ISSN 1330–0016
CODEN FIZBE7

INHERENTLY RELATIVISTIC QUANTUM THEORY PART IV. QUANTIONIC THEOREMS

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Dedicated to Professor Kseno Ilakovac on the occasion of his 70th birthday

Received 16 January 2002; Accepted 18 February 2002
Online 6 April 2002

Having completed in Part III of the present work the development of the quantionic algebra, we derive, in this final part, the algebraic theorems needed for its application to physics. The most important of these theorems are those related to the quantionic norm. The quantionic norm generalizes the norm of state vectors in standard quantum mechanics, and has immediate physical interpretations suggested by its geometric properties.

PACS numbers: 02.10.Jf, 03.65.-w UDC 530.145
Keywords: quaternion, octonion, division algebra, quantion, quantum, unification

1. *Introduction*

In this paper we complete the development of the algebra of quantions to a points that suffices as background for the next step — the study of quantionic differential equations and of their physical interpretations.

All concepts and identities defining the quantionic algebra have been derived in Part III of the present work [1], and our first task is to organize these results in a self-contained section (Sect. 2) sufficiently complete to serve as reference for further work. Readers initially willing to accept on faith the proof, distributed over the three previous articles, Refs. [2], [3] and [1], that *the quantionic algebra is the unique generalization of the field of complex numbers compatible with some very abstract principles of quantum mechanics* may postpone the reading of these articles and begin with the present paper without lacunas in the logic.[1]

[1]For these readers we point out that while the quantionic algebra is also relativistic, this property was not axiomatically imposed. The search for a generalization of quantum theory compatible

2. *The algebra* $\mathbb{D}$

The underlying linear space of the quantionic algebra $\mathbb{D}$ is the complex linear Minkowski space

$$M^4(\mathbb{C}) = M^4 \oplus iM^4,$$

in which a real vector

$$\Omega \in M^4,$$

referred to as *the structure vector*, has been distinguished in addition to the already existing structures. Thus, $\mathbb{D}$ is equipped with three invariant geometric objects. In the order of decreasing generality, they are (1) the Levi-Civitta pseudo-tensor, which exists in every linear space, (2) the metric tensor, which characterizes relativity, and, (3) the structure vector, which brings in quantum structures.

The linear space $\mathbb{D}$ becomes an algebra (the quantionic algebra) with the adjunction of a product, β, i.e., a bilinear operator

$$\beta : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$$

defined in terms of the three distinguished objects. Before we write its definition, we need to make some important remarks concerning the invariance groups.

The metric in Minkowski space is invariant under the continuous group $SO(1,3)$ of Lorentz transformations and under the discrete group $\mathcal{CPT}$ (time-reversal $\mathcal{T}$, parity $\mathcal{P}$, and their product $\mathcal{C} = \mathcal{PT}$). In special relativity, Lorentz covariance is guaranteed by the formalism of tensor algebra, but with respect to the discrete group, invariance is either expressed in a particular coordinate system, or ignored. In the quantionic algebra, however, both groups are equally fundamental. It was the need to make the discrete transformations coordinate-independent that led to the introduction of the structure vector Ω. Thus, for $f \in \mathbb{D}$, the discrete group is defined by the mappings

$$\left.\begin{array}{c} \mathcal{C} : f \longmapsto f^*, \\ \mathcal{P} : f \longmapsto 2\,(\Omega, f)\,\Omega - f, \\ \mathcal{T} : f \longmapsto \mathcal{CP} \equiv \mathcal{PC} = 2\,(\Omega, f^*)\,\Omega - f^*. \end{array}\right\} \qquad (1)$$

Since manifest covariance and the existence of the distinguished vector Ω are mutually incompatible, the formalism of the quantionic algebra can respect one or the other, but not both simultaneously. If preference is given to the continuous group, we refer to the basis vectors as the *Lorentz frame*, and to the formalism as *covariant.* If the discrete group is emphasized, we refer to the frame and formalism as *structural.* When there is a choice, we shall prove theorems in the formalism that minimizes the effort.

with the Hilbert space structure of standard quantum mechanics (abstractly interpreted) led to the quantionic algebra as the *unique* generalization of the field of complex numbers. Its being relativistic is an unexpected theorem — justifying the qualifier "inherent" for a quantionic unification of relativity and quantum theory.

2.1. *The product β in covariant form*

The *covariant expression* for the product β is

$$f\beta g \stackrel{\text{def}}{=} (\Omega, f)\, g + (\Omega, g)\, f - (f, g)\, \Omega - {}^{*}(\mathrm{i}\Omega \wedge f \wedge g)\,, \tag{2}$$

where $f, g \in \mathbb{D}$. A star superscript to the left of a multivector indicates the Hodge duality operator, while $(f, g) \stackrel{\text{def}}{=} \eta_{\alpha\beta} f^{\alpha} g^{\beta}$ is the scalar product in Minkowski space. (As much as possible, we avoid writing the tensor indices.) Written out in tensorial components, the product β reads

$$(f\beta g)^{\rho} \stackrel{\text{def}}{=} (\Omega, f)\, g^{\rho} + (\Omega, g)\, f^{\rho} - (f, g)\, \Omega^{\rho} - \mathrm{i}\eta^{\rho\tau} \varepsilon_{\tau\alpha\beta\gamma} \Omega^{\alpha} f^{\beta} g^{\gamma}. \tag{3}$$

Equivalently,

$$(f\beta g)^{\rho} = B^{\rho}_{\alpha\beta} f^{\alpha} g^{\beta}, \tag{4}$$

where $B^{\rho}_{\alpha\beta}$ is the tensor

$$B^{\rho}_{\alpha\beta} \stackrel{\text{def}}{=} \delta^{\rho}_{\alpha} \Omega_{\beta} + \delta^{\rho}_{\beta} \Omega_{\alpha} - \eta_{\alpha\beta} \Omega^{\rho} - \mathrm{i}\eta^{\rho\tau} \Omega^{\sigma} \varepsilon_{\rho\sigma\alpha\beta}. \tag{5}$$

This definition of the algebraic product is unconditionally associative,

$$(f\beta g)\, \beta h \equiv f\beta\, (g\beta h)\,,$$

i.e., it is associative for an arbitrary selection of the structure vector Ω. But for the algebra to have a unit, the structure vector must be time-like and of unit length, i.e.,

$$(\Omega, \Omega) = 1, \tag{6}$$

in which case Ω itself plays the role of the algebraic unit. Indeed, if condition (6) is satisfied, substitution of Ω for g in the expression (2) yields

$$\Omega\beta f \equiv f. \tag{7}$$

Given a time-like structure vector Ω, one can select the Lorentz frame so as to have $\Omega^0 > 0$. Hence, in the linear space $\mathbb{D}$, where there is no physically predetermined direction of time, Ω is future-pointing by definition.

