J_3 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}. \tag{33}$$

The eigenvalues of the operator $\vec{J}\cdot\vec{n}$ could be $h=\pm1, 0$. As opposed to the spin-1/2 case, one has $9=3^2$ linear independent matrices forming the complete set. They can be chosen from the following set of the ten symmetric matrices

$$J_{00} = \mathbf{1}, \quad J_{0i} = J_{i0} = \vec{J}_i, \tag{34}$$

$$J_{ij} = \vec{J}_i\vec{J}_j + \vec{J}_j\vec{J}_i - \delta_{ij}. \tag{35}$$

The condition $J_{\mu\mu}=0$ eliminates one of the $J_{\mu\nu}$ matrices (e.g., J_{00}). Following the main points of the preceding discussion, consider relations between left and right spinors. The following form of the Ryder-Burgard relation:

$$\phi_R^{\pm,0}(\overset{\circ}{p}{}^{\mu}) = \mathrm{e}^{i\alpha_{\pm,0}}\phi_L^{\pm,0}(\overset{\circ}{p}{}^{\mu}), \quad \phi_L^{\pm,0}(\overset{\circ}{p}{}^{\mu}) = \mathrm{e}^{-i\alpha_{\pm,0}}\phi_R^{\pm,0}(\overset{\circ}{p}{}^{\mu}) \tag{36}$$

[11] Of course, it is possible to choose the so-called 'orthogonal' basis $(\vec{J}_i)_{jk} = -i\epsilon_{ijk}$ because they are connected by the unitary matrix $\vec{J}^{isotr.} = \mathcal{U}\vec{J}^{orth.}\mathcal{U}^{-1}$:

$$\mathcal{U} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&-i&0\\0&0&-\sqrt{2}\\-1&-i&0\end{pmatrix}. \tag{32}$$

is very similar to the first form of the relation in the spin-1/2 case. In the process of deriving this relation, we used that any tensor can be expanded in a direct product of two vectors. The equation obtained by using the Wigner postulate (item 1, $m^2 \neq 0$)

$$\left[\gamma_{\mu\nu} p^\mu p^\nu - m^2 \mathcal{T}\right] \Psi(p^\mu) = 0 \tag{37}$$

in the case $\alpha_{\pm,0} = 0$, is identical to the Weinberg equation and, after taking into account $\alpha_{\pm,0} = \pm\pi$, with the modified equation obtained by Ahluwalia [6] within the framework of the FNBWW-type quantum field theory [16,12].

As for the second form (connecting $\phi_{L,R}$ and $\phi^*_{L,R}$), essentially different relations are found when comparing to the spin-1/2 case. Expanding the $\mathcal{B}$ matrix, $\phi_R(\mathring{p}^\mu) = \mathcal{B}\phi^*_L(\mathring{p}^\mu)$, in the other complete set[12], namely, $J_{\mu\nu}\Theta_{[1]}$ one obtains

$$\phi_R^{\pm,0}(\mathring{p}^\mu) = e^{i\beta_{\mp,0}}\,\Theta_{[1]}\,[\phi_L^{\pm,0}(\mathring{p}^\mu)]^*, \quad \phi_L^{\pm,0}(\mathring{p}^\mu) = e^{i\beta_{\mp,0}}\,\Theta_{[1]}\,[\phi_R^{\pm,0}(\mathring{p}^\mu)]^*. \tag{39}$$

This fact is connected with another property of the Wigner operator: $\Theta_{[j]}\Theta_{[j]} = (-1)^{2j}$. As a result, we obtain

$$\Psi(p^\mu) = \begin{pmatrix} \phi_R(p^\mu) \\ \phi_L(p^\mu) \end{pmatrix} = e^{i\beta_{\pm,0}} \begin{pmatrix} 0 & \Theta_{[1]}\mathcal{K} \\ \Theta_{[1]}\mathcal{K} & 0 \end{pmatrix} \begin{pmatrix} \phi_R(p^\mu) \\ \phi_L(p^\mu) \end{pmatrix} = \Gamma_5 S^c_{[1]} \Psi(p^\mu), \tag{40}$$

provided that the charge conjugation operator $S^c_{[1]}$ is chosen as in Refs. 6, 23 and 10, i.e., in accordance with the FNBWW construct.

Let us pay attention to other ways for the description of a particle of arbitrary spin. In general, it is possible to choose another representation of the Lorentz group for the description of higher spin particles (see Ref. 7c). It is interesting to note that the well-known Dirac-Fierz-Pauli equation for any spin has been rewritten in Ref. 26 (see also my recent work [27]) in the form that is very similar to the spin-1/2 case:

$$\alpha^\mu \partial_\mu \Phi = +m\Upsilon, \tag{41a}$$
$$\overline{\alpha}^\mu \partial_\mu \Upsilon = -m\Phi, \tag{41b}$$

where $\overline{\alpha}^\mu = \alpha_\mu$ are the matrices which satisfy all the algebraic relations that the Pauli 2×2 matrices σ^μ do, except for the completeness. The object Φ belongs to the $(j,0)\oplus(j-1,0)$ representation of the Lorentz group and the Υ, to the $(j-1/2,1/2)$ representation. Does there exist the analog of the Ryder-Burgard relation which could be proposed in the framework of the Dirac-Fierz-Dowker construct for any spin?

[12] The explicit form of the Wigner operator $\Theta_{[1]}$ for spin 1 has been given in Refs. 10 and 26 in the isotropic basis:

$$\Theta_{[1]} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{38}$$

2. Conclusion

In this work, an attempt was made to explain how all possible relations between the basis vectors of the different representation spaces (e.g., between right- $\phi_R(\overset{\circ}{p}^\mu)$ and left-handed spinors $\phi_L(\overset{\circ}{p}^\mu)$ that are known to become interchanged under parity conjugation [9]), define dynamical equations. It was found that, from a mathematical viewpoint, the well-known equations are the particular cases only. The analysis reveals that the choice of spinorial basis in the $(j,0)\oplus(0,j)$ representation space has a profound significance for the dynamical evolution of the physical systems.

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VAŽNOST SPINORNIH BAZA U RELATIVISTIČKOJ KVANTNOJ MEHANICI

Raspravljaju se problemi odabira spinornih baza u reprezentaciji $(j,0)\oplus(0,j)$. Pokazuje se da taj odabir ima duboku važnost u relativističkoj kvantnoj teoriji i u fizičkim primjenama. S metodološkog stanovišta, ta je činjenica u uskoj vezi s važnom dinamičkom ulogom koju imaju vremensko-prostorne simetrije u svim vrstama uzajamnih djelovanja.