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INEQUALITIES ASSOCIATED WITH THE BAXTER NUMBERS

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Abstract. The Baxter numbers B_n enumerate a lot of algebraic and combinatorial objects such as the bases for subalgebras of the Malvenuto-Reutenauer Hopf algebra and the pairs of twin binary trees on n nodes. The Turán inequalities and higher order Turán inequalities are related to the Laguerre-Pólya ($\mathcal{L}$-$\mathcal{P}$) class of real entire functions, and the $\mathcal{L}$-$\mathcal{P}$ class has a close relation with the Riemann hypothesis. The Turán type inequalities have received much attention. In this paper, we are mainly concerned with Turán type inequalities, or more precisely, the log-behavior, and the higher order Turán inequalities associated with the Baxter numbers. We prove the Turán inequalities (or equivalently, the log-concavity) of the sequences $\{B_{n+1}/B_n\}_{n\geqslant 0}$ and $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$. Monotonicity of the sequence $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$ is also obtained. Finally, we prove that the sequences $\{B_n/n!\}_{n\geqslant 2}$ and $\{B_{n+1}B_n^{-1}/n!\}_{n\geqslant 2}$ satisfy the higher order Turán inequalities.

1. Introduction

Baxter permutations were introduced by Glen Baxter [2] while studying fixed points of the composite of commuting functions. These permutations are enumerated by Baxter numbers B_n, whose formula was found by Chung, Graham, Hoggatt, and Kleiman [9] as

$$B_n = \sum_{k=1}^{n} \frac{2}{n(n+1)^2} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}. \tag{1.1}$$

A combinatorial proof of (1.1) was later given by Viennot [39]. For convenience, let $B_0 = 1$. The first few terms of $\{B_n\}_{n\geqslant 0}$ are 1, 1, 2, 6, 22, 92, 422, 2074, 10754, 58202, see Sloane [35, A001181].

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Many of algebraic and combinatorial objects have been found to be enumerated by the Baxter numbers since they were introduced, such as the bases for subalgebras of the Malvenuto-Reutenauer Hopf algebra [31], the pairs of twin binary trees on n nodes [22, 23], the diagonal rectangulations of an $n \times n$ grid [26], and the bases of the Baxter-Cambrian Hopf algebra [5]. Besides, many bijections related to the Baxter permutations and various objects counted by the Baxter numbers were given, including alternating Baxter permutations [10], non-intersecting paths [21], simple walks in Weyl chambers [11], and Baxter tree-like tableaux [1].

In this paper, we mainly study some important inequalities, or, more precisely, the log-behavior and *higher order Turán inequalities* associated with the Baxter numbers. Recall that a sequence $\{a_n\}_{n\geqslant 0}$ of real numbers is said to satisfy the *Turán inequalities* or to be *log-concave* if

$$a_n^2 - a_{n-1}a_{n+1} \geqslant 0 \tag{1.2}$$

for every $n \geqslant 1$. It is said to be *log-convex* if $a_n^2 - a_{n-1}a_{n+1} \leqslant 0$ for every $n \geqslant 1$. A log-convex sequence $\{a_n\}_{n\geqslant 0}$ is called *log-balanced* if $\{a_n/n!\}_{n\geqslant 0}$ is log-concave, see Došlić [17]. Clearly, for a log-balanced sequence $\{a_n\}_{n\geqslant 0}$, there holds $1 \leqslant a_{n-1}a_{n+1}/a_n^2 \leqslant 1 + 1/n$.

The Turán inequalities (1.2) are also called Newton's inequalities [13, 12, 29]. A real sequence $\{a_n\}_{n\geqslant 0}$ is said to satisfy the *higher order Turán inequalities* or the cubic Newton inequalities if for all $n \geqslant 1$,

$$4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_n a_{n+2}) - (a_n a_{n+1} - a_{n-1}a_{n+2})^2 \geqslant 0, \tag{1.3}$$

see [15, 33, 29]. The Turán inequalities and the higher order Turán inequalities are related to the Laguerre-Pólya class of real entire functions, see [38, 15]. A real entire function

$$\psi(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

is said to belong to the Laguerre-Pólya class, denoted by $\psi \in \mathcal{L}\text{-}\mathcal{P}$, if

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} (1 + x/x_k) e^{-x/x_k},$$

where c, β, x_k are real, $\alpha \geqslant 0$, m is a nonnegative integer and $\sum x_k^{-2} < \infty$. If a real entire function $\psi \in \mathcal{L}\text{-}\mathcal{P}$, then its Maclaurin coefficients satisfy (1.2), see [12, 14]. Moreover, Dimitrov [15] proved that the higher order Turán inequalities (1.3), an extension of (1.2), is also a necessary condition for $\psi \in \mathcal{L}\text{-}\mathcal{P}$.

The $\mathcal{L}\text{-}\mathcal{P}$ class has close relation with the Riemann hypothesis. Let ζ and Γ denote the Riemann zeta-function and the gamma-function, respectively.

The Riemann ξ-function is defined by

$$\xi(iz) = \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\pi^{-\frac{z}{2}-\frac{1}{4}}\Gamma\left(\frac{z}{2}+\frac{1}{4}\right)\zeta\left(z+\frac{1}{2}\right),$$

see Boas [4]. It is well known that the Riemann ξ-function is an entire function of order 1 and can be rewritten as

$$\frac{1}{8}\,\xi\Big(\frac{x}{2}\Big) = \sum_{k=0}^{\infty}(-1)^k\,\hat{b}_k\,\frac{x^{2k}}{(2k)!}, \tag{1.4}$$

where

$$\hat{b}_k = \int_0^{\infty} t^{2t}\Phi(t)dt \quad \text{and} \quad \Phi(t) = \sum_{n=0}^{\infty}(2n^4\pi^2e^{9t} - 3n^2\pi e^{5t})\exp(-n^2\pi e^{4t}),$$

see Pólya [30]. Set $z = -x^2$ in (1.4). Then one obtains an entire function of order 1/2, denoted by $\xi_1(z)$, that is,

$$\xi_1(z) = \sum_{k=0}^{\infty}\frac{k!}{(2k)!}\hat{b}_k\frac{z^k}{k!}.$$

So the Riemann hypothesis holds if and only if $\xi_1(z) \in \mathcal{L}\text{-}\mathcal{P}$. See [13, 15, 8] for more details.