2.2. *The product β in structural form*

What we refer to as *the structural frame* consists of the structure vector Ω taken as a basis vector, and of the hyperplane Π orthogonal to Ω. Since Ω is time-like, Π is a 3-dimensional Euclidean space. We write an arbitrary quantion $f \in \mathbb{D}$ in the form

$$\mathbb{D}f = F\Omega + \vec{f}, \tag{8}$$

where F and $\vec{f}$ are the projections of f on Ω and Π, respectively, i.e., $F \stackrel{\text{def}}{=} (\Omega, f) \in \mathbb{C}$, and $\vec{f} \stackrel{\text{def}}{=} (f - F\Omega) \in \Pi$. Let us write f and g in the expression (2) in the form (8) and expand,

$$f\beta g = \left(F\Omega + \vec{f}\right)\beta\left(G\Omega + \vec{g}\right) = FG\Omega + F\vec{g} + G\vec{f} + \vec{f}\beta\vec{g}.$$

To compute $\vec{f}\beta\vec{g}$, we note that $\left(\Omega, \vec{f}\right) = 0$ by orthogonality, $\left(\vec{f}, \vec{g}\right) = -\vec{f} \cdot \vec{g}$ by the definition $\eta = (+1, -1, -1, -1)$ of the metric tensor, and $^*\left(\Omega \wedge \vec{f} \wedge \vec{g}\right) = -\vec{f} \times \vec{g}$. The source of the minus sign is evident from the expression (3), i.e., from $\eta^{\rho\tau} \varepsilon_{\tau\alpha\beta\gamma}\Omega^\alpha f^\beta g^\gamma$, since the indices $\rho\tau$ are in Π, i.e., space-like. Hence,

$$\vec{f}\beta\vec{g} = \left(\vec{f} \cdot \vec{g}\right)\Omega + i\vec{f} \times \vec{g}, \tag{9}$$

and, finally,

$$f\beta g = \left(FG + \vec{f} \cdot \vec{g}\right)\Omega + F\vec{g} + G\vec{f} + i\vec{f} \times \vec{g}. \tag{10}$$

In the special case of $\vec{f} = \vec{g} = 0$, we have $f = F\Omega$, $g = G\Omega$, and $f\beta g = FG\Omega$. Hence, the field $\mathbb{C}$ of complex numbers is a substructure of the quantionic algebra $\mathbb{D}$ — as it should be, since $\mathbb{D}$ is meant to be a generalization of $\mathbb{C}$.

3. *The quantionic norm*

Complex conjugation, indicated by an asterisk, is essential to what follows. We first note that it does not apply only to quantions, but also to the product β, as it is a complex tensor, (5). Its symmetric part being real and antisymmetric part imaginary, taking the complex conjugate of β is equivalent to taking the reverse of the product, i.e., $f\beta^* g \equiv g\beta f$. The implication

$$(f\beta g)^* \equiv g^*\beta f^* \tag{11}$$

is analogous to Hermitian conjugation of operators.

We define the quantionic norm as

$$A(u) \stackrel{\text{def}}{=} u^*\beta u. \tag{12}$$

It is a quantion by construction, but a real one, i.e., a vector in the linear Minkowski space.

The presence of the exterior vector product in the definition of the algebraic product β implies $u\beta u^* \neq u^*\beta u$, leaving two options for the definition of the norm. The reason for selecting the expression (12) is for compatibility with the standard practice of writing linear operators to the left of the vectors they act upon. This will become evident in the section on quantionic Hilbert space.

The next theorem is fundamental to the physical interpretation of quantions. It generalizes the positive definiteness of the norm of complex numbers.

Theorem 1 *For every quantion $u \in \mathbb{D}$, the norm $u^*\beta u$ is a future-pointing time-like or null vector in the real linear Minkowski space.*

Proof. (a) Reality, i.e.,

$$(u^*\beta u)^* = u^*\beta u, \tag{13}$$

is an immediate consequence of relation (11).

(b) A vector is future-pointing if its component in the direction Ω is positive. Computing the norm, $w \in M^4$ of $u \in \mathbb{D}$ using relation (10), one obtains

$$\begin{aligned} W\Omega + \vec{w} &= (U^*\Omega + \vec{u}^*)\,\beta\,(U\Omega + \vec{u}) \\ &= (U^*U + \vec{u}^* \cdot \vec{u})\,\Omega + U^*\vec{u} + U\vec{u}^* + i\vec{u}^* \times \vec{u}. \end{aligned} \tag{14}$$

Clearly, the $\Omega-$component $(U^*U + \vec{u}^* \cdot \vec{u})$ is positive definite, proving the future-pointing orientation.

(c) To show that the product $w = u^*\beta u$ cannot be space-like, we compute the scalar product $(w,w) = WW - \vec{w} \cdot \vec{w}$ from the expression (14):

$$(w,w) = (U^*U + \vec{u}^* \cdot \vec{u})^2 - (U^*\vec{u} + U\vec{u}^* + \mathrm{i}\vec{u}^* \times \vec{u})^2$$

By elementary vector-algebraic computations, one transforms this expression into the form

$$(w,w) = (u,u)^*\,(u,u) \equiv \|(u,u)\|^2 \geq 0, \tag{15}$$

proving the assertion. ■

We now observe that the proof of Theorem 1 exhibits one additional property of the algebra $\mathbb{D}$, sufficiently interesting to be cast as a corollary. To state it conceptually, we introduce the following notations for the two norms defined in the quantionic algebra (in these expressions, C^+ is the cone of future-pointing time-like and null vectors in the real linear Minkowski space M^4, and $\mathbb{R}^+$ the set of non-negative real numbers):

1. The bilinear *metric norm function* M, based on the scalar product in Minkowski space:

$$M\,(u) \stackrel{\text{def}}{=} (u,u) \equiv \eta_{\alpha\beta} u^\alpha u^\beta \in \mathbb{C}. \tag{16}$$

2. The sesquilinear *algebraic norm function* A, which generalizes the complex norm:

$$A\,(u) \stackrel{\text{def}}{=} u^*\beta u \in C^+ \subset M^4. \tag{17}$$

3. The fourth order *quantionic norm function* N, which combines the two:

$$N\,(u) \stackrel{\text{def}}{=} (u,u)^*\,(u,u) \in \mathbb{R}^+. \tag{18}$$

Written out in full as $(u^*\beta u, u^*\beta u) = (u,u)^*\,(u,u)$, relation (15) can now be stated as:

Corollary 2 *The algebraic and metric norms commute,*

$$N\,(u) = M\,(A\,(u)) = A\,(M\,(u))\,. \tag{19}$$

To compute the inverse of a quantion u, i.e., $v = u^{-1}$, we substitute the expression (2) into the reciprocity condition $u\beta v = \Omega$,

$$(\Omega,u)\,v + (\Omega,v)\,u - (u,v)\,\Omega - \mathrm{i}\,{}^*\,(\Omega\wedge u\wedge v) = \Omega, \tag{20}$$

and take the scalar product of both sides with u :

$$V\,(u,u) = U. \tag{21}$$

Splitting equation (20) into its scalar and vector parts, one obtains

$$\begin{aligned} UV + \vec{u}\cdot\vec{v} &= 1, \\ U\,\vec{v} + V\,\vec{u} + \mathrm{i}\,\vec{u}\times\vec{v} &= 0. \end{aligned}$$

The second of these equations implies that $\vec{u}$ and $\vec{v}$ are collinear, and, by relation (21),

$$\vec{v} = -\frac{V}{U}\vec{u} = -\frac{1}{(u,u)}\vec{u}.$$

Since $u^{-1} = v = V\Omega + \vec{v}$, this relation, together with relation (21), yields the result

$$u^{-1} = \frac{1}{(u,u)}\,(U\Omega - \vec{u})\,,$$

but $(U\Omega - \vec{u}) = \mathcal{P}u$, since the parity transformation $\mathcal{P}$ reverses all vectors in Π, but leaves Ω fixed. Hence, in covariant and structural form, respectively,

$$u^{-1} = \mathcal{P}\frac{u}{(u,u)}\,, \tag{22}$$

$$(U\Omega + \vec{u})^{-1} = \frac{U\Omega - \vec{u}}{UU - \vec{u}\cdot\vec{u}}\,. \tag{23}$$

