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CORRECTIONS TO THE HIGH FREQUENCY TAIL OF THE BLACK BODY SPECTRUM DUE TO THREE ALTERNATIVE STATISTICS

CARL WOLF

Department of Physics, Massachusetts College of Liberal Arts, North Adams, Massachusetts 01247, U.S.A.

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By considering three alternative generalizations of the Bose-Einstein statitstics, we calculate the corrections that each of these generalizations have on the black body spectrum at high frequencies.

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1. Introduction

Probably one of the most mysterious and ad-hoc principles of quantum theory is the Pauli principle and the spin-statistics connection [1,2]. Geroch and Horowitz [3] have pointed to the fact that the exclusion principle is not related to the space time properties of field theory, but is a result of topological properties in spin space. Some of original attempts at generalizing Bose-Fermi symmetry go back to Gentile [4] who considers the statistics that allows up to k particles in a state, and attempts by Green [5], Govorkov [6] and Greenberg [7] who generalize the commutation relations to read

$$a_i a_j^+ - q a_j^+ a_i = \delta_{ij} \qquad (q \neq 1) .$$

Experimental limits on such violations are extremely small [8,9]. More recently attempts to generalize Bose-Fermi symmetry include the work of Wu [10] and Haldane [11] who developed a statistics interpolating between Bose and Fermi statistics, the work of Medvedev [12] who ascribed the Fermi-Bose nature to each particle and the work of Tsallis [13] who generalized Bose statistics to take into account the multi-fractal relation of energy levels and embodied notions of self-similarity and scale invariance. In three previous studies [14–16], we have calculated the modified black body spectrum that each of these approaches implies. The purpose of the present note is to compare the high frequency tail of each of these modified approaches with the intent of comparing with the more precise measurements of the black body radiation in the future.

2. *Modification of the high frequency tail of the black body energy distribution*

We begin by briefly discussing the statistics introduced by Haldane (Ref. [11]). Here we write for the number of ways of realizing N_i particles in g_i cells

$$\omega_i = \frac{[g_i + (N_i - 1)(1-\alpha)]\,!}{N_i\,!\,[g_i - \alpha N_i - (1-\alpha)]\,!}\,. \tag{1}$$

For the entropy

$$S = k \ln \pi_i \omega_i\,. \tag{2}$$

In Eq. (1), if $\alpha = 0$, we have Bose-Einstein (BE) statistics, if $\alpha = 1$, we have Fermi statistics. If we maximize Eq. (2) with the constraints $\Sigma N_i = const$, $\Sigma N_i \varepsilon_i = const$, we find for small α (small deviation from BE statistics) (Ref. [14])

$$N_i = \frac{g_i(1+\alpha)}{\exp(h\nu/\tau) - 1} - \frac{g_i \exp(h\nu/\tau)\alpha}{(\exp(h\nu/\tau)-1)^2} \qquad (\mu = 0 \text{ for chemical potential})\,. \tag{3}$$

where $\tau = kT$ and $\varepsilon_i = h\nu$ is the photon energy. From Eq. (3) and $g_i = 8\pi p^2 \mathrm{d}p/h^3$(per unit volume) $= 8\pi\nu^2\mathrm{d}\nu/c^3$, we have for the energy per unit frequency range

$$\mathrm{d}U(\nu) = \frac{8\pi h\nu^3 \mathrm{d}\nu}{c^3}\left[\frac{(1+\alpha)}{\exp(h\nu/\tau)-1} - \alpha\frac{\exp(h\nu/\tau)}{(\exp(h\nu/\tau)-1)^2}\right]. \tag{4}$$

We now consider $h\nu/(kT) > 1$. If we expand Eq. (4) in powers of $\exp(-h\nu/\tau)$, we obtain

$$\mathrm{d}U(\nu) = \frac{8\pi h\nu^3 \mathrm{d}\nu}{c^3}\mathrm{d}\nu\left[1 + (1-\alpha)\mathrm{e}^{-h\nu/(kT)} + (1-2\alpha)\mathrm{e}^{-2h\nu/(kT)} + (1-3\alpha)\mathrm{e}^{-3h\nu/(kT)}\right]\mathrm{e}^{-h\nu/(kT)}\,. \tag{5}$$

Equation (5) gives a precise high-frequency expansion of $\mathrm{d}U(\nu)$ in powers of $\mathrm{e}^{-h\nu/(kT)}$ and the parameter α.