For the deep relation stated above, the higher order Turán inequalities have received much attention. Many combinatorial sequences have been proved to satisfy (1.3). By using the Hardy-Ramanujan-Rademacher formula, Chen, Jia, and Wang [8] proved that the partition function $p(n)$ satisfies the higher order Turán inequalities for $n \geqslant 95$, and hence confirm a conjecture of Chen [6]. Wang [40] gave a sufficient condition to (1.3) and obtained some higher order Turán inequalities for combinatorial sequences $\{b_n/n!\}_{n\geqslant 0}$ where b_n are the Motzkin numbers, the Fine numbers, the Franel numbers of order 3 and the Domb numbers. Hou and Li [25] presented a different sufficient condition on the higher order Turán inequalities for $n \geqslant N$ and also developed an algorithm to find the number N.

The objective of this paper is to study the log-behavior and the higher order Turán inequalities associated with the Baxter numbers. It should be mentioned that the log-convexity of the sequence $\{B_n\}_{n\geqslant 0}$ was first established by Došlić and Veljan [20, Theorem 4.18] in 2002 (noting the received date) with a *calculus method* developed by the two authors. See [19] for more applications of the calculus method. Moreover, Došlić [17, Section 5] first showed the log-balancedness of the sequence $\{B_n\}_{n\geqslant 0}$ by using a robust analytical method developed by the author, which clearly implies the log-concavity of the sequence $\{B_n/n!\}_{n\geqslant 0}$.

As noted by Liu and Wang [27, P. 459], motivated by the results of [18, 20, 19, 17] and other literature, they developed an algebraic approach for dealing with the log-convexity of sequences $\{z_n\}_{n\geqslant 0}$ satisfying a three-term recurrence, such as $\mathfrak{a}_n z_{n+1} = \mathfrak{b}_n z_n \pm \mathfrak{c}_n z_{n-1}$.

By applying the result given in [27, Theorem 3.1] and the following recurrence relation

$$(n+3)(n+4)B_{n+1} = (7n^2+21n+12)B_n + 8n(n-1)B_{n-1}, \quad n \geqslant 1, \tag{1.5}$$

which is due to Richard L. Ollerton as was recorded at OEIS [35], one can also obtain the log-convexity of the sequence $\{B_n\}_{n\geqslant 0}$. Since B_n is a sum of hypergeometric terms on k, it is natural to prove (1.5) using Zeilberger's algorithm [42]. The detailed proof is omitted here.

In this paper, we prove the log-concavity (or, equivalently, the Turán inequalities) of the sequences $\{B_{n+1}/B_n\}_{n\geqslant 0}$ and $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$ in Theorems 2.1 and 3.1, respectively. Monotonicity of the sequence $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$ is also obtained. At last, in Section 4, we prove that the sequences $\{B_n/n!\}_{n\geqslant 2}$ and $\{B_{n+1}B_n^{-1}/n!\}_{n\geqslant 2}$ satisfy the higher order Turán inequalities. Most of our proofs depend on certain bounds for B_{n+1}/B_n.

2. Log-concavity of $\{B_{n+1}/B_n\}_{n\geqslant 0}$

The aim of this section is to prove the log-concavity of the ratio sequence $\{B_{n+1}/B_n\}_{n\geqslant 0}$.

Theorem 2.1. *The sequence $\{B_{n+1}/B_n\}_{n\geqslant 0}$ is strictly log-concave. That is,*

$$\left(\frac{B_{n+1}}{B_n}\right)^2 > \frac{B_n}{B_{n-1}} \cdot \frac{B_{n+2}}{B_{n+1}}, \quad n \geqslant 1. \tag{2.1}$$

In order to prove Theorem 2.1, we need a set of bounds for B_{n+1}/B_n. Based on the asymptotic expansions of the P-recursive sequences given by Birkhoff and Trjitzinsky [3] and developed by Wimp and Zeilberger [41], Hou and Zhang [24] proved a criterion for determining whether a P-recursive sequence $\{a_n\}_{n\geqslant N}$ is r-log-convex for sufficiently large N. Their algorithm also gives bounds of the ratio a_{n+1}/a_n for $n \geqslant N$.

The bounds we proved in Lemma 2.2 are initially found by applying the algorithm of Hou and Zhang [24]. Then we adjust some coefficients in the bounds such that the computation in the proofs of our main results are more concise.

Lemma 2.2. *Let $r_n = B_{n+1}/B_n$. Then for $n \geqslant 1$, it holds that*

$$\hat{u}(n) < r_n < \hat{v}(n), \tag{2.2}$$

where

$$\hat{u}(n) = 8 - \frac{32}{n} + \frac{416}{3n^2} - \frac{15904}{27n^3} + \frac{2454}{n^4} - \frac{10450}{n^5},$$
$$\hat{v}(n) = 8 - \frac{32}{n} + \frac{416}{3n^2} - \frac{15904}{27n^3} + \frac{2463}{n^4}.$$

PROOF. Note that $\hat{u}(n) < 0$ for $1 \leqslant n \leqslant 4$, and

$$\hat{u}(n) = \frac{2(n-5)(108n^4 + 108n^3 + 2412n^2 + 4108n + 53669) + 254540}{27n^5} > 0$$

for $n \geqslant 5$. Thus, we shall use mathematical induction to prove (2.2) for $n \geqslant 5$. With the help of a computer, it is easy to check that the values of $\{r_n - \hat{u}(n)\}_{n=1}^5$ are

$$\frac{228754}{27}, \frac{96401}{432}, \frac{18553}{729}, \frac{811717}{152064}, \frac{1218857}{776250},$$

which are all positive, and the values of $\{r_n - \hat{v}(n)\}_{n=1}^5$ are

$$-\frac{53639}{27}, -\frac{44917}{432}, -\frac{12878}{729}, -\frac{372727}{76032}, -\frac{1388101}{776250},$$

which are all negative. Therefore, (2.2) holds for $1 \leqslant n \leqslant 5$.