Hence, the inverse exists everywhere in $\mathbb{D}$ except on the complex null cone, $\mathcal{N}$, defined by the equation $(u,u) = 0$. $\mathcal{N}$ is also the kernel of the quantionic norm — in the sense that the fourth-order norm N, relation (18), vanishes exactly on $\mathcal{N}$. This suggests the following terminology:

Definition 3 *The quantions $q \in \mathcal{N}$, which have no inverse, are called* ***singular****. All other quantions are said to be* ***regular****.*

Being implicit, the singularity condition $(u,u) = 0$ is often impractical. To parametrize the algebraic variety $\mathcal{N}$, we first write q in terms of two real scalars, x, y, and two real vectors, $\vec{a}, \vec{b}$:

$$q = Q\Omega + \vec{q} = (x + iy)\,\Omega + \left(\vec{a} + \mathrm{i}\vec{b}\right). \tag{24}$$

The complex equation $(u,u) = 0$ is then equivalent to two real equations,

$$x^2 - y^2 = \vec{a}^{\,2} - \vec{b}^{\,2}, \tag{25}$$

$$xy = \vec{a} \cdot \vec{b}. \tag{26}$$

Hence, for some real number s, equation (25) yields

$$x^2 = \frac{1}{2}\left(s + \left(a^2 - b^2\right)\right), \tag{27}$$

$$y^2 = \frac{1}{2}\left(s - \left(a^2 - b^2\right)\right). \tag{28}$$

With equation (26), one obtains the solution for s :

$$s^2 = \left(a^2 + b^2\right)^2 - 4\left(\vec{a} \times \vec{b}\right)^2. \tag{29}$$

This completes the parametrization of $\mathcal{N}$ by arbitrary complex 3-vectors $\vec{q}$.

We note that the expression (23) is evidently true in the special case of complex numbers, characterized by $\vec{u} = 0$. We also note that β-multiplying relation (23) on both sides by $(U\Omega + \vec{u})$ yields $\Omega = \Omega$ — on the left side by definition, on the right side identically.

4. The polar form of quantions

The algebra of quantions generalizes the field of complex numbers structurally (it is not a field like the complex numbers only because the kernel $\mathcal{N}$ is non-trivial). The two structures are also similar in their Cartesian representations. Indeed, both are direct squares of a linear metric space,

$$\mathbb{C} = \mathbb{R} \times \mathbb{R},$$

$$\mathbb{D} = M^4 \times M^4.$$

Thus, a complex number is represented by a pair of arbitrary real numbers, $z = x + \mathrm{i}y$, a quantion by a pair of arbitrary vectors in Minkowski space, $q^\rho = r^\rho + \mathrm{i}s^\rho$.

It is now natural to inquire if the representation $z = re^{i\phi}$ of complex numbers also generalizes to quantions. Specifically, given an arbitrary quantion $q \in \mathbb{D}$, the question is whether two 4-vectors $r, s \in M^4$ exist such that

$$q = E\,(\mathrm{i}s)\,\beta r, \tag{30}$$

where $E\,(is)$ is some quantionic generalization of the exponential function. The exponential function has several properties that characterize it. The property we take over as defining for $E\,(\mathrm{i}s)$ is that it be of unit norm, i.e., we define $E\,(\mathrm{i}s)$ as the most general solution of the equation

$$E\,(-\mathrm{i}s)\,\beta E\,(\mathrm{i}s) = \Omega. \tag{31}$$

Since the product β is not symmetric, the expression $r\beta E\,(\mathrm{i}s)$ would give different solutions for r and s. We begin with the form (30) for convenience.

To compute the quantionic radius vector r, we take the algebraic norm of q :

$$\begin{aligned} q^*\beta q &= (E\,(\mathrm{i}s)\,\beta r)^*\,\beta\,(E\,(\mathrm{i}s)\,\beta r) \\ &= r\beta E\,(-\mathrm{i}s)\,\beta E\,(\mathrm{i}s)\,\beta r = r\beta r. \end{aligned}$$

Hence, formally,

$$r = \sqrt{q^*\beta q}. \tag{32}$$

The polar representation problem is now reduced to three sub-problems to be solved separately. They are related to the quantionic exponential functions, the quantionic square root, and the relationship between the various polar forms which stem from the non-commutativity of the product β.

4.1. The quantionic exponential function

The solution to the first problem is given by the following theorem:

Theorem 4 *The most general quantion $E \in \mathbb{D}$ of unit norm, $E^*\beta E = \Omega$ is of the form*

$$E = e^{i\chi}\,(\cos\phi\,\Omega + \mathrm{i}\,\sin\phi\,\vec{n})\,, \tag{33}$$

where $\vec{n}$ is an arbitrary unit vector, $\vec{n}\cdot\vec{n} = 1$.

Proof. We begin with a general quantion E written in structural form,

$$E = A\Omega + \vec{a},$$

where $A = x + \mathrm{i}y$, and $\vec{a} = \vec{v} + \mathrm{i}\vec{w}$, with $x, y \in \mathbb{C}$, and $\vec{v}, \vec{w} \in \Pi$. Expanding $E^*\beta E = \Omega$ by relation (10), one concludes that $\vec{v}$ and $\vec{w}$ are parallel, which further implies that the complex 3-vector $\vec{a}$ is of the special form $\vec{a} = z\,\vec{n}$, where $z \in \mathbb{C}$ and $\vec{n} \cdot \vec{n} = 1$, i.e.,

$$E = A\Omega + z\,\vec{n}.$$

The same expansion further yields

$$\begin{aligned} A^*A + z^*z &= 1, \\ A^*z + Az^* &= 0. \end{aligned}$$

The parametric solutions of these equations are

$$\begin{aligned} A &= e^{\mathrm{i}\alpha} \cos\phi, \\ z &= e^{\mathrm{i}\beta} \sin\phi, \end{aligned}$$

subject to the condition $e^{-\mathrm{i}\alpha} e^{\mathrm{i}\beta} + e^{\mathrm{i}\alpha} e^{-\mathrm{i}\beta} = 0$. Rearranged, this yields the expression (33) for arbitrary χ and ϕ. ■

Formally, the expression (33) can be written as

$$E = e^{\mathrm{i}\chi} e^{\mathrm{i}\phi\,\vec{n}}, \tag{34}$$

where

$$e^{\mathrm{i}\phi\,\vec{n}} = \sum_{k=0}^{\infty} \frac{(\mathrm{i}\phi)^k}{k!} \vec{n}^k.$$

The powers of $\vec{n}$, viewed as a quantion, are defined with respect to the product β. Thus,

$$\vec{n}^2 \stackrel{\mathrm{def}}{=} \vec{n}\beta\vec{n} = \vec{n} \cdot \vec{n}\,\Omega = \Omega,$$

which leads to the Euler formula (33) for the exponential function. We can now expand the function $E\,(is)$ in relation (30),

$$E\,(\mathrm{i}s) = E\,(\mathrm{i}S\Omega + \mathrm{i}\vec{s}) = e^{\mathrm{i}(S\Omega + \vec{s})} = e^{\mathrm{i}S\Omega} \beta e^{\mathrm{i}\vec{s}} = e^{\mathrm{i}S} e^{\mathrm{i}\vec{s}}.$$

Comparison with the expression (34) yields the interpretations for the variables χ and ϕ,

$$\begin{aligned} \chi &= S, \\ \phi &= \sqrt{\vec{s} \cdot \vec{s}}. \end{aligned}$$

Thus, the exponential function $E\,(\mathrm{i}s)$ is periodic with period 2π in the direction of the structure vector Ω, and it is also periodic with period 2π in every radial direction in the Euclidean space Π. The first is the periodicity of the ordinary complex phase factor, the radial periodicity is a new concept.

