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ON THE COEFFICIENTS OF A CLASS OF TERNARY CYCLOTOMIC POLYNOMIALS

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Abstract. A cyclotomic polynomial $\Phi_n(x)$ is said to be flat if its nonzero coefficients involve only ± 1. In this paper, for odd primes $p < q < r$ with $q \equiv 1 \pmod p$ and $9r \equiv \pm 1 \pmod{pq}$, we prove that $\Phi_{pqr}(x)$ is flat if and only if $p = 5$, $q \geq 41$, and $q \equiv 1 \pmod 5$.

1. Introduction

Let

$$\Phi_n(x) = \prod_{\substack{(m,n)=1 \\ 1\leq m\leq n}} (x - \zeta_n^m),$$

where $\zeta_n = e^{2\pi i/n}$. This polynomial is called the nth cyclotomic polynomial. It is a classical and neoteric problem in number theory to investigate the arithmetic properties of cyclotomic polynomials. Since $\deg(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = \phi(n) = \deg \Phi_n(x)$, it follows that $\Phi_n(x)$ is the irreducible polynomial for ζ_n. Also, $\Phi_n(x) \in \mathbb{Z}[x]$ since the coefficients are rational and also are algebraic integers.

The first few cyclotomic polynomials are

$$\Phi_1(x) = x - 1, \quad \Phi_2(x) = x + 1,$$
$$\Phi_3(x) = x^2 + x + 1, \quad \Phi_4(x) = x^2 + 1.$$

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All these have coefficients ±1 and 0; however, this is not true in general. By choosing n with many prime factors one can obtain arbitrarily large coefficients.

Let $a(n,j)$ denote the coefficient of x^j in the nth cyclotomic polynomial $\Phi_n(x)$ and $A(n) = \max\{|a(n,j)| : 0 \le j \le \phi(n)\}$ be the height of $\Phi_n(x)$. Cyclotomic polynomials of unit height are called $flat$. Since

$$\Phi_n(x) = \Phi_{\text{radical}(n)}(x^{n/\text{radical}(n)}) \text{ and } \Phi_{2n}(x) = \pm\Phi_n(-x),$$

we can restrict the study of $A(n)$ to the case $n = pqr\cdots$, where $3 \le p < q < r < \cdots$ are primes. It is clear that

$$a(p,j) = \begin{cases} 1 & \text{if } 0 \le j \le p-1; \\ 0 & \text{otherwise,} \end{cases}$$

(1.1)

$$a(pq,j) = \begin{cases} 1 & \text{if } j = up + vq \text{ for } 0 \le u \le s-1, 0 \le v \le t-1; \\ -1 & \text{if } j = up + vq + 1 \text{ for } 0 \le u \le q-s-1, 0 \le v \le p-t-1; \\ 0 & \text{otherwise,} \end{cases}$$

where s and t are unique positive integers $1 \le s < q$, $1 \le t < p$ such that $pq + 1 = ps + qt$, see e.g. Lam and Leung [8] or Thangadurai [10]. Thus all the prime cyclotomic polynomials $\Phi_p(x)$ and binary cyclotomic polynomials $\Phi_{pq}(x)$ are flat.

The problem of flat ternary cyclotomic polynomials—flat polynomials of the form $\Phi_{pqr}(x)$, where p, q, and r are distinct odd primes—seems tantalizingly accessible but remains open. In 2009, Broadhurst proposed some conjectures about flat ternary cyclotomic polynomials, see [7, Conjecture] or [4, Conjecture 4.4.3]. Let $p < q < r$ be odd primes such that $zr \equiv \pm1 \pmod{pq}$, where z is a positive integer. At present, the flatness of $\Phi_{pqr}(x)$ in special cases $1 \le z \le 8$ has been considered, see [1, 2, 3, 5, 6, 9, 11, 12, 13, 14, 15, 16]. In this paper, we study the flatness of $\Phi_{pqr}(x)$ in the special case when $q \equiv 1 \pmod p$ and $z = 9$, and establish the following result.

THEOREM 1.1. *Let $p < q < r$ be odd primes such that $q \equiv 1 \pmod p$ and $9r \equiv \pm1 \pmod{pq}$. Then $\Phi_{pqr}(x)$ is flat if and only if $p = 5$, $q \ge 41$, and $q \equiv 1 \pmod 5$.*

2. PRELIMINARIES

Before starting our proof of the theorem, the following lemmas will be useful. Let $p < q$ be odd primes such that $q = kp+1$. Then, for $pq+1 = ps+qt$, we have $s = q - k$, $t = 1$. Hence it follows immediately from (1.1) that the following holds.