Assume that (2.2) is valid for $n-1 \geqslant 5$. We aim to prove it for $n \geqslant 6$. For this purpose, note that by (1.5) it holds that

$$r_n = \frac{b_n}{a_n} + \frac{c_n}{a_n \cdot r_{n-1}}, \tag{2.3}$$

where

$$a_n = (n+3)(n+4),$$
$$b_n = 7n^2 + 21n + 12,$$
$$c_n = 8n(n-1).$$

Thus, by (2.3) and the inductive hypothesis, we have that

$$r_n - \hat{u}(n) > \frac{b_n}{a_n} + \frac{c_n}{a_n \cdot \hat{v}(n-1)} - \hat{u}(n) = \frac{f_1}{27n^5(n+3)(n+4)f_2},$$

where

$$\begin{aligned} f_1 &= 43497n^7 - 5809365n^6 + 241492004n^5 - 1235704128n^4 \\ &\quad + 2920346230n^3 - 5037170424n^2 + 12024249066n + 295339948200, \\ f_2 &= 216n^4 - 1728n^3 + 7632n^2 - 26848n + 87229. \end{aligned}$$

We first show that $f_1 > 0$. For this purpose, set $f_1 = f_{11} + f_{12}$ where

$$\begin{aligned} f_{11} &= 43497n^7 - 5809365n^6 + 241492004n^5 - 1235704128n^4 \\ &= n^4[(n-6)(43497n^2 - 5548383n + 208201706) + 13506108)], \end{aligned}$$

and

$$
\begin{aligned}
f_{12} &= 2920346230n^3 - 5037170424n^2 + 12024249066n + 295339948200 \\
&= 2920346230(n-2)n^2 + 803522036n^2 + 12024249066n + 295339948200.
\end{aligned}
$$

Clearly, $f_{11} > 0$, $f_{12} > 0$ for $n \geqslant 6$. Thus $f_1 = f_{11} + f_{12} > 0$ for $n \geqslant 6$. Notice that

$$f_2 = 72n(n-6)(3n^2 - 6n + 70) + 3392n + 87229 > 0, \quad n \geqslant 6.$$

It follows that

$$r_n - \hat{u}(n) > \frac{f_1}{27n^5(n+3)(n+4)f_2} > 0$$

for $n \geqslant 6$.

It remains to prove that $r_n - \hat{v}(n) < 0$ for $n \geqslant 6$. By the inductive hypothesis, $r_{n-1} > \hat{u}(n-1) > 0$ for $n-1 \geqslant 5$. Therefore by (2.3) it holds that

$$r_n - \hat{v}(n) < \frac{b_n}{a_n} + \frac{c_n}{a_n \cdot \hat{u}(n-1)} - \hat{v}(n) = -\frac{f_3}{27n^4(n+3)(n+4)f_4},$$

where

$$
\begin{aligned}
f_3 = (n-6)(&1215n^6 + 29537847n^5 - 18982066n^4 + 397104580n^3 \\
&+ 1399763449n^2 + 10254978749n + 56257059186) + 190254876300,
\end{aligned}
$$

and

$$f_4 = (n-6)(108n^4 - 324n^3 + 2736n^2 - 824n + 51973) + 127270.$$

It is clear that $f_3 > 0$, $f_4 > 0$ for $n \geqslant 6$. Hence we obtain that

$$r_n - \hat{v}(n) < -\frac{f_3}{27n^4(n+3)(n+4)f_4} < 0$$

for $n \geqslant 6$. This completes the proof. ◻

We are now ready to give a proof of Theorem 2.1.

Proof of Theorem 2.1. Recall that $r_n = B_{n+1}/B_n > 0$ for $n \geqslant 0$. Thus, the inequality (2.1) is equivalent to

$$\frac{r_n^2}{r_{n-1}r_{n+1}} > 1 \tag{2.4}$$

for $n \geqslant 1$. By Lemma 2.2,

$$0 < \hat{u}(n) < r_n, \quad n \geqslant 5,$$

and

$$0 < r_n < \hat{v}(n), \quad n \geqslant 1.$$

Hence we obtain that

$$\frac{r_n^2}{r_{n-1}r_{n+1}} > \frac{(\hat{u}(n))^2}{\hat{v}(n-1)\cdot\hat{v}(n+1)}, \quad n \geqslant 5.$$

For $1 \leqslant n \leqslant 4$, it is easy to check that the values of $\{r_n^2/(r_{n-1}r_{n+1})-1\}_{n=1}^4$ are

$$\frac{1}{3}, \frac{5}{22}, \frac{89}{1242}, \frac{11167}{280841},$$

which means that (2.4) is true for $1 \leqslant n \leqslant 4$. It is sufficient to prove that

$$\frac{(\hat{u}(n))^2}{\hat{v}(n-1)\cdot\hat{v}(n+1)} > 1, \qquad n \geqslant 5. \tag{2.5}$$