4.2. The quantionic square root

Given an arbitrary quantion $q \in \mathbb{D}$, we are to compute the square root (32), i.e., determine the vector $r \in M^4$ such that

$$r\beta r = q^* \beta q. \tag{35}$$

The solutions for regular and singular quantions being different, we consider them separately.

The case of regular quantions:

By Theorem 1 and relation (18), one can write, without loss of generality,

$$\begin{aligned} q^*\beta q &= \sqrt{N(u)}\left(\cosh\omega\,\Omega + \sinh\omega\,\vec{m}\right) \\ &= \left(R^2 + \vec{r}^2\right)\Omega + 2R\vec{r}, \end{aligned}$$

for some parameter ω and unit vector $\vec{m}$. This equation immediately yields two sets of solutions,

$$\left.\begin{aligned} R &= \sqrt[4]{N(u)}\cosh\tfrac{\omega}{2}, \\ \vec{r} &= \sqrt[4]{N(u)}\sinh\tfrac{\omega}{2}\,\vec{m}. \end{aligned}\right\} \tag{36}$$

$$\left.\begin{aligned} R &= \sqrt[4]{N(u)}\sinh\tfrac{\omega}{2}, \\ \vec{r} &= \sqrt[4]{N(u)}\cosh\tfrac{\omega}{2}\,\vec{m}. \end{aligned}\right\} \tag{37}$$

As in the case of complex numbers, the negative signs of square roots are subsumed in the phase factor $E(\mathrm{i}s)$. To determine which of these solutions applies to a given quantion q, consider the scalar product (q^*, q) :

$$(q^*, q) = (Q^*\Omega + \vec{q}^*, Q\Omega + \vec{q}) = Q^*Q - \vec{q}^* \cdot \vec{q}.$$

Clearly, if $(q^*, q) > 0$, the solution is (36), if $(q^*, q) < 0$, it is (37).

Once the quantionic radius vector r has been found according to the algorithm outlined above, the phase factor $E(\mathrm{i}s)$ in the expression (30) is given as

$$E(\mathrm{i}s) = q\beta r^{-1}, \tag{38}$$

the reciprocal being defined by relation (22). Thus, for regular quantions, the polar form is uniquely defined.

The case of singular quantions:

By definition, the norm $q^*\beta q$ of a singular quantion is a null vector, and, by (35), so is $r\beta r$, i.e., $(r\beta r, r\beta r) = 0$. In the structural frame,

$$\begin{aligned} &\left(\left(R^2 + \vec{r}^2\right)\Omega + 2R\vec{r}, \left(R^2 + \vec{r}^2\right)\Omega + 2R\vec{r}\right) \\ = \ &\left(R^2 + \vec{r}^2\right)^2 - 4R^2\vec{r}^2 = \left(R^2 - \vec{r}^2\right)^2 = 0, \end{aligned}$$

implying that r itself is a null vector. Hence, its most general form is

$$r = R\,(\Omega + \vec{m})\,,$$

where $\vec{m}$ is an arbitrary unit vector.

From relations (34) and (30) written in the form $r = E^* \beta q$, we get

$$R\Omega + R\vec{m} = e^{-\mathrm{i}\chi}\,(\cos\phi\Omega - \mathrm{i}\sin\phi\,\vec{n})\,\beta\,(Q\Omega + \vec{q})\,.$$

Writing $e^{-\mathrm{i}\chi}Q$ and $e^{-\mathrm{i}\chi}\vec{q}$ in terms real variables, $e^{-\mathrm{i}\chi}Q = x+\mathrm{i}y$, and $e^{-\mathrm{i}\chi}\vec{q} = \vec{a}+\mathrm{i}\vec{b}$,

$$R\Omega + R\vec{m} = (\cos\phi\Omega - \mathrm{i}\sin\phi\,\vec{n})\,\beta\left((x+\mathrm{i}y)\,\Omega + \vec{a} + \mathrm{i}\vec{b}\right),$$

the vector part of this equation yields two real equations,

$$\begin{aligned} R\vec{m} &= \cos\phi\,\vec{a} + \sin\phi\,\vec{n}\times\vec{a} + y\sin\phi\,\vec{n}, \\ 0 &= \cos\phi\,\vec{b} + \sin\phi\,\vec{n}\times\vec{b} - x\sin\phi\,\vec{n}. \end{aligned}$$

The second equation implies

$$\begin{aligned} \vec{n} &= \vec{b}_0, \\ b\cos\phi &= x\sin\phi. \end{aligned} \tag{39}$$

Hence,

$$\cos\phi = \frac{x}{\sqrt{b^2+x^2}}, \tag{40}$$

$$\sin\phi = \frac{b}{\sqrt{b^2+x^2}}. \tag{41}$$

Substitution into the first equation yields

$$R\vec{m} = \frac{1}{\sqrt{b^2+x^2}}\left(x\,\vec{a} + y\,\vec{b} + \vec{b}\times\vec{a}\right).$$

Taking the square of both sides, one computes R and $\vec{m}$:

$$R = \sqrt{a^2+y^2}, \tag{42}$$

$$\vec{m} = \frac{x\,\vec{a} + y\,\vec{b} + \vec{b}\times\vec{a}}{\sqrt{(b^2+x^2)\,(a^2+y^2)}}. \tag{43}$$

With x and y given by relations (27) to (29), after adjustment for the rotation $e^{\mathrm{i}\chi}$, this completes the algorithm for the quantionic radius vector and phase factor of singular quantions.

4.3. The general polar expressions

The polar expression $E\,(\mathrm{i}s)\,\beta r$ is the simplest, as its algebraic norm is $r\beta r$, but it is not necessarily the only one that might appear in computations. Due to the non-commutativity of the product, $r\beta E\,(\mathrm{i}s)$ is a different expression, while the most general form is $E\,(\mathrm{i}s)\,\beta r\beta E\,(\mathrm{i}t)$. To cover all cases, it suffices to derive the exchange rule understood as follows: Given two 4-vectors r and s, find the 4-vectors r' and s' satisfying the equation

$$E\,(\mathrm{i}s)\,\beta r = r'\beta E\,(\mathrm{i}s')\,. \tag{44}$$

Since the complex phase factor $e^{\mathrm{i}\chi}$ commutes with all quantions, it suffices to derive the exchange rule for the purely quantionic phase factor $e^{\mathrm{i}\phi\,\vec{n}}$. This rule has a simple geometric interpretation:

Theorem 5 *Under a permutation of factors, the quantion $q = \exp\,(\mathrm{i}\phi\,\vec{n})\,\beta r$ remains invariant if the quantionic radius vector r is simultaneously rotated by the angle 2ϕ around the vector $\vec{n}$, i.e.,*

$$\exp\,(\mathrm{i}\phi\,\vec{n})\,\beta\,(R\Omega + \vec{r}) = (R\Omega + \vec{r}')\,\beta\exp\,(\mathrm{i}\phi\,\vec{n})\,, \tag{45}$$

where

$$\vec{r}' = \cos 2\phi\,\vec{r} - \sin 2\phi\,\vec{n}\times\vec{r} + (1-\cos 2\phi)\,(\vec{n}\cdot\vec{r})\,\vec{n}. \tag{46}$$

