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EXACT SOLUTIONS OF A NON-POLYNOMIAL INTERACTION

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Exact solvability of a non-polynomial interaction is investigated. The relation to the existence of a double-well potential is discussed, with a presentation of a new type of exact solutions.

1. Introduction

The interaction under consideration has the form:

$$V(A,B,k,b;x) = \frac{A}{x^2} + Bx^2 + \frac{kx^2}{x^2+b}, \tag{1}$$

where A, B, k and b are the parameters. It obviously reduces to two well known cases of mathematical physics:

(a) for $k = 0$, it corresponds to the harmonic oscillator problem,

(b) for $k \neq 0$ and $A = 0$, it corresponds to the case of a two-parameter non-polynomial interaction

$$V(\lambda,g;x) = x^2 + \frac{\lambda x^2}{gx^2+1}, \tag{2}$$

with the following replacements:

$$\lambda = \frac{k}{b}, \qquad b = \frac{1}{g} \qquad \text{and} \qquad B = 1. \tag{3}$$

It may be noted that in recent years the second case acquired a special status being the basis of a considerable volume of literature because of its potential applications in various fields, ranging from laser physics to elementary particle theory [1-3].

For an overview of the actual situation, the list of references in Ref. 4 is recommended. The present discussion will be limited to some aspects of exact solvalibility of the Schrödinger equation.

We shall consider a generalisation of the two-parameter case by introducing two other parameters so that the ensuing discussion will involve three or four parameters.

In the first part of the presentation, a formulation adapted to the problem is introduced leading to the construction of exact solutions. It will be shown that corresponding to each state $|n\rangle$ (n is the quantum number), these solutions can always be derived from a specific constraint among the parameters, or, in other words, these solutions must be eigenfunctions of a specific interaction V_n. After the reduction to the case of two parameters using (3), they serve to check the exactness of the solutions proposed by other authors who follow various different approaches [5-8].

When two states $|n\rangle$ and $|m\rangle$, $n \neq m$, are eigenfunctions of the same interaction of type $V_{m,n}$, they constitute a "doublet". In the second part, the construction of these doublets will be outlined, followed by a discussion on a relationship between these exact doublets and the existence of a double-well structure of the interaction $V_{m,n}$.

Generalisation of the problem with inclusion of a fourth parameter will be considered in the last part where it is shown that the present formulation allows the derivation of new solutions of different types.

2. Formulation

Let

$$H\psi_n = E_n\psi_n, \qquad H = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - V(x), \tag{4}$$

and write the eigenfunction in the form:

$$\psi_n = \mathrm{e}^{-\int u \mathrm{d}s} \sum_{p=0}^{n} a_p f^{r+p}, \tag{5}$$

where r is an arbitrary parameter, $\{a_p\}$ represents the set of unknown coefficients of the expansion, the function $u(x)$ is defined as:

$$u(x) = \frac{2\alpha}{f'} + \beta\frac{f}{f'},$$

α and β are constants and $f(x) = x^2 + b$. Define the potential $V(\alpha, \beta, k, b; x)$ as

$$V(\alpha, \beta, k, b, x) = u^2 - u' + \frac{kx^2}{x^2 + b}. \tag{6}$$

Its connection with (1) is then $V(\alpha, \beta, k, b; x) = V(A, B, k, b; x) + D$, where

$$D = \beta(\alpha + \beta b/2 - 1/2), \quad A = (\alpha + a)(\alpha + a + 1), \quad a = \beta b/2 \quad \text{and} \quad B = \beta^2/4.$$

The two-parameter potential (2) can be obtained with the transformation

$$\beta = 2, \quad k = \frac{\lambda}{g}, \quad b = \frac{1}{g} \quad \text{and} \quad \alpha = -a \quad \text{or} \quad \alpha = -(a+1), \tag{3'}$$

so that the results obtained for (6) can always be compared to the ones corresponding to (2), which is then considered as a special case of it.

