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SEMI-CLASSICAL GRAVITATIONAL EFFECTS NEAR COSMIC STRING IN LYRA GEOMETRY

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In recent past, W. A. Hiscock [Phys. Lett. B 188 (1987) 317] studied the semi-classical gravitational effects near cosmic strings. He obtained the vacuum expectation value of the stress-energy tensor of an arbitrary collection of conformal massless free-quantum fields (scalar, spinor and vectors) in the presence of a static, cylindrically symmetric cosmic string. With this stress-energy tensor, we study the semi-classical gravitational effects of a cosmic string in the context of Lyra geometry.

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1. Introduction

In the past two decades, it has been proposed that the spontaneous breaking of symmetries in grand unified theories during phase transitions in the early Universe could lead to the formation of such structures as domain walls, cosmic strings and monopoles [1]. Cosmologists are interested in defects as possible sources for the density perturbations which seeded galaxy formation. One of these defects, the cosmic string, is particularly interesting as it is capable of producing observational effects and also may play an important role in the large-scale structure formation of the Universe [2].

In last few decades, there has been considerable interest in alternative theories of gravitation. The most important among them are scalar-tensor theories proposed by Lyra [4] and by Brans-Dicke [4]. Lyra [4] proposed a modification of Riemannian geometry by introducing a gauge function in to the structureless manifold that bears a close resemblance to Weyls geometry.

In general relativity, Einstein succeeded in geometrising gravitation by identifying the metric tensor with the gravitational potentials. In the scalar-tensor theory of Brans-Dicke, on the other hand, scalar field remains alien to the geometry. Lyra's geometry is more in keeping with the spirit of Einsteins principle of geometrisation, since both the scalar and tensor fields have more-or-less intrinsic geometrical significance. In the consecutive investigations, Sen [5] and Sen and Dunn [5] proposed a new scalar-tensor theory of gravitation and constructed an analog of the Einstein field equation based on Lyra's geometry which in normal gauge may be written as

$$F_{ik} \equiv R_{ik} - \frac{1}{2}g_{ik}R + \frac{3}{2}\phi_i\phi_k - \frac{3}{4}g_{ik}\phi_m\phi^m = -8\pi T_{ik}\,, \tag{1}$$

Where ϕ_i is the displacement vector and other symbols have their usual meaning as in Riemannian geometry.

Halford [6] has pointed out that the constant displacement field ϕ_i in Lyra's geometry plays the role of cosmological constant Λ in the normal general-relativistic treatment. According to Halford, the present theory predicts the same effects within observational limits, as far as the classical solar system tests are concerned, as well as tests based on the linearised form of field equations. Soleng [7] has pointed out that the constant displacement field in Lyra's geometry will either include a creation field and be equal to Hoyles creation field cosmology or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term. Subsequent investigations were done by several authors in scalar tensor theory and cosmology within the framework of Lyra geometry [8]. In classical theory, if the Lagrangian is conformally invariant, the trace of the energy stress tensor vanishes, but in the corresponding quantized theory, it acquires a trace during renormalization. This trace anomaly is a geometrical scalar containing derivatives of the metric tensor. The trace of the vacuum stress energy for a conformally coupled massless free field is given by the anomaly, which is

$$T^a_a = \frac{1}{2880\pi^2}\left[aC_{ijkl}C^{ijkl} + b\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right) + c\,\Box R + dR^2\right].$$

The constants a, b, c and d depend on the conformal scalar field under consideration and other symbols have their usual meaning as in Riemannian geometry.

In 1987, Hiscock [3] studied semi-classical gravitational effects near cosmic strings. He has considered the vacuum expectation value (VEV) of the stress tensor of an arbitrary collection of conformal massless free quantum fields(scalar, spinor and vectors) as the source term in the background space-time of a static cylindrically symmetric cosmic string. Taking these non-zero vacuum expectation value of the stress-energy tensors as a source, he solved the semi-classical Einstein equations for the quantum perturbations (to the first order in $\hbar$) of the metric. In a recent work, we have studied semi-classical gravitational effects around global monopole in Lyra geometry [9]. In this paper, we discuss semi-classical gravitational effects near cosmic string with constant displacement vector based on Lyra geometry in normal gauge, i.e. displacement vector $\phi_i = (\beta, 0, 0, 0)$, where β is a constant, and look whether the semi-classical gravitational effects near the cosmic string shows any significant properties due to the introduction of the gauge field in the Riemannian geometry. We have taken the same vacuum expectation value of the stress-energy tensor as obtained by Hiscock [3] and set equal to the modified Einstein's equations to solve for the quantum perturbations (to the first order in $\hbar$) of the metric.