By Lemma 2.2, we get that

$$\frac{(\hat{u}(n))^2}{\hat{v}(n-1)\cdot\hat{v}(n+1)} - 1 = \frac{f_5}{n^{10}f_2f_6},$$

where $f_2 = 216n^4 - 1728n^3 + 7632n^2 - 26848n + 87229 > 0$ for $n \geqslant 6$, which has been shown in the proof of Lemma 2.2, and

$$\begin{aligned} f_5 = {} & (n-23)(373248n^{14} + 2881008n^{13} - 2163024n^{12} + 24719472n^{11} + 18013968n^{10} \\ & + 4560284715n^9 + 82298410437n^8 + 1926154728903n^7 + 44427203227329n^6 \\ & + 1021582551830975n^5 + 23496190881398905n^4 + 540412815701543079n^3 \\ & + 12429494906472774753n^2 + 285878382543804079083n \\ & + 6575202798470104429509) + 151229664364892010501207, \\ f_6 = {} & 72(n-1)(n-2)(3n^2+9n+55) + 440n + 45773, \end{aligned}$$

which are easy to verify with the aid of a computer. Clearly, $f_5 > 0$ for $n \geqslant 23$ and $f_6 > 0$ for $n \geqslant 0$. Combining the fact that $f_2 > 0$ for $n \geqslant 6$, we arrive at the result that $(\hat{u}(n))^2/(\hat{v}(n-1)\cdot\hat{v}(n+1)) > 1$ for $n \geqslant 23$, which implies that (2.4) holds for $n \geqslant 23$.

Note that, although the inequality (2.5) is not true for $5 \leqslant n \leqslant 22$, it does not affect the validity of (2.4). It is easy to check that (2.4) holds for $1 \leqslant n \leqslant 22$ by using a computer. Thus, (2.4) is proved for all $n \geqslant 1$. This completes the proof. $\square$

3. Log-concavity and monotonicity of $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$

Motivated by Firoozbakht's conjecture on the strictly decreasing property of the sequence $\{\sqrt[n]{p_n}\}_{n\geqslant 1}$, where p_n is the n-th prime number (see [32, p.185]), Sun [36, 37] studied the monotonicity of many combinatorial sequences of this type, which has received much attention.

The goal of this section is to prove the log-concavity and monotonicity of the sequence $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$. The first main result of this section is as follows.

THEOREM 3.1. *The sequence* $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$ *is strictly log-concave. That is,*

$$\left(\sqrt[n]{B_n}\right)^2 > \sqrt[n-1]{B_{n-1}} \cdot \sqrt[n+1]{B_{n+1}}, \qquad n \geqslant 2. \tag{3.1}$$

To prove Theorem 3.1, we shall apply a criterion given by Chen, Guo, and Wang [7].

THEOREM 3.2 ([7, Theorem 3.1]). *Let* $\{S_n\}_{n\geqslant 0}$ *be a positive sequence. If the sequence* $\{S_{n+1}/S_n\}_{n\geqslant N}$ *is log-concave and*

$$\frac{\sqrt[N+1]{S_{N+1}}}{\sqrt[N]{S_N}} > \frac{\sqrt[N+2]{S_{N+2}}}{\sqrt[N+1]{S_{N+1}}} \tag{3.2}$$

for some positive integer N*, then the sequence* $\{\sqrt[n]{S_n}\}_{n\geqslant N}$ *is strictly log-concave.*

PROOF OF THEOREM 3.1. By Theorem 2.1, the sequence $\{B_{n+1}/B_n\}_{n\geqslant 0}$ is strictly log-concave. Set $N = 1$. Since

$$\frac{\sqrt{B_2}}{B_1} - \frac{\sqrt[3]{B_3}}{\sqrt{B_2}} = \frac{2-\sqrt[3]{6}}{\sqrt{2}} > 0,$$

by Theorem 3.2, it follows that the sequence $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$ is strictly log-concave as desired. □

By using the log-concavity of $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$, we obtain the monotonicity of $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$.

THEOREM 3.3. *The sequence* $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$ *is strictly increasing.*

PROOF. Recall that for a real sequence $\{S_n\}_{n\geqslant 0}$ with positive numbers, it was shown that

$$\liminf_{n\to\infty} \frac{S_{n+1}}{S_n} \leqslant \liminf_{n\to\infty} \sqrt[n]{S_n}, \quad \text{and} \quad \limsup_{n\to\infty} \sqrt[n]{S_n} \leqslant \limsup_{n\to\infty} \frac{S_{n+1}}{S_n},$$

see Rudin [34, Subsection 3.37]. The above inequalities imply a well-known criterion, that is, if $S_{n+1}/S_n \to C$ as n goes to infinity, then $\sqrt[n]{S_n} \to C$ as n tends to infinity, where C is a finite real number.

By [20, Theorem 4.18], we have

$$\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = 8,$$

which can also be obtained by using Lemma 2.2 together with the well-known Squeeze theorem. Therefore

$$\lim_{n\to\infty} \sqrt[n]{B_n} = 8.$$

Consequently,

$$\lim_{n\to\infty} \frac{\sqrt[n+1]{B_{n+1}}}{\sqrt[n]{B_n}} = 1.$$

By Theorem 3.1, we see that the sequence $\{\sqrt[n+1]{B_{n+1}}/\sqrt[n]{B_n}\}_{n\geqslant 1}$ is strictly decreasing. Hence we obtain that

$$\frac{\sqrt[n+1]{B_{n+1}}}{\sqrt[n]{B_n}} > 1$$

for all $n \geqslant 1$. This completes the proof. □

4. Higher order Turán inequalities related to B_n

This section is devoted to the study of higher order Turán inequalities related to the Baxter numbers. Numerical computation suggests that the sequences $\{B_n\}_{n\geqslant 0}$, $\{B_{n+1}/B_n\}_{n\geqslant 0}$, and $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$ do not satisfy the higher order Turán inequalities (1.3), however the sequences $\{B_n/n!\}_{n\geqslant 2}$, $\{B_{n+1}B_n^{-1}/n!\}_{n\geqslant 2}$, and $\{\sqrt[n]{B_n}/n!\}_{n\geqslant 2}$ do. The first main result of this section is as follows.