Proof. The expansions of $\exp\,(\mathrm{i}\phi\,\vec{n})\,\beta r$ and of $r'\beta\exp\,(\mathrm{i}\phi'\,\vec{n}')$ in the structural frame are

$$\begin{aligned} E\beta r &= (\cos\phi\,R + \mathrm{i}\sin\phi\,\vec{n}\cdot r)\,\Omega + \mathrm{i}\sin\phi\,R\vec{n} + (\cos\phi\,\vec{r} - \sin\phi\,\vec{n}\times\vec{r})\,, \\ r'\beta E' &= (\cos\phi'\,R' + \mathrm{i}\sin\phi'\,\vec{n}'\cdot r')\,\Omega + \mathrm{i}\sin\phi'\,R'\vec{n}' + (\cos\phi'\vec{r}' + \sin\phi'\vec{n}'\times\vec{r}')\,. \end{aligned}$$

Term-wise comparisons yield immediately

$$\begin{aligned} \vec{n}' &= \vec{n}, \\ \phi' &= \phi, \\ R' &= R, \end{aligned}$$

and the following equations for $\vec{r}'$:

$$\begin{aligned} \vec{n}\cdot\vec{r} &= \vec{n}\cdot\vec{r}', \\ \cos\phi\,\vec{r} - \sin\phi\,\vec{n}\times\vec{r} &= \cos\phi\,\vec{r}' + \sin\phi\,\vec{n}\times\vec{r}'. \end{aligned}$$

The first of these equations means that the projection of $\vec{r}$ on $\vec{n}$ remains unaffected, while the second represents a rotation by 2ϕ in the 2-plane orthogonal to $\vec{n}$, i.e.,

$$\vec{r}' = (\vec{n}\cdot\vec{r})\,\vec{n} + \cos 2\phi\ (\vec{r} - (\vec{n}\cdot\vec{r})\,\vec{n}) - \sin 2\phi\ \vec{n}\times(\vec{r} - (\vec{n}\cdot\vec{r})\,\vec{n})\,, \tag{47}$$

which is equivalent to the expression (46). ■

Being algebraic, the transformation (47) is a single rotation, but we note, anticipating future developments, that if $\phi = \omega t$ for a constant angular velocity $\omega\vec{n}$, *the permutation of factors in the polar representation of quantions is equivalent to a precession at twice the angular velocity.*

4.4. Quantionic gauge transformations

Another role played by the quantionic phase factor $e^{\mathrm{i}s} = e^{\mathrm{i}\chi}e^{\mathrm{i}\phi\vec{n}}$ is that of quantionic gauge transformation, as the mapping

$$q \longmapsto e^{\mathrm{i}s}\beta q$$

preserves the algebraic norm. Indeed,

$$\left(e^{\mathrm{i}s}\beta q\right)^* \beta \left(e^{\mathrm{i}s}\beta q\right) = q^*\beta e^{-\mathrm{i}s}\beta e^{\mathrm{i}s}\beta q = q^*\beta q,$$

in complete analogy with the complex gauge transformations of standard quantum mechanics. The totality of gauge factors forms a group. The reason is that the product of two gauge factors is a unit quantion,

$$\left(e^{\mathrm{i}s}\beta e^{\mathrm{i}t}\right)^* \beta \left(e^{\mathrm{i}s}\beta e^{\mathrm{i}t}\right) = e^{-\mathrm{i}t}\beta e^{-\mathrm{i}s}\beta e^{\mathrm{i}s}\beta e^{\mathrm{i}t} = e^{-\mathrm{i}t}\beta\,\Omega\,\beta e^{\mathrm{i}t} = \Omega,$$

which implies that for every pair of 4-vectors, $s, t \in M^4$, there exists a 4-vector, $p \in M^4$, uniquely defined by s, t up to periodicities, such that

$$e^{\mathrm{i}s}\beta e^{\mathrm{i}t} = e^{\mathrm{i}p}.$$

The quantionic gauge group is non-Abelian, since, in general, $e^{\mathrm{i}s}\beta e^{\mathrm{i}t} \neq e^{\mathrm{i}t}\beta e^{\mathrm{i}s}$.

5. The tetrads

In this section we reformulate the quantionic algebra $\mathbb{D}$ in terms of basis tetrads (Vierbeins, or “repères mobiles”), which are well suited for defining quantionic fields in curved space. Two types of tetrads are common in general relativity, the orthonormal tetrad, and the Neumann-Penrose null-tetrad. We shall express the quantionic algebra in both, and then point out a re-interpretation of quantions suggested by the orthonormal tetrad.

The orthonormal tetrad.

Let $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ be three real orthonormal vectors in the subspace Π. With Ω, they form an orthonormal tetrad $\{\Omega, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ in $\mathbb{D}$. Their mutual products

$$\begin{aligned}
\vec{e}_i\beta\vec{e}_i &= \Omega \\
\vec{e}_i\beta\vec{e}_j &= \mathrm{i}\vec{e}_i \times \vec{e}_j = \mathrm{i}\vec{e}_k \quad (\textit{cyclically})
\end{aligned}$$

can be expressed as a multiplication table:

β	Ω	$\vec{e}_1$	$\vec{e}_2$	$\vec{e}_3$
Ω	Ω	$\vec{e}_1$	$\vec{e}_2$	$\vec{e}_3$
$\vec{e}_1$	$\vec{e}_1$	Ω	$\mathrm{i}\vec{e}_3$	$-\mathrm{i}\vec{e}_2$
$\vec{e}_2$	$\vec{e}_2$	$-\mathrm{i}\vec{e}_3$	Ω	$\mathrm{i}\vec{e}_1$
$\vec{e}_3$	$\vec{e}_3$	$\mathrm{i}\vec{e}_2$	$-\mathrm{i}\vec{e}_1$	Ω

(48)

The null tetrad.

The Neuman-Penrose null tetrad consists of two pairs of null vectors [9]. We take the following definitions,

$$\left.\begin{aligned} k &= \tfrac{1}{2}\,(\Omega + \vec{e}_3) \\ l &= \tfrac{1}{2}\,(\Omega - \vec{e}_3) \\ m &= \tfrac{1}{2}\,(\vec{e}_1 + \mathrm{i}\vec{e}_2) \\ m^* &= \tfrac{1}{2}\,(\vec{e}_1 - \mathrm{i}\vec{e}_2) \end{aligned}\right\} \tag{49}$$

which differ from the original definition in the coefficients. The reason is that one cannot simultaneously maximally simplify both the Minkowski scalar product and the quantionic product. Introduced in relativity, the original definition was concerned only with the scalar product. In the quantionic algebra, both the scalar and algebraic product are important, but the latter takes precedence. The coefficient 1/2 simplifies it, as tabulated in (51). The scalar products are

$$\left.\begin{gathered} (k,k) = (l,l) = (m,m) = (m^*,m^*) = 0\,, \\ (k,m) = (k,m^*) = (l,m) = (l,m^*) = 0\,, \\ (k,l) = -\,(m,m^*) = \tfrac{1}{2}\,. \end{gathered}\right\} \tag{50}$$

The quantionic products follow from the definitions (49) and the table (48). They are expressed in the multiplication table:

β	k	l	m	m^*
k	k	0	m	0
l	0	l	0	m^*
m	0	m	0	k
m^*	m^*	0	l	0

(51)

Since this table has only half as many non-vanishing entries as table (48), the null tetrad could have computational advantages over the orthogonal tetrad.

Quantions and the 2×2 general linear group.