2. Basic equations

The form of the entire vacuum stress-energy tensors as obtained by Hiscock [3] are

$$\langle T_a^b \rangle = D\hbar r^{-4}[1,\, 1,\, -3,\, 1]\,, \tag{2}$$

where the constant $D = [(1-4\mu)^{-4} - 1]/(1440\pi^2)$ and μ is the mass per unit length of the string.

In our consideration, the vacuum expectation values of the stress tensors of the quantum fields can be set equal to F_{ik} in the semi-classical approach to the quantum theory of gravity

$$F_{ik} = -8\pi \langle T_{ik} \rangle\,. \tag{3}$$

Here the geometrical units are used with $G = c = 1$ and $\hbar \approx 2.612 \times 10^{-66}$ cm^2.

For a cosmic string, the space time is static, cylindrically symmetric. One can write the corresponding line element as

$$\mathrm{d}s^2 = A(r)(-\mathrm{d}t^2 + \mathrm{d}z^2) + A(r)\mathrm{d}r^2 + r^2 B(r)\mathrm{d}\theta^2\,. \tag{4}$$

The field equations (3) for the metric (4) are

$$\frac{1}{2}\frac{A'B'}{A^2B} + \frac{A'^{\,2}}{4A^3} + \frac{A'}{rA^2} + \frac{3}{4}\,\frac{1}{A}\,\beta^2 = -\frac{8\pi D\hbar}{r^4}\,, \tag{5}$$

$$-\frac{A'^{\,2}}{4A^3} + \frac{A''}{A^2} + \frac{3}{4}\,\frac{1}{A}\,\beta^2 = \frac{24\pi D\hbar}{r^4}\,, \tag{6}$$

$$-\frac{A'^{\,2}}{A^3} + \frac{1}{2}\frac{A''}{A^2} + \frac{1}{2}\frac{B''}{B^2} + \frac{B'}{rAB} - \frac{3}{4}\frac{B'^{\,2}}{AB^2} - \frac{3}{4}\,\frac{1}{A}\beta^2 = -\frac{8\pi D\hbar}{r^4} \tag{7}$$

(‘$'$’ indicates differentiation w.r.t. r).

3. Solutions in the weak-field approximations

Under the weak-field approximation, one can write

$$A(r) = 1 + f(r) \qquad \text{and} \qquad B(r) = 1 + g(r)\,. \tag{8}$$

Here the functions f and g should be computed to the first order in $\hbar$ and β^2.

Under these weak-field approximations, the field equations take the following forms

$$\frac{f'}{r} + \frac{3}{4}\beta^2 = -\frac{8\pi D\hbar}{r^4}\,, \tag{9}$$

$$f'' + \frac{3}{4}\beta^2 = \frac{24\pi D\hbar}{r^4}\,, \tag{10}$$

$$\frac{1}{2}f'' + \frac{1}{2}g'' + \frac{g'}{r} - \frac{3}{4}\beta^2 = -\frac{8\pi D\hbar}{r^4}\,, \tag{11}$$

From the above equations, we get the following solution for f and g

$$f = \frac{4\pi D\hbar}{r^2} - \frac{3}{8}\beta^2 r^2 \,, \tag{12}$$

$$g = -\frac{20\pi D\hbar}{r^2} - \frac{3}{8}\beta^2 r^2 \,. \tag{13}$$

Thus in the weak-field approximation, the string metric in Lyra's geometry, considering semi-classical gravitational effects, takes the following form

$$\mathrm{d}s^2 = \left[1 + \frac{4\pi D\hbar}{r^2} - \frac{3}{8}\,\beta^2 r^2\right](-\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2) + r^2\left[1 - \frac{20\pi D\hbar}{r^2} + \frac{3}{8}\,\beta^2 r^2\right]\mathrm{d}\theta^2 \,. \tag{14}$$

4. Gravitational effects on test particles

Let us now consider a relativistic particle of mass m, moving in the gravitational field of the string described by Eq. (14) using the formalism of Hamilton and Jacobi (H - J). In our case, the H - J equation is [10]

$$\frac{1}{A(r)}\left[-\left(\frac{\partial S}{\partial t}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 + \left(\frac{\partial S}{\partial r}\right)^2\right] + \frac{1}{B(r)}\left(\frac{\partial S}{\partial \theta}\right)^2 + m^2 = 0 \,, \tag{15}$$

where $A = 1 + 4\pi D\hbar/r^2 - (3/8)\beta^2 r^2$ and $B = 1 - 20\pi D\hbar/r^2 + (3/8)\beta^2 r^2$.

In order to solve the particle differential equation, let us use the separation of variables for the H - J function S as follows [10],

$$S(r, z, q, t) = -Et + S_1(r) + J\theta + Mz \,. \tag{16}$$

Here the constants E and J can be identified as the energy and angular momentum of the particle and M is the momentum along the z-direction.