3. *Exact solutions*

Replacing (5) into (4), and after some simple algebra, the following three-term recursion relation can be established:

$$A_p a_p + B_{p-1} a_{p-1} + C_{p-2} a_{p-2} = 0, \tag{7}$$

in which

$$\begin{aligned} A_p &= -4b(r+p)(r+p-1) \\ B_{p-1} &= 2(r+p-1)[2(r+p-2)+1-2\alpha]+kb \\ C_{p-2} &= E-[k+D+2\beta(r+p-2)]. \end{aligned} \tag{8}$$

For an expansion of order n, it will be necessary that:

$$\text{(a)} \quad C_n = 0, \qquad \text{(b)} \quad a_{n+1} = 0. \tag{9}$$

Note that

$$a_n = (-1)^n \left[\prod_{s=0}^{n} \frac{1}{A_s} \right] \Delta_n. \tag{10}$$

Δ_n is the determinant defined by

$$\Delta_n = \begin{vmatrix} B_0 & A_1 & 0 & \cdots & & & \\ C_0 & B_1 & A_2 & 0 & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & \cdot & A_{n-1} & 0 & \\ & & & \cdot & C_{n-1} & B_n & \end{vmatrix}. \tag{11}$$

The condition of type (b) means that $\Delta_n = 0$, which in turn imposes a constraint of the type $k_n(\alpha,\beta,k,b)$, binding the four parameters together. This implies that to each state $|n\rangle$ must correspond a potential $V_n(\alpha,\beta,k,b,x)$. If $a_0 = 1$, we have:

$$a_1 = -\frac{B_0}{A_1}, \qquad a_2 = \frac{1}{A_2 A_1}\left[B_0 B_1 - C_0 A_1\right], \tag{12}$$

$$a_3 = -\frac{1}{A_3A_2A_1}\left[B_2(B_0B_1 - C_0A_1) - C_1B_2A_2\right]\ldots.$$

Taking $r = 1$, the exact solutions can then be written as:

For $n = 0$,

$$E_0 = k + D + 2\beta, \quad kb = -2(1-2\alpha), \quad \psi_0 = x^{-(\alpha+\beta b/2)}(x^2+b)\mathrm{e}^{-\beta x^2/4}. \tag{13}$$

For $n = 1$,

$$E_1 = k + D + 4\beta, \qquad kb = -(4+3\gamma) + \sqrt{(4+3\gamma)^2 - 8\gamma(\gamma+2) - 16\beta b}, \tag{14}$$

$$\psi_1 = x^{-(\alpha+\beta b/2)}(x^2+b)(1+a_1 f)\mathrm{e}^{-\beta x^2/4}, \quad \gamma = 1-2\alpha.$$

For $n = m$,

$$E_m = k + D + 2(m-1)\beta, \quad \psi_m = x^{-(\alpha+\beta b/2)}(x^2+b)\sum_{s=0}^{m} a_s f^s. \tag{15}$$

In order to verify the consistency of the method under consideration, the simplest way is to return to the special case of two parameters by making the use of transfomations (3′). The interested reader can then check that the results (13) and (14) found here are in exact agreement with those obtained in other papers quoted above.

As these solutions are actually well known, it does not seem necessary to carry on further the discussion in this direction. It is more important to underline the following remarks:

1) The assumed positive values of the quantity b as well as the normalization of the eigenfunctions require that

$$\alpha < 0, \quad |\alpha| > \frac{\beta b}{2}, \quad \alpha + \frac{\beta b}{2} = integer, \quad \beta > 0.$$

2) Let $\alpha + \beta b/2 = -m, \quad m = 0, 1, 2, \ldots$. The above solutions have been derived for the cases $m = 0, 1$, which means that to each state $|n\rangle$, there may be two types of solutions depending on the choice of m, the even and the odd solutions, respectively. However, the situation becomes qualitativelly different for $m > 1$, since

$$V(\alpha, \beta, k, b : x) = \frac{m(m-1)}{x^2} + \frac{\beta^2 x^2}{4} + \frac{kx^2}{x^2+b}. \tag{16}$$

3) When $b \to \infty$ and $A_m \to 0$, so that (7) reduces to a two terms recursion relation which can be solved exactly, corresponding to a combination of a harmonic plus inverse harmonic term for the interaction [9].