We note that the multiplication table (48) is essentially the same as for the Pauli matrices. Thus, for the identifications $\Omega = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\vec{e}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$\vec{e}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $\vec{e}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the product β is represented by matrix multiplication, and, consequently, the quantionic algebra $\mathbb{D}$ is isomorphic to the algebra of 2×2 complex matrices. Due to the identification $\Omega = I$, this isomorphism is valid only once the structure vector Ω has been fixed. This precludes the Lorentz covariance.

A special case of this isomorphism, valid also for fixed Ω, is expressed by the following theorem:

Theorem 6 *The group of quantionic gauge transformations* $e^{\mathrm{i}\phi\vec{n}}$ *is isomorphic to the unitary group* $SU(2)$ *.*

Proof. Both groups being 3-parametric, it remains to be shown is that the product

$$\begin{aligned} e^{\mathrm{i}\phi\vec{n}}\beta e^{\mathrm{i}\psi\vec{m}} &= (\cos\phi\Omega + \mathrm{i}\sin\phi\vec{n})\,\beta\,(\cos\psi\Omega + \mathrm{i}\sin\psi\vec{m}) \\ &= (\cos\phi\cos\psi - \sin\phi\sin\psi\,(\vec{n}\cdot\vec{m}))\,\Omega \\ &\quad +\mathrm{i}\,(\sin\phi\cos\psi\vec{n} + \cos\phi\sin\psi\vec{m}) \\ &\quad -\mathrm{i}\sin\phi\sin\psi\,(\vec{n}\times\vec{m}) \end{aligned} \tag{52}$$

is formally the same as the product of 2×2 unitary matrices. Writing an arbitrary traceless 2×2 Hermitian matrix in the form ϕH, where $H^2 = I$, the matrix H can be represented as a real linear combination of Pauli matrices, $H(\vec{n}) = \vec{n}\cdot\vec{\sigma}$, where $\vec{n}\cdot\vec{n} = 1$. This establishes the one-to-one correspondence

$$e^{\mathrm{i}\phi\vec{n}} \longleftrightarrow e^{\mathrm{i}\phi H(\vec{n})}. \tag{53}$$

Expanding the exponential, one obtains

$$\exp\left[\mathrm{i}\phi H(\vec{n})\right] = \cos\phi I + \mathrm{i}\sin\phi H(\vec{n}),$$

and similarly for $\exp\left[\mathrm{i}\phi H(\vec{m})\right]$. Next, computing the product $H(\vec{n})\,H(\vec{m})$ of the two matrices using the Pauli expansion, one obtains

$$H(\vec{n})\,H(\vec{m}) = (\vec{n}\cdot\vec{m})\,I + \mathrm{i}H(\vec{n}\times\vec{m})$$

which immediately yields

$$\begin{aligned} e^{\mathrm{i}\phi H(\vec{n})}e^{\mathrm{i}\psi H(\vec{m})} &= (\cos\phi\cos\psi - \sin\phi\sin\psi\,(\vec{n}\cdot\vec{m}))\,I \\ &\quad +\mathrm{i}\,(\sin\phi\cos\psi H(\vec{n}) + \cos\phi\sin\psi H(\vec{m})) \\ &\quad -\mathrm{i}\sin\phi\sin\psi H(\vec{n}\times\vec{m})\;. \end{aligned} \tag{54}$$

The products (52) and (54) being formally identical, the one-to-one mapping (53) is a group isomorphism. ■

This theorem suggests that the gauge factor $e^{\mathrm{i}\phi\vec{n}}$ might be spin related. Should this prove to be true, the spin group $SU(2)$, which appears in quantum mechanics as the unitary representation of the rotation group $SO(3)$, would appear in the quantionic algebra as a gauge group that preserves the algebraic norm.

Finally, as a curiosity, we point out that the quantionic exponential function can also be expressed as the quantionic product of two unit space-like vectors:

Theorem 7 *Let $\vec{n}_1$ and $\vec{n}_2$ be two unit vectors in the subspace Π, let ϕ be the oriented angle from $\vec{n}_1$ to $\vec{n}_2$, and let $\vec{n}$ be a unit vector orthogonal to both $\vec{n}_1$ and $\vec{n}_2$. Then*

$$\begin{aligned} e^{\mathrm{i}\phi\vec{n}} &= \cos\phi\,\Omega + i\sin\phi\,\vec{n} = \vec{n}_1\beta\vec{n}_2\,, \\ e^{-\mathrm{i}\phi\vec{n}} &= \cos\phi\,\Omega - i\sin\phi\,\vec{n} = \vec{n}_2\beta\vec{n}_1\,. \end{aligned}$$

If the vectors $\vec{n}_1$ and $\vec{n}_2$ are given, the exponential function is uniquely defined. If the exponential function is given, the two vectors are defined only up to rotation in their common plane.

Proof. Consider the product

$$\vec{n}_1\beta\vec{n}_2 = \vec{n}_1\cdot\vec{n}_2\,\Omega + \mathrm{i}\vec{n}_1\times\vec{n}_2 = \cos\phi\,\Omega + \mathrm{i}\sin\phi\,\vec{n}$$

For given ϕ and $\vec{n}$, it still leaves arbitrary the orientation of $\vec{n}_1$ in the plane perpendicular to $\vec{n}$. ∎

6. *Quantionic Hilbert spaces of finite dimension*

Having developed the quantionic algebra $\mathbb{D}$ as the unique relativistic generalization of the field $\mathbb{C}$ of complex numbers, we must verify that a generalization of Hilbert space can actually be constructed over the algebra $\mathbb{D}$. The following brief outline of approaches to unification puts this construction in perspective.

For the last seven decades, physics has been encapsulated in two mutually incompatible mathematical structures: Non-relativistic Hilbert space quantum mechanics, and relativistic space-time — both its local structure (linear Minkowski space), and its manifold structure (Riemannian space of general relativity). Their incompatibility manifests itself in the non-existence of a single mathematical structure unifying both theories in their present form. If one takes the minimalist approach aimed at generalizing only one of the two theories in the expectation that the other one will "fall into place", three approaches to the structural unification problem suggest themselves:

(1) *Generalize the structure of Hilbert space.* This idea could never take off. As Steven Weinberg points out, the Hilbert space structure is so rigid that any modification destroys it completely. Allowing this, one would be starting *ab initio*, rather than generalizing Hilbert space.

(2) *Generalize the underlying number system of Hilbert space while preserving its structure.* This ideas has been thoroughly investigated in the transition from complex numbers to quaternions — which proved not to lead to unification. The other division algebras and the Clifford and Grassmann algebras are even less adapted to this task.

(3) *Generalize the space-time structure.* Locally, all one can do is increase the number of space-time dimensions in the expectation of retrieving the linear Minkowski space later. A mathematically structural but physically partial solution of this type is Penrose's twistor approach. It was obtained by generalizing the affine Minkowski space to the six-dimensional embedding space of its conformal compactification.

The present work follows the approach (2) outlined above, but respects a lesson of the past — which tells us that if a new number system for Hilbert space exists, it is not to be found by trial and error among the known mathematical structures. We have shown that such new number system, $\mathbb{D}$, exists by extracting it from the structure of quantum theory itself. Moreover, it is unique, and relativistic. We shall now verify that it supports a Hilbert space structure — even though this is practically evident from its properties.

Initially disregarding the topological issues related to infinite-dimensional spaces, it suffices to consider the finite-dimensional case. As we shall see, the transition from the field $\mathbb{C}$ of complex numbers to the richer structure of the quantionic algebra $\mathbb{D}$ does not invalidate any of the standard structures relevant to quantum mechanics. It is necessary, however, to accept the idea that the norm of a vector is no longer a real number, as in standard Hilbert space, but a real Minkowski vector — and leave it formally at that. This modification does not affect the algebraic properties of operators. As for physical interpretations, they cannot be convincingly extracted from the algebra of quantions alone. They follow from quantionic differential equations (work in progress), and are not needed in the sequel.