Hence the expression for $S_1(r)$ is

$$S_1(r) = \epsilon \int \left[E^2 - M^2 + m^2 A - \frac{AJ^2}{Br^2}\right]^{1/2} \mathrm{d}r \,.$$

Here $\epsilon = \pm 1$.

Hence the trajectory of the particle is given by [10]

$$t = \epsilon \int E\left[E^2 - M^2 + m^2 A - \frac{AJ^2}{Br^2}\right]^{-1/2} \mathrm{d}r \,, \tag{17}$$

$$\theta = \epsilon \int \frac{AJ}{Br^2}\left[E^2 - M^2 + m^2 A - \frac{AJ^2}{Br^2}\right]^{-1/2} \mathrm{d}r \,, \tag{18}$$

$$z = \epsilon \int M \left[E^2 - M^2 + m^2 A - \frac{AJ^2}{Br^2} \right]^{-1/2} \mathrm{d}r \,. \tag{19}$$

Thus the radial velocity of the particle is given by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{E} \left[E^2 - M^2 + m^2 A - \frac{AJ^2}{Br^2} \right]^{1/2} \,. \tag{20}$$

The turning points of the trajectory are given by $\mathrm{d}r/\mathrm{d}t = 0$ and as a consequence the potential curves are

$$\frac{E}{m} = \left[\frac{M^2}{m^2} + \frac{AJ^2}{r^2 m^2 B} - A \right]^{1/2} \,. \tag{21}$$

The extrema of the potential curve are the solutions of the equation

$$\frac{3}{4}\beta^2 r^7 + 2r^2 \left[\frac{J^2}{m^2} + 4\pi D\hbar \right] - 96\pi D\hbar \frac{J^2}{m^2} = 0 \,. \tag{22}$$

As an algebraic equation of odd degree, it has at least one real root, so it is possible to have a bound orbit. Hence a trajectory of the particle can be trapped by the string. Thus a cosmic string, with semi-classical effects taken into consideration in Lyra geometry, may have a gravitational field which is attractive in nature.

5. *Geodesics*

We shall now study the bending of light in the above field. The equation for the path of a light beam is [11]

$$\left(\frac{\mathrm{d}X}{\mathrm{d}\theta} \right)^2 = aX^2 + bX - c \,, \tag{23}$$

where $A\mathrm{d}t/\mathrm{d}p = l$, $A\mathrm{d}z/\mathrm{d}p = k$, $r^2 B\mathrm{d}\theta/\mathrm{d}p = h$ and $r = 1/U$, where p is the affine parameter along the light path and $X = U^2$, $a = [48\pi D\hbar(l^2/k^2) - 1]$, $b = [(l^2/h^2) - (k^2/h^2) + \frac{3}{4}\beta^2]$ and $c = [\frac{3}{2}\beta^2(l^2/h^2)]$ (here we neglect the term $U^6\hbar$).

Then we get

$$\frac{1}{\sqrt{a}} \ln \left[X + \frac{b}{2a} \right] + \sqrt{\left[X + \frac{b}{2a} \right]^2 - \frac{b^2 - 4ac}{4a^2}} = 2\theta \,. \tag{24}$$

To $U = 0$ corresponds

$$2\theta = \frac{1}{\sqrt{a}} \ln \left(\frac{b}{2a} + \sqrt{\frac{c}{a}} \right) \,. \tag{25}$$

Hence the angle of bending of light is given by the expression

$$\Delta = \pi \left[1 - \frac{1}{\sqrt{a}} \ln \left(\frac{b}{2a} + \sqrt{\frac{c}{a}} \right) \right] \,. \tag{26}$$

6. Summary

In this paper, we have studied the semi-classical gravitational effects near cosmic string in Lyra geometry. We have assumed the same vacuum expectation value of the stress-energy tensor as obtained by Hiscock [3]. We have shown that cosmic string, with semi-classical effects into consideration, in Lyra geometry may have gravitational field which is attractive in nature. When the gauge function is switched off i.e. when $\beta = 0$, then we return to the earlier general relativity solution of Hiscock. We also discuss the trajectory of photons in our space-time.

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POLUKLASIČNI GRAVITACIJSKI EFEKTI BLIZU KOZMIČKE STRUNE U LYRINOJ GEOMETRIJI

Nedavno je W. A. Hiscock [Phys. Lett. B 188 (1987) 317] proučavao poluklasične gravitacijske efekte u blizini kozmičkih struna. Izveo je vakuumsku očekivanu vrijednost za tenzor naprezanja-energije za proizvoljan skup konformnih, bezmasenih i slobodnih kvantnih polja (skalara, spinora i vektora) u prisustvu statičke cilindrično-simetrične kozmičke strune. S tim tenzorom proučavamo poluklasične gravitacijske efekte kozmičke strune u okviru Lyrine geometrije.