4) For $m = 0,1$, one may wonder what would be the conditions such that the interaction (6) may present a double-well structure? From the relation

$$\frac{\mathrm{d}V}{\mathrm{d}x} = x\left[\frac{\beta^2}{2} + \frac{2kb}{(x^2+b)^2}\right]$$

it can be shown that this condition must be:

$$x(-\infty,+\infty), \quad \beta > 0, \quad b > 0, \quad k < 0 \quad \text{and} \quad \sqrt{|k|} > \frac{\beta\sqrt{b}}{2}. \tag{17}$$

4. Exactly solvable doublet

Consider two exact solutions ψ_n and ψ_q. They correspond to two different interactions V_n and V_q since their construction depends on two different constraints

$$F_n(\alpha,\beta,k,b) = 0 \quad \text{and} \quad F_q(\alpha,\beta,k,b) = 0. \tag{18}$$

The states $|n\rangle$ and $|q\rangle$ are components of a doublet when they are eigenstates of the same potential.

In order to construct this potential, it can be noted from (18) that it will always be possible to eliminate one of the parameters from one of the constraints and use it in the second one resulting in a reduced type of constraint with one parameters less, for instance:

$$f_{n,q}(k(\alpha,\beta,b),\beta,b) = 0.$$

If this equation can be solved exactly, then a constraint between the remaining parameters can be obtained from which the whole set of exact solutions for the "doublet" can be inferred.

For convenience in the ensuing discussion, these doublets will be designated by a pair of numbers using the following notation $[(n,m);(q,m)]$, the first and second number referring to the order of the excited state and the choice of the number m.

Example I. The simplest doublet is $[(0,1);(1,0)]$. Noting that the two choices for the parameter m above corresponds to the relations

$\gamma = b\beta + 1$ for the first choice, and

$\gamma = b\beta + 3$ for the second one.

For this case, the reduced constraint is simple:

$$f_{0,1}(\beta,b) = \beta b - 3 = 0,$$

leading to the following solutions:

$$b = \tfrac{3}{\beta}\,, \qquad k = -4\beta\,,$$
$$E_0 = -\tfrac{1}{2}\beta\,, \qquad \psi_0 \approx x(x^2+\tfrac{3}{\beta})\mathrm{e}^{-\beta x^2/4}\,,$$
$$E_1 = +\tfrac{1}{2}\beta\,, \qquad \psi_1 \approx (x^2+\tfrac{3}{\beta})(x^2-\tfrac{3}{\beta})\mathrm{e}^{-\beta x^2/4}\,,$$

where β is arbitrary but must be positive. These results show the following:

- The existence of a double-well structure is found, since the condition (17) is always satisfied here.
- Using the transformation ($3'$), the reader may verify that the above results are in exact agreement with those obtained by other authors with other types of formulation, for instance supersymmetrisation or expansion in term of the variable [10–15].

Example II. The doublet to be considered here is $[(0,1);(2,0)]$. The reduced constraint is

$$f_{0,2}(\beta,b) = \beta^2 b^2 + 3\beta b - 6 = 0. \tag{19}$$

The solutions are: $+1.37\beta$ and $-4,372\beta$. As both β and b are assumed to be positive, only the first solution is retained. Hence, the solutions are:

$$k = -6.3725\beta,$$

$$E_0 = -2.8725\beta, \quad \psi_0 \approx x(x^2 + \frac{1.3722}{\beta})e^{-\beta x^2/4},$$

$$E_2 = +.1275\beta, \quad \psi_2 \approx (x^2 + \frac{1.3722}{\beta})\left(x^4 - \frac{12}{\beta}x^2 + \frac{24.1729}{\beta^2}\right) e^{-\beta x^2/4}.$$

- Again, the condition (17) remains valid, implying the existence of a double-well structure of the interaction.
- For the special case $\beta = 2$, an agreement with some authors [11,13] is obtained.