Theorem 4.1. *The sequence $\{B_n/n!\}_{n\geqslant 2}$ satisfies the higher order Turán inequalities. That is,* (1.3) *holds for $a_n = B_n/n!$. More precisely, for $n \geqslant 3$,*

$$\begin{aligned}4\left(\frac{B_n^2}{n!^2}-\frac{B_{n-1}}{(n-1)!}\frac{B_{n+1}}{(n+1)!}\right)\left(\frac{B_{n+1}^2}{(n+1)!^2}-\frac{B_n}{n!}\frac{B_{n+2}}{(n+2)!}\right)\\ > \left(\frac{B_n}{n!}\frac{B_{n+1}}{(n+1)!}-\frac{B_{n-1}}{(n-1)!}\frac{B_{n+2}}{(n+2)!}\right)^2.\end{aligned} \tag{4.1}$$

As mentioned in [6] and [8], for $n = 1$, the polynomial in (1.3) reduces to

$$\begin{aligned}I(a_0,a_1,a_2,a_3) &= 4(a_1^2-a_0a_2)(a_2^2-a_1a_3)-(a_1a_2-a_0a_3)^2\\ &= 3a_1^2a_2^2-4a_1^3a_3-4a_0a_2^3-a_0^2a_3^2+6a_0a_1a_2a_3\end{aligned}$$

which is clearly an invariant of the cubic binary form

$$a_3x^3+3a_2x^2y+3a_1xy^2+a_0y^3.$$

Notice that $27I(a_0,a_1,a_2,a_3)$ is the discriminant of the cubic polynomial $a_3x^3+3a_2x^2+3a_1x+a_0$. For a real cubic polynomial, the discriminant is positive if and only if the three zeros are real and distinct [28]. See also [16]. Thus, as a consequence of Theorem 4.1, we get the following result.

Theorem 4.2. *For any integer $n \geqslant 3$, the polynomials*

$$P_n(x) = \frac{B_{n+3}}{(n+3)!}x^3+\frac{3B_{n+2}}{(n+2)!}x^2+\frac{3B_{n+1}}{(n+1)!}x+\frac{B_n}{n!}$$

have only real and distinct zeros.

We shall apply a criterion given by Hou and Li [25] to prove Theorem 4.1.

THEOREM 4.3 ([25, Theorem 5.2]). *Let $\{a_n\}_{n\geqslant 0}$ be a real sequence with positive numbers. Let*

$$d(x, y) = 4(1-x)(1-y) - (1-xy)^2.$$

If there exist an integer N, and two functions $g(n)$ and $h(n)$ such that for all $n \geqslant N$,

(i) $g(n) < a_{n-1}a_{n+1}/a_n^2 < h(n)$;
(ii) $d(g(n), g(n+1)) > 0$, $d(g(n), h(n+1)) > 0$, $d(h(n), g(n+1)) > 0$, $d(h(n), h(n+1)) > 0$,

then $\{a_n\}_{n\geqslant N}$ satisfies the higher order Turán inequalities.

In order to use Theorem 4.3, the first step is to find appropriate bounds for $B_{n-1}B_{n+1}/B_n^2$. It should be mentioned that Theorem 4.3 is applicable to any positive real sequences $\{a_n\}_{n\geqslant 0}$. In particular, for a P-recursive sequence $\{a_n\}_{n\geqslant 0}$, Hou and Li [25] gave an algorithm, called the "HT Algorithm", to find the number N and the bounds $g(n)$ and $h(n)$. This algorithm has been implemented in a `Mathematica` package `P-rec.m` by Hou. For example, the Baxter numbers satisfy the recurrence (1.5) with initial values $B_0 = 1, B_1 = 1, B_2 = 2$. Running the command

$$\mathtt{HT}[(\mathtt{n}+3)(\mathtt{n}+4)\mathtt{N} - (7\mathtt{n}^2+21\mathtt{n}+12) - 8\mathtt{n}(\mathtt{n}-1)\mathtt{N}^{-1}, \mathtt{n}, \mathtt{N}, \{1,1,2\}, 4]$$

outputs the result $\left\{\left\{1+\frac{3}{n^2}, 1+\frac{5}{n^2}\right\}, 753\right\}$, which means that

$$1 + \frac{3}{n^2} < \frac{B_{n-1}B_{n+1}}{B_n^2} < 1 + \frac{5}{n^2} \tag{4.2}$$

for $n \geqslant 753$. In fact the above bounds are valid for $n \geqslant 11$. To be rigorous, we show a simple proof.

LEMMA 4.4. *The inequalities in* (4.2) *hold for all $n \geqslant 11$.*

PROOF. Let $r_n = B_{n+1}/B_n$. By Lemma 2.2, it is easy to derive that

$$0 < u(n) < r_n < v(n), \quad n \geqslant 5, \tag{4.3}$$

where

$$u(n) = 8 - \frac{32}{n} + \frac{137}{n^2} - \frac{589}{n^3} \qquad \text{and} \qquad v(n) = 8 - \frac{32}{n} + \frac{140}{n^2}. \tag{4.4}$$

As will be seen, the bounds given in (4.3) will make the proof more concise, although they are not sharper than those provided in Lemma 2.2.