We next consider the ambiguous statistics of Medvedev (Ref. [12]). For N_j particles, we consider $N_j - k$ Fermion states and k Boson states. For the number of ways of realizing k Bosons and $N_j - k$ Fermions, where k varies from 0 to N_j, we have [17]

$$\omega_j = \sum_{k=0}^{N_j} \frac{g_j\,!}{(N_j-k)\,!\,(g_j - N_j + k)\,!}\,\frac{(g_j + k - 1)\,!}{(g_j - 1)\,!\,k\,!}\,\frac{N_j\,!}{(N_j-k)\,!\,k\,!}\,P_b^k P_f^{N_j-k}\,. \tag{6}$$

Here P_b is the probability of a particle being a Boson and P_f the probability of a particle being a Fermion. We then construct $\omega = \pi\omega_j$ and maximize $S = k\ln\omega$, subject to $\sum N_j = const$ and $\sum N_j\varepsilon_j = const$, and the result is [12]

$$\frac{N_j}{g_j} = \frac{P_f + P_b}{(x+P_f)(x-P_b)} \left[1 + \sqrt{\frac{(P_f - P_b)^2(x+P_f)(x-P_b)}{(P_f+P_b)^2 x(x+P_f-P_b)}}\right], \tag{7}$$

where $x = \exp((\varepsilon_j - \mu)/\tau)$, μ is the chemical potential and $\tau = kT$. If we let $P_b = 1 - \varepsilon$ and $P_f = \varepsilon$, then after the expansion of Eq. (7) to the order ε, we find [15]

$$\frac{N_j}{g_j} = \frac{1}{x-1} - \frac{\varepsilon}{x-1}\left(\frac{1}{x} + \frac{1}{x-1}\right) + \frac{2\sqrt{\varepsilon}}{x-1}\left(1 + \frac{1}{4(x-1)} - \frac{1}{4x}\right). \tag{8}$$

In the limit $h\nu/(kT) \gg 1$, Eq. (8) becomes

$$\frac{N_j}{g_j} \cong \mathrm{e}^{-h\nu/(kT)} - 2\varepsilon\,\mathrm{e}^{-2h\nu/(kT)} + 2\sqrt{\varepsilon}\;\mathrm{e}^{-h\nu/(kT)} - \dots .$$

For the energy per unit frequency, we have

$$\mathrm{d}U(\nu) \cong \frac{8\pi h\nu^3}{c^3}\left[\mathrm{e}^{-h\nu/(kT)}\left(1+2\sqrt{\varepsilon}\right) - 2\varepsilon\,\mathrm{e}^{-2h\nu/(kT)} - \dots\right]\mathrm{d}\nu\,. \tag{9}$$

Note that Eq. (9) gives a diminuation in the factor $\mathrm{e}^{-2h\nu/(kT)}$, while Eq. (5) gives a positive coefficient of the factor $\mathrm{e}^{-2h\nu/(kT)}$.

For the third modification of statistics, we consider the non-extensive statistics of Tsallis [13] that applies to any system with a long-range gravitational influence or to a system with non-Markovian memory. Since the cosmic microwave background may retain some memory of times when matter and radiation were strongly coupled, this might lead to a Tsallis like statistics for photons [18]. Starting with the expression

$$S = \frac{kN}{q-1}\left(\sum P_i - \sum P_i^q\right), \qquad \text{with} \qquad P_i = \frac{N_i}{N} \tag{10}$$

for the entropy of N particles, we find after using the constraints

$$\sum N_i = N \qquad \text{and} \qquad \sum N_i\varepsilon_i = U\,,$$

the following expressions for the number of oscillators with energy ε_I [16]