A remark: It should be mentioned that some other possibilities, for instance in Example I, are:

$$[(00);(11)]\,, \qquad [(01);(11)]\,, \qquad [(00);(11)]\,.$$

While the first two cases do not lead to any solution, the third one may be significant (real solution) in which the quantity b must be negative.

This alternative situation, which persists for other excited states is, however, not of concern for the moment and will be developed later on.

Generalisation. The use of the relation $\alpha + \frac{1}{2}\beta b = m$ in this formulation provides a natural way to extend the discussion from a more general point of view. Obviously, if $m \neq 0,1$, the solutions developed in the first part correspond to the cases (16). Extension means that the quantity γ must be written as:

$$\gamma = \beta b + 2m + 1 \qquad \text{or} \quad \gamma = \beta b + 2m + 3 \qquad (m \to m+1). \tag{20}$$

With this generalisation, the problem is subject to a radical qualitative shift, since the interaction (16) is now a double-well separated by an infinite barrier. However, the line of reasoning stil remains valid from the mathematical point of view so that this new situation is also exactly solvable but with appearance of a new type of solution referred to as solution of type II.

Exact solutions of type I. The constraint corresponding to a state $|n\rangle$ can be written as

$$k_n(\gamma,\beta,b,k) = 0,$$

where γ is given by (20).

Example I. Following exactly the same procedure with $\gamma = \beta b + 2m + 1$, the solutions are:

$$k = -2(\beta + \frac{2m+1}{b}), \qquad b \text{ is arbitrary}, \quad E_0 = (m - \frac{1}{2})\beta - 2\frac{2m+1}{b},$$

$$\psi_0 \approx x^m (x^2 + b) \mathrm{e}^{-\beta x^2/4}. \tag{21}$$

This is the case (8) discussed in Ref. 11.

Example II. For the first excited state

$$k = -\left[3\beta + \frac{7+6m}{b} + \sqrt{\Delta}\right],$$

$$\Delta = \beta^2 + \frac{2\beta(2m-3)}{b} + \frac{4m(m+5)+25}{b^2},$$

$$E_1 = \frac{3}{2}\beta + k, \qquad \psi_1 \approx x^m (x^2+b)(x^2+A_1)\mathrm{e}^{-\beta x^2/4}, \tag{22}$$

$$A_1 = -\frac{1}{8}\left[\beta + \frac{2m+5+\sqrt{\Delta}}{b}\right].$$

Remarks:

(a) Since β and b are assumed to be positive, the quantity A_1 must be negative, indicating that this state represents effectively the first excited state (existence of a single node).

(b) It can be verified that for the special case $m = 0$, and after using the reduction (3′) which leads to the type of interaction (2),

$$\lambda = \frac{k}{b} = -(6+7m) + \sqrt{[25g^2 - 12g + 4]}$$

which, of course, is a well known result.

(c) The same line of reasoning remains valid for higher excited states.

Exact solutions of type II. Consider now the couple $\{(0,m+1); (n,m)\}$. The reason for the change of notation will be clarified below. From the first part above, we know that the exact eigenfunctions must take the form

$$\psi_0 \approx x^{m+1}(x^2+b)e^{-\beta x^2/4},$$

$$\psi_n \approx x^m(x^2+b)\left(\sum_p a_p f^p\right)e^{-\beta x^2/4}.$$

The exact solutions depend on the constraints

$$k_0(\gamma,\beta,k,b) = 0 \qquad k_n(\gamma,\beta,k,b) = 0 \tag{23}$$

given above.

Following the same procedure as in the second part, one can derive the reduced constraint

$$f_{0,n}(\beta,b,m) = 0. \tag{24}$$

Solving the last equation leads to a certain relationship between the parameters β, b and m from which the whole set of eigenfunctions and eigenvalues are derived.