It follows from (4.3) that

$$\frac{B_{n-1}B_{n+1}}{B_n^2} = \frac{r_n}{r_{n-1}} < \frac{v(n)}{u(n-1)}$$

for $n \geqslant 6$. Direct computation gives that

$$\frac{v(n)}{u(n-1)} - \left(1 + \frac{5}{n^2}\right) = -\frac{5n^3 - 522n^2 + 673n - 3690}{n^2(8n^3 - 56n^2 + 225n - 766)},$$

which is negative for $n \geqslant 104$. We have verified that $r_n/r_{n-1} < 1 + 5/n^2$ for $1 \leqslant n \leqslant 103$. Hence,

$$\frac{B_{n-1}B_{n+1}}{B_n^2} < 1 + \frac{5}{n^2}, \qquad n \geqslant 1. \tag{4.5}$$

Notice that

$$\frac{B_{n-1}B_{n+1}}{B_n^2} = \frac{r_n}{r_{n-1}} > \frac{u(n)}{v(n-1)}$$

for $n \geqslant 2$. Direct computation shows that

$$\frac{u(n)}{v(n-1)} - \left(1 + \frac{3}{n^2}\right) = \frac{5n^3 - 751n^2 + 775n - 589}{4n^3(2n^2 - 12n + 45)},$$

which is positive for $n \geqslant 150$. It has been checked that $r_n/r_{n-1} > 1 + 3/n^2$ for $11 \leqslant n \leqslant 149$. Therefore,

$$\frac{B_{n-1}B_{n+1}}{B_n^2} > 1 + \frac{3}{n^2}, \qquad n \geqslant 11. \tag{4.6}$$

Combining (4.5) and (4.6) leads to the desired result, namely that (4.2) holds for $n \geqslant 11$. This completes the proof. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $a_n = B_n/n!$ for $n \geqslant 1$. Then, by Lemma 4.4, we have that

$$\frac{n}{n+1}\left(1 + \frac{3}{n^2}\right) < \frac{a_{n-1}a_{n+1}}{a_n^2} < \frac{n}{n+1}\left(1 + \frac{5}{n^2}\right)$$

for $n \geqslant 11$. Set

$$g(n) = \frac{n}{n+1}\left(1 + \frac{3}{n^2}\right) \qquad \text{and} \qquad h(n) = \frac{n}{n+1}\left(1 + \frac{5}{n^2}\right).$$

Thus, the conditions in (i) of Theorem 4.3 are satisfied.

It remains to verify the four inequalities in (ii) of Theorem 4.3. By a direct computation we have that

$$\begin{aligned} d(g(n), g(n+1)) &= \frac{4(n^5 - 6n^4 + 9n^3 + 4n^2 - 12n - 36)}{n^2(n+1)^4(n+2)^2} \\ &= \frac{4[(n-4)(n^4 - 2n^3 + n^2 + 8n + 20) + 44]}{n^2(n+1)^4(n+2)^2}, \end{aligned}$$

which is positive for $n \geqslant 4$. Similarly, we obtain that

$$\begin{aligned} d(g(n), h(n+1)) &= \frac{4(n^5 - 11n^4 + 25n^3 + 6n^2 - 12n - 81)}{n^2(n+1)^4(n+2)^2} \\ &= \frac{4[(n-8)(n^4 - 3n^3 + n^2 + 14n + 100) + 719]}{n^2(n+1)^4(n+2)^2}, \end{aligned}$$

which is positive for $n \geqslant 8$,

$$d(h(n), g(n+1)) = \frac{4(n^5 - 9n^4 + 11n^3 - 20n^2 - 60n - 100)}{n^2(n+1)^4(n+2)^2}$$
$$= \frac{4[(n-9)(n^4 + 11n^2 + 79n + 651) + 5759]}{n^2(n+1)^4(n+2)^2},$$

which is positive for $n \geqslant 9$, and

$$d(h(n), h(n+1)) = \frac{4(n^5 - 12n^4 + 43n^3 - 24n^2 - 80n - 225)}{n^2(n+1)^4(n+2)^2}$$
$$= \frac{4[(n-7)(n^4 - 5n^3 + 8n^2 + 32n + 144) + 783]}{n^2(n+1)^4(n+2)^2},$$

which is positive for $n \geqslant 7$. Consequently, the conditions in (ii) of Theorem 4.3 hold for $n \geqslant 11$.

It follows from Theorem 4.3 that the sequence $\{B_n/n!\}_{n\geqslant 11}$ satisfies the higher order Turán inequalities. For $3 \leqslant n \leqslant 10$, it is easy to check that inequality (1.3) still holds for $a_n = B_n/n!$. Thus, we have that (4.1) holds true for $n \geqslant 3$. This completes the proof. $\square$

Note that Theorem 4.1 and Theorem 4.5 stated below can also be proved by using a criterion given by Wang [40, Theorem 2.1], in which case one should examine the signs of eight values of four functions of n.

The second main result of this section is as follows.

THEOREM 4.5. *Let B_n be defined by* (1.1), *and $r_n = B_{n+1}/B_n$. Then the sequence $\{r_n/n!\}_{n\geqslant 2}$ satisfies the higher order Turán inequalities. That is, for $n \geqslant 3$,*

$$(4.7)\qquad 4\left(\frac{r_n^2}{n!^2} - \frac{r_{n-1}}{(n-1)!}\frac{r_{n+1}}{(n+1)!}\right)\left(\frac{r_{n+1}^2}{(n+1)!^2} - \frac{r_n}{n!}\frac{r_{n+2}}{(n+2)!}\right)$$
$$> \left(\frac{r_n}{n!}\frac{r_{n+1}}{(n+1)!} - \frac{r_{n-1}}{(n-1)!}\frac{r_{n+2}}{(n+2)!}\right)^2.$$

In order to prove Theorem 4.5, we need the following lemma.

