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A STUDY OF FOUR AND HIGHER-DIMENSIONAL COSMOLOGICAL MODELS IN LYRA GEOMETRY

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Exact cosmological solutions for spherically symmetric models, both in four and higher dimensions are obtained within the framework of Lyra geometry. It is observed that there is no singularity at finite past in our four-dimensional model. For the five-dimensional model, the diminision of extra dimension with the evolution of the Universe is exhibited. The physical behaviour of the models is examined in vacuum (for the four-dimensional case) and in the presence of perfect fluids (for both four- and five-dimensional models).

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1. *Introduction*

In the last few decades, there has been a considerable interest in alternative theories of gravitation. The most important among them are the scalar-tensor theories proposed by Lyra [1] and by Brans and Dicke [1].

Lyra proposed a modification of the Riemannian geometry by introducing a gauge function into the structureless manifold that bears a close resemblance to Weil's geometry. In general relativity, Einstein succeded in geometrising gravitation by identifying the metric tensor with the gravitational potentials. In the scalar-tensor theory of Brans-Dicke, on the other hand, the scalar field remains alien to the geometry. Lyra's geometry is more in keeping with the spirit of Einstein's principle of geometrisation, since both the scalar and tensor fields have more or less intrinsic geometrical significance. In the subsequent investigations, Sen [2] and Sen and Dunn [3] proposed a new scalar-tensor theory of gravitation and constructed an analog of the Einstein's field equation based on Lyra geometry, which in normal gauge may be written as

$$R_{ik} - \frac{1}{2} g_{ik} R + \frac{3}{2} \phi_i \phi_k - \frac{3}{4} \phi_m \phi^m = -8P\, T_{ik} \,, \qquad (1)$$

where ϕ_i is the displacement vector and other symbols have their usual meaning as in the Riemannian geometry. According to Halford [4], the present theory predicts the same effects within observational limits, as far as the classical solar-system tests are concerned, as well as tests based on the linearised form of field equations.

Subsequent investigations were done by several authors in scalar-tensor theory and cosmology within the framework of Lyra geometry [5].

Soleng [6] has pointed out that the cosmologies based on Lyra's manifold with constant gauge vector ϕ_i will either include a creation field and be equal to Hoyle's creation field cosmology, or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term.

In this work, we present cosmological models within the framework of Lyra geometry.

In Sect. 2, we study the Friedmann-Robertson-Walker (FRW) models of the Universe with curvature parameter $K = 0$.

The unification of gravitational forces with other forces in Nature is not possible in the usual 4-dimensional space-time. So, higher-dimensional theory may be useful to meet this challenge in quantum-field theory [7]. This idea is particularly important in the field of cosmology, since one knows that our Universe was much smaller in its early stage than it is today, and the present 4-D stages could have been preceded by a higher-dimensional one at early times [7].

It is argued that the extra dimensions are not observable at the present time, owing to their size-bearing, assumed to be of the order of the Planck length, but perhaps they may have been relevant for the very early Universe.

As higher-dimensional theory is important at the early Universe, so it is interesting to study higher-dimensional cosmological model in Lyra geometry. In Sect. 3, we consider the five-dimensional perfect-fluid cosmological model based on Lyra's geometry.

The article ends with a short discussion in Sect. 4.

2. *Four-dimensional spherically-symmetric model*

The metric ansatz for the model is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{\lambda(t)} (\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\phi^2) \,. \qquad (2)$$

We take a perfect-fluid form for the energy-momentum tensor

$$T_{ik} = (P + \rho)U_i U_k - P g_{ik}\,, \tag{3}$$

together with the comoving coordinates $U_i U^i = 1$.

The equation of state for the fluid is taken as

$$P = m\rho \qquad (0 \le m \le 1)\,. \tag{4}$$

The time-like displacement vector is taken as

$$\phi_i = (\beta(t), 0, 0, 0)\,. \tag{5}$$

Now, the field equations can be set up and one obtains

$$\frac{3}{4}\,\dot{\lambda}^2 - \frac{3}{4}\,\beta^2 = \chi\rho\,, \tag{6}$$

$$\ddot{\lambda} + \frac{3}{4}\,\dot{\lambda}^2 + \frac{3}{4}\,\beta^2 = -\chi P\,. \tag{7}$$

Case I: Empty Universe

In this case

$$P = \rho = 0\,. \tag{8}$$

Solving the above field equations, we get

$$\mathrm{e}^{\lambda} = A(t - t_0)^{2/3} \quad \text{and} \quad \beta^2 = \frac{4}{9}\,\frac{1}{(t - t_0)^2}\,. \tag{9}$$

where A and t_0 are constants of integration. The scalar of expansion is

$$\theta = \frac{3}{2}\,\dot{\lambda} = \frac{1}{(t - t_0)}\,. \tag{10}$$

Here the gauge function β was large in the beginning, but decreases with the evolution of the model. We also see that at $t \to \infty$, the expansion ceases.

Case II: Matter-filled Universe

In this case, we take that the displacement vector is constant, i.e.,

$$\beta = const\,.$$

From Eqs. (4), (6) and (7), we get

$$\ddot{\lambda} + B\dot{\lambda}^2 = D\,, \tag{11}$$

where $B = \frac{3}{4}(m+1)$ and $D = \frac{3}{4}(m-1)\beta^2$. Solving Eq. (11), we get

$$\mathrm{e}^{\lambda} = \left[\cosh aB(t-t_0)\right]^{1/B}, \tag{12}$$

where $A^2 = \frac{D}{B}$, and t_0 is the integration constant.