$$N_i = \frac{N}{\mathrm{e}}\,\mathrm{e}^{\frac{\mu_0-\varepsilon_i}{\tau}} + \alpha\frac{N}{\mathrm{e}}\left[\frac{1}{2\tau^2}\,\mathrm{e}^{\frac{\mu_0-\varepsilon_i}{\tau}}\frac{\sum \mathrm{e}^{\frac{\mu_0-\varepsilon_i}{\tau}}(\mu_0-\varepsilon_i)^2}{\sum \mathrm{e}^{\frac{\mu_0-\varepsilon_i}{\tau}}} - \frac{\mathrm{e}^{\frac{\mu_0-\varepsilon_i}{\tau}}(\mu_0-\varepsilon_i)^2}{2\tau^2}\right], \tag{11}$$

where μ_0 is the zeroth-order chemical potential, $\tau = kT$, $\alpha = (q-1)$ where $q \neq 1$ is the Tsallis parameter. When we evaluate $< \varepsilon >$ (average energy of an oscillator with the frequency ν), we obtain from Eq. (11)

$$< \varepsilon > = \frac{\hbar\omega}{\mathrm{e}^{\hbar\omega/\tau} - 1} - \frac{1}{8}\alpha\frac{(\hbar\omega)^3}{\tau^2} + \frac{1}{8}\alpha\frac{(\hbar\omega)^2}{\tau} \,. \tag{12}$$

The vacuum energy was subtracted. In deriving Eq. (12), we replaced the sum over all oscillators by an integral over N_i $[\varepsilon_i = (i+1/2)\hbar\omega]$. We also assume $\hbar\omega/\tau >> 1$. If instead we assume that just the first excited state contributes to $< \varepsilon >$, we obtain

$$< \varepsilon > = \frac{\hbar\omega}{\mathrm{e}^{\frac{\hbar\omega}{\tau}} - 1} + \alpha\left(1 - \mathrm{e}^{-\frac{\hbar\omega}{\tau}}\right)\frac{(\hbar\omega)^3}{\tau^2}\left[\frac{1}{4}\frac{\left(1 + 3\mathrm{e}^{-\frac{\hbar\omega}{\tau}}\right]\mathrm{e}^{-\frac{\hbar\omega}{\tau}}}{1 + \mathrm{e}^{\frac{\hbar\omega}{\tau}}} - \frac{3}{4}\mathrm{e}^{-\frac{\hbar\omega}{\tau}}\right]. \tag{13}$$

The dominant contribution to Eq. (13) is

$$< \varepsilon > = \frac{\hbar\omega}{\mathrm{e}^{\frac{\hbar\omega}{\tau}} - 1} - \alpha\frac{\mathrm{e}^{-\frac{\hbar\omega}{\tau}}(\hbar\omega)^3}{\tau^2} \qquad (\omega = 2\pi\nu) \,. \tag{14}$$

For the energy distribution per unit frequency, we obtain from Eq. (14)

$$\mathrm{d}U(\nu) = \frac{8\pi\nu^2\mathrm{d}\nu}{c^3}\left[\frac{\hbar\omega}{\mathrm{e}^{\frac{\hbar\omega}{\tau}} - 1} - \alpha\frac{\mathrm{e}^{-\frac{\hbar\omega}{\tau}}(\hbar\omega)^3}{\tau^2}\right]. \tag{15}$$

Actually, Eq. (14) is a better approximation to $< \varepsilon >$ at high ν since if $h\nu/\tau \gg 1$, we expect only the lowest states to contribute in Eq. (11).

3. Conclusion

In Eq. (5), Eq. (9) and Eq. (15), we obtained corrections to the usual black-body energy-density formulae due to the Haldane statistics, the ambiguous statistics of Medvedev and the Tsallis statistics, respectively. Another modification would be due to the possible finite number of oscillator components leading to an average oscillator energy of

$$< \varepsilon > = \frac{\sum_{n=0}^{N}\left(n + \frac{1}{2}\right)\hbar\omega\mathrm{e}^{-(n+\frac{1}{2})\frac{\hbar\omega}{\tau}}}{\sum \mathrm{e}^{-(n+\frac{1}{2})\frac{\hbar\omega}{\tau}}} = \frac{\hbar\omega}{2} + \hbar\omega\left(\frac{x + 2x^2 + 3x^3 + 4x^4}{1 + x + x^2 + x^3 + x^4}\right)$$

for $N = 4$, where $x = \mathrm{e}^{-\frac{\hbar\omega}{\tau}}$. The deviation from the usual B.E. energy would be

$$\delta\varepsilon = \hbar\omega\left(\frac{x + 2x^2 + 3x^3 + 4x^4}{1 + x + x^2 + x^3 + x^4} - \frac{x}{1 - x}\right) = \frac{-5h\nu\ \mathrm{e}^{-5h\nu(\hbar\omega/\tau)}}{(1 - x)(1 + x + x^2 + x^3 + x^4)} \,.$$