LEMMA 4.6. *Let $r_n = B_{n+1}/B_n$. Then for $n \geqslant 2$, we have that*

$$(4.8)\qquad \frac{n^2(n^2+2n+4)}{(n^2+5)(n+1)^2} < \frac{r_{n-1}r_{n+1}}{r_n^2} < \frac{n^2(n^2+2n+6)}{(n^2+3)(n+1)^2}.$$

PROOF. It is easy to check that (4.8) is valid for $2 \leqslant n \leqslant 10$. For $n \geqslant 11$, by Lemma 4.4, we have that

$$1 + \frac{3}{(n+1)^2} < \frac{r_{n+1}}{r_n} < 1 + \frac{5}{(n+1)^2},$$

and

$$\frac{n^2}{n^2+5} < \frac{r_{n-1}}{r_n} < \frac{n^2}{n^2+3}.$$

It follows that (4.8) holds true for $n \geqslant 11$. This completes the proof. □

We proceed to prove Theorem 4.5.

Proof of Theorem 4.5. Let $a_n = r_n/n!$ where $r_n = B_{n+1}/B_n$ for $n \geqslant 1$. By Lemma 4.6, we have that

$$\frac{n}{n+1}\cdot\frac{n^2(n^2+2n+4)}{(n^2+5)(n+1)^2} < \frac{a_{n-1}a_{n+1}}{a_n^2} < \frac{n}{n+1}\cdot\frac{n^2(n^2+2n+6)}{(n^2+3)(n+1)^2}$$

for $n \geqslant 2$. Set

$$g(n) = \frac{n^3(n^2+2n+4)}{(n^2+5)(n+1)^3} \qquad \text{and} \qquad h(n) = \frac{n^3(n^2+2n+6)}{(n^2+3)(n+1)^3}.$$

Therefore, the conditions in (i) of Theorem 4.3 are satisfied.

It remains to verify the conditions in (ii) of Theorem 4.3. By a direct computation, we obtain that

$$\begin{aligned} &d(g(n), g(n+1)) \\ &\quad = \frac{4}{(n^2+5)^2(n+1)^3(n^2+2n+6)^2(n+2)^6}\big(n^{14}+20n^{13}+217n^{12} \\ &\qquad + 1569n^{11}+8343n^{10}+34045n^9+109634n^8+282077n^7 \\ &\qquad + 579185n^6+936561n^5+1156632n^4+1033910n^3+617220n^2 \\ &\qquad +217400n+34800\big), \end{aligned}$$

$$\begin{aligned} &d(g(n), h(n+1)) \\ &\quad = \frac{4}{(n^2+5)^2(n+1)^3(n^2+2n+4)(n+2)^6}\big(n^{12}+6n^{11}+33n^{10} \\ &\qquad + 143n^9+669n^8+2894n^7+9933n^6+25744n^5+47130n^4 \\ &\qquad +56675n^3+41650n^2+16900n+3000\big), \end{aligned}$$

$$\begin{aligned} &d(h(n), g(n+1)) \\ &\quad = \frac{4}{(n^2+3)^2(n+1)^3(n^2+2n+6)(n+2)^6}\big(n^{12}+10n^{11}+67n^{10} \\ &\qquad + 385n^9+1867n^8+6508n^7+16267n^6+31156n^5+46264n^4 \\ &\qquad +49173n^3+33126n^2+12348n+2088\big), \end{aligned}$$

and

$$\begin{aligned} &d(h(n), h(n+1)) \\ &\quad = \frac{4}{(n^2+3)^2(n+1)^3(n^2+2n+4)^2(n+2)^6} \big(n^{14} + 8n^{13} + 61n^{12} \\ &\qquad + 307n^{11} + 1295n^{10} + 4481n^9 + 12610n^8 + 29313n^7 + 57181n^6 \\ &\qquad + 93487n^5 + 122784n^4 + 118356n^3 + 74376n^2 + 26496n + 4320\big), \end{aligned}$$

which are all positive for $n \geqslant 0$. Thus, the conditions in (ii) of Theorem 4.3 are satisfied for $n \geqslant 2$.

By Theorem 4.3, the sequence $\{r_n/n!\}_{n\geqslant 2}$ satisfies the higher order Turán inequalities (4.7). This completes the proof. $\square$

At last, we propose a conjecture.

Conjecture 4.7. *The sequence $\{\sqrt[n]{B_n}/n!\}_{n\geqslant 2}$ satisfies the higher order Turán inequalities.*

If one can find appropriate bounds for $\sqrt[n+1]{B_{n+1}}/\sqrt[n]{B_n}$ in a form of rational functions of n, then Conjecture 4.7 will be solved by applying Theorem 4.3 or Wang's criterion [40, Theorem 2.1].

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References

[1] J.-C. Aval, A. Boussicault, M. Bouvel, O. Guibert and M. Silimbani, *Baxter tree-like tableaux*, arXiv.2108.06212v1, 2021.

[2] G. Baxter, *On fixed points of the composite of commuting functions*, Proc. Amer. Math. Soc. **15** (1964), 851–855.

[3] G. D. Birkhoff and W.J. Trjitzinsky, *Analytic theory of singular difference equations*, Acta Math. **60** (1933), 1–89.

[4] R. P. Boas Jr., Entire functions, Academic Press Inc., New York, 1954.

[5] G. Chatel and V. Pilaud, *Cambrian Hopf algebras*, Adv. Math. **311** (2017), 598–633.

[6] W. Y. C. Chen, *The spt-function of Andrews*, in: Surveys in Combinatorics 2017, eds. Claesson, A., Dukes, M., Kitaev, S., Manlove, D., Meeks, K., London Math. Soc. Lecture Note Ser. **440**, Cambridge Univ. Press, Cambridge, 2017, 141–203.