Example. Consider the couple $\{(0,m+1),(1,m)\}$ for which the reduced constraint reads:

$$f_{0,1}(\beta,b,m) = \beta b-(2m+3) = 0,$$

so that the set of solutions is:

$$k = -4\beta, \qquad b = \frac{2m+3}{\beta},$$

$$E_0 = -\frac{2m-1}{2}\beta, \qquad \psi_0 \approx x^{m+1}(x^2+\frac{2m+3}{\beta})e^{-\beta x^2/4}, \tag{25}$$

$$E_1 = \frac{2m+1}{2}\beta, \qquad \psi_0 \approx x^m(x^2+\frac{2m+3}{\beta})(x^2+\frac{2m-3}{\beta})e^{-\beta x^2/4}.$$

One should note:

- For the special case $m = 0$, these results reduce exactly to those obtained in the second part (Example I). This is expected since the removal of the singular term $m(m\pm 1)/x^2$ requires ψ_0 and ψ_1 to be the two components of a “doublet”.

- For $m > 0$, they are not components of a doublet since they are eigenfunctions of two different interactions. Therefore, ψ_0 and ψ_1 must be interpreted as eigenfunctions corresponding the potential (16) with $m(m \pm 1)$.

These types of solutions are new since, at least to our knowledge, they have not yet been mentioned in the literature.

The procedure can be continued to higher states for which the reduced constraint corresponds to an n-th order algebraic equation. For instance, with the couple $\{(0, m+1), (2, m)\}$, for which the reduced constraint is

$$y^2 + (3 + \frac{4m}{3})y - [2m(\frac{2}{3}m + 3) + 6] = 0,$$

the exact solutions are then

$$A = \frac{64}{9}m^2 + 32m + 33, \qquad B = \sqrt{A} - (3 + \frac{4}{3}m),$$

$$k = -2\beta \left[1 + \frac{2m+3}{B} \right],$$

$$E_0 = -4\beta \frac{2m+3}{B}, \qquad E_2 = E_0 + 3\beta,$$

with the analytic expression of the solutions:

$$\psi_0 \approx x^{m+1}(x^2 + \frac{B}{2\beta})e^{-\beta x^2/4}, \tag{26}$$

$$\psi_1 \approx x^m(x^2 + \frac{B}{2\beta})(1 + a_1 f + a_2 f^2)e^{-\beta x^2/4},$$

where the coefficients a_1 and a_2 can be inferred from (12):

$$a_1 = -\frac{\beta}{4}\frac{2m+3}{B}, \qquad a_2 = \frac{0.5092\beta^2}{B^2}(\frac{1}{2}B - \frac{2}{3}m - 1).$$

Again, note that for the special case $m = 0$, they reduce exactly to the results given above for the doublet [(0,1);(2,0)].

It should be noted that as m must be a positive integer, it can be considered as a dynamical parameter. For example, in the 3-dimensional space, it may be identified with the usual orbital quantum number l so that for the state $|n\rangle$, $s \leq l$, and the solution must be multiplied by the usual spherical harmonics $Y_l^s(\theta, \phi)$, i.e., with a $(2l+1)$-fold degeneracy, and $x[0, \infty]$.

On the other hand, m can also be considered as a static parameter and may take any positive integer value with no degeneracy.

5. Summary

It has been shown that:

(a) The inclusion of an arbitrary positive parameter β may serve to make more flexible the use of the solutions.

(b) For $m = 0$, two exactly solvable doublets have been constructed implying the existence of a double-well structure of the corresponding potential. In fact, we have also verified numerically that this property does remain valid for doublets with $n = 3, 4$ and 5.

(c) For $m \neq 0$, two types of exact solutions have been found and are given in Eqs. (21) and (22) for type I and in Eqs. (25) and (26) for type II.

It may be useful to observe the fact that the present unified approach can, in some sense, be regarded as part of a more global treatment of quasi-exactly solvable problems which will be presented elsewhere.

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EGZAKTNA RJEŠENJA NEPOLINOMNOG MEĐUDJELOVANJA

Istražuje se egzaktno rješavanje nepolinomnog međudjelovanja. Raspravlja se problem dvojamnog potencijala i predstavljaju novi tipovi egzaktnih rješenja.