6.1. The quantionic Hilbert space

Definition 8 *An n-dimensional* ***quantionic Hilbert space*** *is the set of all vectors*

$$|\Psi\rangle = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} \in \mathbb{D}^{\ltimes}$$

whose components are quantions.

Each component being a 4-vector, we may write $|\Psi\rangle^{\rho}$ if we want to display the tensor index.

Definition 9 *The **adjunct** of a quantionic vector is defined as*

$$\langle\Psi| = \begin{pmatrix} q_1^* & q_2^* & \cdots & q_n^* \end{pmatrix},$$

i.e., as a simultaneous transposition and complex conjugation of the components.

Definition 10 *The* norm *of a quantionic vector $|\Psi\rangle$ is defined by the sesquilinear form*

$$\langle\Psi|\,\beta\,|\Psi\rangle \overset{\text{def}}{=} \sum_{k=1}^{n} (q_k^* \beta q_k) = \sum_{k=1}^{n} A\,(q_k)\,. \tag{55}$$

Theorem 11 *The norm of a quantionic vector is a future-pointing time-like or null vector.*

Proof. By theorem 1, the summands $A\,(q_k) = q_k^* \beta q_k$ are future pointing time-like or null real Minkowski vectors. Due to the convexity of the future null cone, the norm $\langle\Psi|\,\beta\,|\Psi\rangle$, being a sum of such vector, can ever be a space-like vector. Moreover, it is a null vector only if all components of $|\Psi\rangle$ are proportional to some singular quantion (the proportionality coefficients being real numbers and quantionic gauge factors). ■

Expressing $A\,(q_k)$ in the structural frame, i.e., $A\,(q_k) = P_k\Omega + \vec{p}_k$, the time-like norm of the quantionic vector $|\Psi\rangle$ is

$$\langle\Psi|\,\beta\,|\Psi\rangle = P\Omega + \vec{p} = \sum_{k=1}^{n} P_k\Omega + \sum_{k=1}^{n} \vec{p}_k.$$

Since $P_k > 0$ for every k, it follows that $P > 0$, but the vectors $\vec{p}_k$ may well sum up to zero. This special case appears to be important enough to warrant a name:

Definition 12 *A quantionic vector $|\Psi\rangle$ is said to be **special** if its norm is in the direction of the structure vector Ω,*

$$\langle\Psi|\,\beta\,|\Psi\rangle = P\Omega,$$

i.e., if it is a scalar in the structural frame.

In standard Hilbert space, all vectors may be viewed as special, since their norms are scalars.

6.2. *The quantionic unitary matrices*

Definition 13 *Transformations of the form*

$$q'_j = \sum_{k=1}^{n} u_{jk} \beta q_k, \tag{56}$$

*where the matrix elements u_{jk} are quantions, are referred to as **linear quantionic transformations.***

The transformations relevant to quantum mechanics are those that preserve transition probabilities. Let us consider only the unitary transformations, which preserve the amplitudes of transition probabilities:

$$\langle U\Phi | \, \beta \, | U\Psi \rangle = \langle \Phi | \, \beta \, | \Psi \rangle \, . \tag{57}$$

In components,

$$\begin{aligned} \sum_{j=1}^{n} \left(p_j^* \beta q_j \right) &= \sum_{j=1}^{n} \left(\sum_{k=1}^{n} u_{jk} \beta p_k \right)^* \beta \left(\sum_{l=1}^{n} u_{jl} \beta q_l \right) \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} p_k^* \beta \left(\sum_{j=1}^{n} u_{jk}^* \beta u_{jl} \right) \beta q_l \, . \end{aligned}$$

This is an identity in Φ and Ψ if and only if

$$\sum_{j=1}^{n} u_{jk}^* \beta u_{jl} = \delta_{kl} \Omega, \tag{58}$$

or, symbolically, $U^{*T} \beta U = I\Omega$, which is formally the same definition of unitarity as in the standard case. The concept of Hermitian conjugation is also the same:

Definition 14 *The **Hermitian conjugate** of a quantionic matrix M is the complex conjugate of its transpose,*

$$M^\dagger \stackrel{\text{def}}{=} M^{*T}. \tag{59}$$

Definition 15 *A quantionic matrix U is **unitary** if its Hermitian conjugate is its bilateral inverse, i.e.,*

$$U \beta U^\dagger = U^\dagger \beta U = I\Omega. \tag{60}$$

Thus we may write

$$U^{-1} = U^\dagger \, . \tag{61}$$

Theorem 16 *The quantionic unitary matrices form a group, the* ***quantionic unitary group.***

Proof. (a) The unit matrix, $I\Omega^\rho$, exists. (Note: the quantionic unit matrix is not I, but $I\Omega^\rho$ — the diagonal matrix all of whose elements are the structure vector Ω^ρ.) (b) The product of quantionic unitary matrices is associative (because this is true of all quantionic matrices). (c) The set of quantionic unitary matrices is stable under the matrix product. For verification, let U and V be arbitrary unitary matrices. Then,

$$\begin{aligned}(U\beta V)^\dagger &= \left(\sum_{i=1}^{n} u_{ji}\beta v_{ik}\right)^\dagger = \left(\sum_{i=1}^{n} u_{ki}\beta v_{ij}\right)^* \\ &= \sum_{i=1}^{n} u_{ki}^*\beta^* v_{ij}^* = \sum_{i=1}^{n} v_{ij}^*\beta u_{ki}^* \\ &= \sum_{i=1}^{n} v_{ji}^\dagger\beta u_{ik}^\dagger = V^\dagger\beta U^\dagger.\end{aligned}$$

Hence,

$$(U\beta V)\,\beta\,(U\beta V)^\dagger = U\beta V\beta V^\dagger\beta U^\dagger = I\Omega,$$

proving the statement. ■

In the complex domain, the rows and columns of a unitary matrix form two bases of orthonormal vectors. The same is true in the quantionic domain:

Theorem 17 *The rows and columns of a unitary quantionic matrix form two sets of special orthonormal vectors.*

Proof. Extracting the k-th column of the matrix $U = (u_{jk})$ as a vector,

$$|k\rangle = \begin{pmatrix} u_{1k} \\ \dots \\ u_{nk} \end{pmatrix},$$

one can rewrite equation (60) as

$$\langle k|\,\beta\,|l\rangle = \delta_{lk}\Omega,$$

proving the assertion. The same conclusion holds if one similarly extracts the rows. ■

6.3. The quantionic gauge transformations

The gauge group in Hilbert space is defined as the subgroup of the unitary group which keeps every ray invariant. The concept generalizes to quantionic Hilbert space. Indeed, consider two rays, labeled by $a = 1, 2$. The most general ray-preserving transformations are

$$|\Psi_a\rangle \to |\Psi'_a\rangle = g_a\beta\,|\Psi_a\rangle\,.$$

The condition

$$\langle\Psi'_a|\Psi'_b\rangle = \langle\Psi_a|\Psi_b\rangle$$

implies

$$\langle g_a\beta\Psi_a|g_b\beta\Psi_b\rangle = \langle\Psi_a|\beta g_a^*\beta g_b\beta|\Psi_b\rangle = \langle\Psi_a|\Psi_b\rangle\,.$$