Lemma 2.1. *Let $p < q$ be odd primes such that $q = kp + 1$. Then*

$$a(pq, j) = \begin{cases} 1 & \text{if } j = up \text{ for some } 0 \le u \le q - k - 1; \\ -1 & \text{if } j = up + vq + 1 \text{ for some } 0 \le u \le k - 1, 0 \le v \le p - 2; \\ 0 & \text{otherwise.} \end{cases}$$

Our main tool is the following technical lemma due to Kaplan [6, Lemma 1], which reduces the computation of the ternary cyclotomic coefficients to that of the binary cyclotomic coefficients.

Lemma 2.2. *Let $p < q < r$ be odd primes. Let $n \ge 0$ be an integer and $f(i)$ be the unique value $0 \le f(i) \le pq - 1$ such that*

$$rf(i) + i \equiv n \pmod{pq}. \tag{2.1}$$

(1) Then

$$\sum_{i=0}^{p-1} a(pq, f(i)) = \sum_{i=q}^{q+p-1} a(pq, f(i)).$$

(2) Put

$$a^*(pq, f(i)) = \begin{cases} a(pq, f(i)) & \text{if } f(i) \le \frac{n}{r}; \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Then

$$a(pqr, n) = \sum_{i=0}^{p-1} a^*(pq, f(i)) - \sum_{i=q}^{q+p-1} a^*(pq, f(i)). \tag{2.3}$$

We also need the periodicity of $A(pqr)$, which implies that the height $A(pqr)$ is determined by $\pm r \pmod{pq}$, see [6].

Lemma 2.3. *Let $3 \le p < q < r$ be primes. Then for any prime $s > q$ such that $s \equiv \pm r \pmod{pq}$, $A(pqr) = A(pqs)$.*

3. Proof of Theorem 1.1

Note that $9r \equiv \pm 1 \pmod{pq}$ implies that $p \ne 3$. In view of Lemma 2.3, it is enough to consider primes $p < q < r$ such that

$$p \ge 5,\ q \equiv 1 \pmod{p}, \text{ and } 9r \equiv 1 \pmod{pq}.$$

The proof will be split into the following three parts according to the value of p.

3.1. *The case when* $p = 5$. (i) By using the PARI/GP system, we have $A(5 \cdot 11 \cdot 269) = A(5 \cdot 31 \cdot 379) = 2$. On noting that $9 \cdot 269 \equiv 1 \pmod{5 \cdot 11}$ and $9 \cdot 379 \equiv 1 \pmod{5 \cdot 31}$, we obtain from Lemma 2.3 that

$$A(5qr) = 2 \text{ if } q = 11, 31, \text{ and } 9r \equiv 1 \pmod{5q}.$$

(ii) Next, we will show that

$$A(5qr) = 1 \text{ if } q \geq 41,\ q \equiv 1 \pmod 5, \text{ and } 9r \equiv 1 \pmod{5q}.$$

Let $q = 5k + 1$. Then k is even and $k \geq 8$. For $5q + 1 = 5s + qt$, we have $s = 4k + 1$, $t = 1$. Thus, in this case, we may rewrite the conclusion of Lemma 2.1 in the following form:

$$a(5q, j) = \begin{cases} 1 & \text{if } j \equiv 0 \pmod 5 \text{ and } 0 \leq j \leq 20k; \\ -1 & \text{if } j \equiv 1 \pmod 5 \text{ and } 1 \leq j \leq 5k-4; \\ -1 & \text{if } j \equiv 2 \pmod 5 \text{ and } 5k+2 \leq j \leq 10k-3; \\ -1 & \text{if } j \equiv 3 \pmod 5 \text{ and } 10k+3 \leq j \leq 15k-2; \\ -1 & \text{if } j \equiv 4 \pmod 5 \text{ and } 15k+4 \leq j \leq 20k-1; \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

For any given $n \in [0, \phi(5qr)]$, it follows from $rf(i) + i \equiv n \pmod{5q}$ that

$$f(i+1) + 9 \equiv f(i) \pmod{5q}. \tag{3.2}$$

Therefore, the quantity $f(i)$ can be regarded as uniquely determined by $f(0)$, where $1 \leq i \leq 4$ and $q \leq i \leq q+4$. Now we give the following tables according to the value of $f(0)$. For simplicity, we may put

$$a_{f(i)} = a(5q, f(i))$$

in the rest of this section. The first row in each table is the inequality about $f(i)$ for $i \in [0,4] \cup [q, q+4]$. The values of $a_{f(i)}$ in the following tables are obtained by using (3.1) and (3.2). We will let $\overline{f(0)}$ denote the unique integer satisfying $\overline{f(0)} \equiv f(0) \pmod 5$ and $0 \leq \overline{f(0)} \leq 4$ in Tables 5, 7, 9, 11, and 16 below.