We also get the following expressions for ρ, σ^2 (shear scalar), θ, H (the Hubble parameter) and Q (the deceleration parameter)

$$\rho = \frac{3}{4}(a^2 \tanh^2 aB(t-t_0) - \beta^2), \tag{13}$$

$$\theta = \frac{3}{2}\tanh aB(t-t_0), \tag{14}$$

$$\sigma^2 = 0, \tag{15}$$

$$H = \frac{a}{2}\tanh aB(t-t_0), \tag{16}$$

$$Q = 2B\,\mathrm{cosech}^2 aB(t-t_0) - 1. \tag{17}$$

3. *Five-dimensional spherically symmetric model*

The line element for the five-dimensional Kaluza-Klein model is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{\lambda(t)}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2) - \mathrm{e}^{\mu(t)}\mathrm{d}y^2. \tag{18}$$

Here the displacement vector is taken as

$$\phi_i = (\beta(t), 0, 0, 0, 0). \tag{19}$$

The field equations (1) for the metric in (18) reduce to

$$\frac{3}{4}(\dot{\lambda}^2 + \dot{\lambda}\dot{\mu}) - \frac{3}{4}\beta^2 = \chi\rho, \tag{20}$$

$$\ddot{\lambda} + \frac{3}{4}\dot{\lambda}^2 + \frac{1}{2}\ddot{\mu} + \frac{1}{4}\dot{\mu}^2 + \frac{1}{2}\dot{\lambda} + \frac{3}{4}\beta^2 = -\chi P, \tag{21}$$

$$\frac{3}{2}(\ddot{\lambda} + \dot{\lambda}^2) + \frac{3}{4}\beta^2 = -\chi P. \tag{22}$$

Here we have three field equations and one equation of state connecting five unknowns, namely λ, μ, ρ, P and β^2. So for the unique solutions, one must asume one more relation connecting them. We take

$$\mu = a\lambda, \tag{23}$$

where a is an arbitrary constant. We obtain the solutions

$$\mathrm{e}^{\lambda} = B(t - t_0)^{1/A}\,, \tag{24}$$

$$\mathrm{e}^{\mu} = B^a (t - t_0)^{a/A}\,, \tag{25}$$

$$\rho = \frac{a^3 + 5a^2 + 6a}{\chi(1-m)(2a-1)A^2}\,\frac{1}{(t-t_0)^2}\,, \tag{26}$$

$$\frac{3}{4}\beta^2 = \frac{E}{(t-t_0)^2}\,, \tag{27}$$

where

$$A = \frac{a^2+2a-3}{2(2a-1)},\ E = \frac{1}{A^2}\left(\frac{3}{4}(a+1) - \frac{D}{1-m}\right), D = \frac{3}{2} + \frac{5}{4}a + \frac{1}{4}a^2 - \left(1 + \frac{1}{2}a\right)A.$$

B and t_0 are integration constants.

For this model, the physical parameters are

$$\theta = \frac{3+a}{2A}\,\frac{1}{t-t_0}\,, \tag{28}$$

$$\sigma^2 = \frac{3(a-1)^2}{32A^2}\,\frac{1}{(t-t_0)^2}\,. \tag{29}$$

4. Discussion

In this article, we have obtained several sets of explicit solutions both in the four-dimensional model and in the higher dimension, within the framework of the Lyra geometry.

In the four-dimensional spherically-symmetric model, both empty Universe and matter filled Universe are studied. For the empty Universe, we see that the Universe starts at an initial epoch at $t = t_0$, which is a point singularity. In this case, the deceleration parameter is constant. For the matter-filled Universe, we find an exact solution of the field equations and note that our space-time is singularity-free. This is unlike the solution in the general-relativity scheme, because it is very well known that the isotropic and homogeneous FRW models must have a universal big-bang singularity if the energy conditions are satisfied [8].

For the five-dimensional model, we note that the Universe starts at an initial epoch $t = t_0$. From Eq. (26), one can see that $\rho \geq 0$ would hold only for $a > \frac{1}{2}$. If $\frac{1}{2} < a < 3$, the initial epoch is a line singularity with the evolution of the Universe, e^{λ} increases while e^{μ} decreases. Thus, the extra dimension becomes insignificant as the time proceeds after the creation, and we are left with the real four-dimensional world. At the initial epoch, all physical parameters θ, σ^2 and ρ diverge. As t gradually increases, θ, σ^2 and ρ decrease, and finally, when $t \to \infty$, θ, σ^2 and ρ vanish.

The gauge function was large in the beginning, but decreases with the evolution of the model, and gradually dies out as $t \to \infty$. So, the concept of the Lyra geometry will not linger for indefinite time.

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STUDIJ ČETIRI- I VIŠE-DIMENZIJSKIH KOZMOLOŠKIH MODELA U LYROVOJ GEOMETRIJI

U okviru Lyrove geometrije izvodimo točna kozmološka rješenja za sferno-simetrične modele u četiri i više dimenzija. U našem 4-dimenzijskom modelu ne nalazi se singularnost u konačnoj prošlosti. U 5-dimenzijskom se modelu pokazuje nestanak dodatne dimenzije razvojem Svemira. Fizička se svojstva modela ispituju u vakuumu (4-dimenzijski model) i u prisustvu perfektne tekućine (za 4- i 5-dimenzijske modele).