[7] W. Y. C. Chen, J. J. F. Guo and L. X. W. Wang, *Infinitely log-monotonic combinatorial sequences*, Adv. in Appl. Math. **52** (2014), 99–120.

[8] W. Y. C. Chen, D. X. Q. Jia and L. X. W. Wang, *Higher order Turán inequalities for the partition function*, Trans. Amer. Math. Soc. **372** (2019), 2143–2165.

[9] F. R. K. Chung, R. L. Graham, V. E. Hoggatt Jr. and M. Kleiman, *The number of Baxter permutations*, J. Combin. Theory Ser. A **24** (1978), 382–394.

[10] R. Cori, S. Dulucq and G. Viennot, *Shuffle of Parenthesis Systems and Baxter Permutations*, J. Combin. Theory Ser. A **43** (1986), 1–22.
[11] J. Courtiel, E. Fusy, M. Lepoutre and M. Mishna, *Bijections for Weyl Chamber walks ending on an axis, using arc diagrams and Schnyder woods*, European J. Combin. **69** (2018), 126–142.
[12] T. Craven and G. Csordas, *Jensen polynomials and the Turán and Laguerre inequalities*, Pacific J. Math. **136** (1989), 241–260.
[13] G. Csordas, T.S. Norfolk and R.S. Varga, *The Riemann hypothesis and the Turán inequalities*, Trans. Amer. Math. Soc. **296** (1986), 521–541.
[14] G. Csordas and R.S. Varga, *Necessary and sufficient conditions and the Riemann hypothesis*, Adv. in Appl. Math. **11** (1990), 328–357.
[15] D.K. Dimitrov, *Higher order Turán inequalities*, Proc. Amer. Math. Soc. **126** (1998), 2033–2037.
[16] D. K. Dimitrov and F.R. Lucas, *Higher order Turán inequalities for the Riemann ξ-function*, Proc. Amer. Math. Soc. **139** (2011), 1013–1022.
[17] T. Došlić, *Log-balanced combinatorial sequences*, Intl. J. Math. Math. Sci. **4** (2005), 507–522.
[18] T. Došlić, D. Svrtan and D. Veljan, *Enumerative aspects of secondary structures*, Discrete Math. **285** (2004), 67–82.
[19] T. Došlić and D. Veljan, *Calculus proofs of some combinatorial inequalities*, Math. Inequal. Appl. **6** (2003), 197–209.
[20] T. Došlić and D. Veljan, *Logarithmic behavior of some combinatorial sequences*, Discrete Math. **308** (2008), 2182–2212.
[21] S. Dulucq and O. Guibert, *Baxter permutations*, Discrete Math. **180** (1998), 143–156.
[22] S. Felsner, É. Fusy, M. Noy and D. Orden, *Bijections for Baxter families and related objects*, J. Combin. Theory Ser. A **118** (2011), 993–1020.
[23] S. Giraudo, *Algebraic and combinatorial structures on pairs of twin binary trees*, J. Algebra **360** (2012), 115–157.
[24] Q.-H. Hou and Z.-R. Zhang, *Asymptotic r-log-convexity and P-recursive sequences*, J. Symbolic Comput. **93** (2019), 21–33.
[25] Q.-H. Hou and G. Li, *Log-concavity of P-recursive sequences*, J. Symbolic Comput. **107** (2021), 251–268.
[26] S. Law and N. Reading, *The Hopf algebra of diagonal rectangulations*, J. Combin. Theory Ser. A **119** (2012), 788–824.
[27] L. L. Liu and Y. Wang, *On the log-convexity of combinatorial sequences*, Adv. in Appl. Math. **39** (2007), 453–476.
[28] G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias, Topics in polynomials: extremal problems, inequalities, zeros, World Scientific Publishing Co., Inc., River Edge, NJ, 1994.
[29] C. P. Niculescu, *A new look at Newton's inequalities*, J. Inequal. Pure Appl. Math. **1** (2000), Article 17, 14.
[30] G. Pólya, *Über die algebraisch-funktionentheoretischcen Untersuchungen von J. L. W. V. Jensen*, Kgl. Danske Vid. Sel. Math.-Fys. Medd. **7** (1927), 3–33.
[31] N. Reading, *Lattice congruences, fans and Hopf algebras*, J. Combin. Theory Ser. A **110** (2005), 237–273.
[32] P. Ribenboim, The Little Book of Bigger Primes, Second Edition, Springer-Verlag, New York, 2004.
[33] S. Rosset, *Normalized symmetric functions, Newton's inequalities and a new set of stronger inequalities*, Amer. Math. Monthly **96** (1989), 815–819.
[34] W. Rudin, Principles of mathematical analysis, third edition, McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976.

[35] N. J. A. Sloane, The On-line encyclopedia of integer sequences, `https://oeis.org`.

[36] Z.-W. Sun, *Conjectures involving arithmetical sequences*, in: Number theory–arithmetic in Shangri-La, World Sci., Singapore, 2013, 244–258.

[37] Z.-W. Sun, New conjectures in number theory and combinatorics, Harbin institute of Technology Press, 2021.

[38] G. Szegö, *On an inequality of P. Turán concerning Legendre polynomials*, Bull. Amer. Math. Soc. **54** (1948), 401–405.

[39] G. Viennot, *A bijective proof for the number of Baxter permutations*, in: Troisième Séminaire Lotharingien de Combinatoire, Le Klebach 1981, 28–29.

[40] L. X. W. Wang, *Higher order Turán inequalities for combinatorial sequences*, Adv. in Appl. Math. **110** (2019), 180–196.

[41] J. Wimp and D. Zeilberger, *Resurrecting the asymptotics of linear recurrences*, J. Math. Anal. Appl. **111** (1985), 162–176.

[42] D. Zeilberger, *The method of creative telescoping*, J. Symbolic Comput. **11** (1991), 195–204.

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