Such a cutting-off of the number of oscillator states might be due to axion-photon oscillations [19], or the strongly coupled phase of QED [20]. Also deviations from the usual black body spectrum of the CBR might be the result of heavy neutrino decays [21], Higgsino decays [22], or the decay of particles with $m_e \approx 1$ eV [23]. Since the CMB temperature is only known to an accuracy of $(2.73 \pm .01)$ K [24], and effects such as reionization and foreground contamination create a margin of error in measurements, it could very well be that the corrections included in Eq. (5), Eq. (9) and Eq. (15) would be hidden by these larger experimental fluctuations.

One application of testing the difference between Eq. (5), Eq. (9) and Eq. (15) was originally proposed by Sunyaev [25,26] by looking for short wavelengths less than the Lyman continuum. The method entails the observation of rarefied, neutral hydrogen in the halo of galaxies and in the bridge between galaxies which absorbs UV radiation, the presence of neutral hydrogen detected by the 21 cm line. The differential density of neutral hydrogen atoms around galaxies can be set in correspondence with the UV flux and input temperatures can be inserted into Eq. (5), Eq. (9) and Eq. (15) to see which is most compatible with the neutral hydrogen absorbing UV radiation in a certain windows. In Fig. 1 below, the deviations from a Bose-Einstein plot are given in the UV region for the three statistics (Haldane, Medvedev and Tsallis) with parameters listed in the graph. It seems that only the statistics of Medvedev enhances the spectrum.

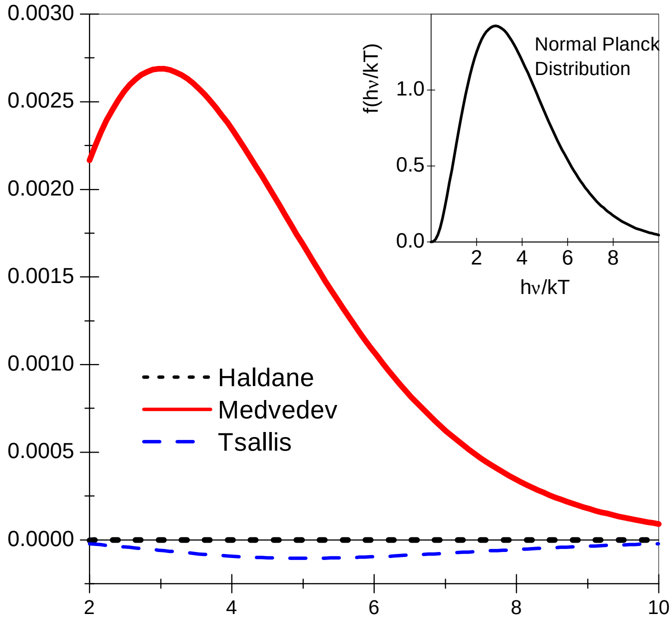

Fig. 1. Additive corrections to the Planck distribution according to the Haldane ($\alpha = 10^{-5}$), Medvedev ($\epsilon = 10^{-6}$) and Tsallis statistics ($\alpha = 10^{-5}$). The common multiplicative factor is $\frac{8\pi h}{c^3} x (kT/h)^4$).

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POPRAVKE VISOKOFREKVENTNOG DIJELA SPEKTRA CRNOG TIJELA PREMA TRIMA INAČNIM STATISTIKAMA

Razmatraju se tri poopćenja Bose-Einsteinove statistike i računaju popravke koje svaka od njih uvodi za spektar crnog tijela u području visokih frekvencija.