Hence, $g_a^*\beta g_b = \Omega$ for all a, b, which further implies that all phase factors are the same,

$$g = e^{\mathrm{i}s}. \tag{62}$$

6.4. Hermitian matrices

Definition 18 *A quantionic matrix H is* ***Hermitian*** *if*

$$H = H^\dagger. \tag{63}$$

In the complex domain, the Hermitian matrices generate unitary transformations. The same is true in the quantionic domain. Thus, for infinitesimal quantionic transformations,

$$U = I\Omega + \mathrm{i}\varepsilon H, \tag{64}$$

where $\varepsilon^2 = 0$, U is unitary,

$$\begin{aligned} U\beta U^\dagger &= (I\Omega + \mathrm{i}\varepsilon H)\,\beta\left(I\Omega - \mathrm{i}\varepsilon H^\dagger\right) \\ &= I\Omega + \mathrm{i}\varepsilon\left(H - H^\dagger\right) = I\Omega. \end{aligned}$$

The same is true for the finite transformations

$$U = \exp(\mathrm{i}\tau H) = \sum_{m=1}^{n} \frac{(\mathrm{i}\tau)}{m!} H^m, \tag{65}$$

where

$$H^m = H\beta H\beta ... \beta H.$$

With every quantionic unitary matrix U are associated two orthonormal bases of special unitary vectors. Taking one of them, $\{|1\rangle, |2\rangle, ..., |n\rangle\}$, we form n new matrices E_k defined as

$$E_k \stackrel{\text{def}}{=} |k\rangle \beta \langle k| . \tag{66}$$

As in the complex domain, they have the properties of states as represented by idempotents (pure density matrices):

Theorem 19 *The matrices E_k are Hermitian, idempotent, of unit trace, and mutually orthogonal.*

Proof. *Hermiticity*:

$$\begin{aligned} E_k^\dagger &= (|k\rangle \beta \langle k|)^\dagger = \left(|k\rangle^* \beta^* \langle k|^*\right)^T = \left(\langle k|^* \beta |k\rangle^*\right)^T \\ &= (|k\rangle \beta \langle k|)^T = (|k\rangle \beta \langle k|) = E_k. \end{aligned}$$

Idempotence:

$$E_k \beta E_k = |k\rangle \beta \langle k| \beta |k\rangle \beta \langle k| = |k\rangle \beta \langle k| = E_k.$$

Unit trace:

$$Tr E_k = Tr\left(|k\rangle \beta \langle k|\right) = \langle k| \beta |k\rangle = \Omega.$$

Orthogonality:

$$E_k \beta E_l = |k\rangle \beta \langle k| \beta |l\rangle \beta \langle l| = \delta_{kl} |k\rangle \beta \langle l| .$$

■

This completes the verification that the transition from $\mathbb{C}$ to $\mathbb{D}$ preserves the structure of Hilbert space concepts.

7. Conclusion

Since the algebra of quantions generalizes the field $\mathbb{C}$ of complex numbers, let us briefly discuss the role of the latter in physics, where, even before the advent of quantum mechanics, they had practical applications in the description of oscillatory systems. The reason is to be found in Euler's formula, $e^{i\phi} = cos\phi + i \sin\phi$, which relates complex numbers to harmonic functions. But in this role, complex numbers are an elegant convenience more than a necessity, even though it might be difficult to formulate some theorems without them (like dispersion relations, or the polology of control systems in engineering). In quantum mechanics, however, complex numbers are essential. Not merely because the wave function is an oscillatory object, but because the classical composition of transition probabilities (real numbers) is replaced in quantum mechanics by the composition of transition amplitudes (complex numbers). This is not a matter of convenience, but a revolution in physics. The results obtained in the present work suggest that it might be possible to extend the concept of amplitude to the relativistic domain.

There seems to be a widespread belief that this is impossible — probably because the superposition principle based on complex numbers has withstood seven decades of attempts at modification [4], even though other algebras do find applications in physics [5–7]. Let us not forget, however, that physicists have been attempting to generalize the number system of quantum mechanics within self-imposed mathematical limitations — the belief that one must stay within existing mathematical structures, and, specifically, within the division algebras — a belief explicitly stated in Dixon's excellent book on the subject [5] (page 7): "Quantum mechanics must rest on a division algebra." Though supported by past experience and heuristic arguments, this is an opinion, not a theorem. The argument favoring division algebras is that a positive definite norm is ostensibly needed for the wave function to have a probabilistic interpretation, and only division algebras have a positive definite norm. But this is an over-reaching conclusion. Indeed, work in progress on quantionic differential equations shows that the norm *must be* a 4-vector to be relativistic. Its physical interpretation is also quite compelling: It represents 4-currents of probability density.

To put it in perspective, we place the quantionic algebra $\mathbb{D}$ in a family we refer to as "the number systems", Table 1. The other members of this family are the division algebras (real numbers, complex numbers, quaternions and octonions). As we saw in the last section, only two properties are apparently needed for an algebra to support a Hilbert space structure. They are associativity, and, for lack of better terminology, "star symmetry", by which we mean that the real and imaginary parts of the algebra are linearly isomorphic. The former is needed for unitary groups to exist, the latter for observables and generators to be linearly isomorphic (justifications are to be found in Part I, [2]). Among all possible mathematical structures, only two enjoy both properties. They are the field $\mathbb{C}$ of complex numbers (a division algebra), and the quantionic algebra $\mathbb{D}$ (not a division algebra).

In Table 1, the first column selects the structure; the next four columns indicate if it is a division algebra, if the product is associative or commutative, and if the underlying linear space is star-symmetric, i.e., if $D(R) = D(I)$. The columns D, $D(R)$ and $D(I)$ indicate, respectively, the total number of real dimensions, the dimensionality of the real subspace, and the dimensionality of the imaginary subspace.

Table 1. The number systems.

St.	Div. a.	Comm.	Assoc.	* symm.	D	$D(R)$	$D(I)$
$\mathbb{R}$	yes	yes	yes	no	1	1	0
$\mathbb{C}$	yes	yes	YES	YES	2	1	1
$\mathbb{Q}$	yes	no	yes	no	4	1	3
$\mathbb{O}$	yes	no	no	no	8	1	7
$\mathbb{D}$	no	no	YES	YES	8	4	4

While some authors assign an *a priori* fundamental importance to division algebras, others to Clifford algebras, our viewpoint is that no algebra is intrinsically distinguished as physical for its aesthetic or other general properties. *Only the specific properties of associativity and star symmetry are relevant.*

We point out, incidentally, that the quantionic product (10) differs from the Clifford product

$$(U\Omega + \vec{u})\,(V\Omega + \vec{v}) = (UV + \vec{u}\cdot\vec{v})\,\Omega + U\vec{v} + V\vec{u}$$

in the term $\mathrm{i}\vec{U} \times \vec{V}$, so that the quantionic algebra is not a Clifford algebra either.

Acknowledgments

The four parts of the present work have been presented at various times in seminars at the Institute Ruđer Bošković and at the University of Zagreb. I am grateful to the audiences for discussions, but in particular to Nikola Zovko and Mladen Martinis.

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SUŠTINSKI RELATIVISTIČKA KVANTNA TEORIJA
IV Dio. KVANTIONSKI TEOREMI

Po završetku razvoja kvantionske algebre u dijelu III ovog niza radova, u ovom konačnom dijelu izvodimo algebarske teoreme koji su potrebni za primjene u fizici. Najvažniji od njih su teoremi koji se odnose na kvantionsku normu. Kvantionska norma poopćava normalizaciju vektora stanja u standardnoj kvantnoj mehanici i ima neposredna tumačenja koja se nameću njenim geometrijskim svojstvima.