	$f(1) > f(2) > f(3) > f(4) > f(q) > f(q+1) > f(q+2) > f(q+3) > f(q+4) > f(0)$									
$f(0)$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$
0	0	0	0	0	0	0	0	0	-1	1
1	0	0	0	0	1	0	0	-1	1	-1
2	0	0	0	0	0	0	-1	1	0	0
3	0	0	0	0	0	-1	1	0	0	0
4	0	0	0	0	-1	1	0	0	0	0
5	0	0	0	0	0	0	0	0	-1	1
6	0	0	0	0	1	0	0	-1	1	-1
7	0	0	0	0	0	0	-1	1	0	0
8	0	0	0	0	0	-1	1	0	0	0

TABLE 1. $0 \leq f(0) \leq 8$

	$f(2) > f(3) > f(4) > f(q) > f(q+1) > f(q+2) > f(q+3) > f(q+4) > f(0) > f(1)$									
$f(0)$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$
9	0	0	0	-1	0	0	0	0	0	1
10	0	0	0	0	1	0	0	-1	1	-1
11	0	0	0	1	0	0	-1	1	-1	0
12	0	0	0	0	0	-1	1	0	0	0
13	0	0	0	0	-1	1	0	0	0	0
14	0	0	0	-1	0	0	0	0	0	1
15	0	0	0	0	1	0	0	-1	1	-1
16	0	0	0	1	0	0	-1	1	-1	0
17	0	0	0	0	0	-1	1	0	0	0

TABLE 2. $9 \leq f(0) \leq 17$

	$f(3) > f(4) > f(q) > f(q+1) > f(q+2) > f(q+3) > f(q+4) > f(0) > f(1) > f(2)$									
$f(0)$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$
18	0	0	0	-1	0	0	0	0	0	1
19	0	0	-1	0	1	0	0	0	1	-1
20	0	0	0	1	0	0	-1	1	-1	0
21	0	0	1	0	0	-1	1	-1	0	0
22	0	0	0	0	-1	1	0	0	0	0
23	0	0	0	-1	0	0	0	0	0	1
24	0	0	-1	0	1	0	0	0	1	-1
25	0	0	0	1	0	0	-1	1	-1	0
26	0	0	1	0	0	-1	1	-1	0	0

TABLE 3. $18 \leq f(0) \leq 26$

	$f(4) > f(q) > f(q+1) > f(q+2) > f(q+3) > f(q+4) > f(0) > f(1) > f(2) > f(3)$									
$f(0)$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$
27	0	0	0	-1	0	0	0	0	0	1
28	0	0	-1	0	1	0	0	0	1	-1
29	0	-1	0	1	0	0	0	1	-1	0
30	0	0	1	0	0	-1	1	-1	0	0
31	0	1	0	0	-1	1	-1	0	0	0
32	0	0	0	-1	0	0	0	0	0	1
33	0	0	-1	0	1	0	0	0	1	-1
34	0	-1	0	1	0	0	0	1	-1	0
35	0	0	1	0	0	-1	1	-1	0	0

TABLE 4. $27 \leq f(0) \leq 35$

	$f(q) > f(q+1) > f(q+2) > f(q+3) > f(q+4) > f(0) > f(1) > f(2) > f(3) > f(4)$									
$\overline{f(0)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
0	0	1	0	0	-1	1	-1	0	0	0
1	1	0	0	-1	0	-1	0	0	0	1
2	0	0	-1	0	1	0	0	0	1	-1
3	0	-1	0	1	0	0	0	1	-1	0
4	-1	0	1	0	0	0	1	-1	0	0

TABLE 5. $36 \leq f(0) \leq q-1$

	$f(q) > f(q+1) > f(q+2) > f(q+3) > f(q+4) > f(0) > f(1) > f(2) > f(3) > f(4)$									
$f(0)$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
q	0	0	0	-1	0	0	0	0	0	1
$q+1$	1	0	-1	0	1	-1	0	0	1	-1
$q+2$	0	-1	0	1	0	0	0	1	-1	0
$q+3$	-1	0	1	0	0	0	1	-1	0	0
$q+4$	0	1	0	0	-1	1	-1	0	0	0
$q+5$	0	0	0	-1	0	0	0	0	0	1
$q+6$	1	0	-1	0	1	-1	0	0	1	-1
$q+7$	0	-1	0	1	0	0	0	1	-1	0
$q+8$	-1	0	1	0	0	0	1	-1	0	0
$q+9$	0	0	0	0	-1	1	0	0	0	0
$q+10$	0	1	0	-1	0	0	-1	0	0	1
$q+11$	1	0	-1	0	1	-1	0	0	1	-1
$q+12$	0	-1	0	1	0	0	0	1	-1	0
$q+13$	-1	0	1	0	0	0	1	-1	0	0
$q+14$	0	0	0	0	-1	1	0	0	0	0
$q+15$	0	1	0	-1	0	0	-1	0	0	1
$q+16$	1	0	-1	0	1	-1	0	0	1	-1
$q+17$	0	-1	0	1	0	0	0	1	-1	0
$q+18$	-1	0	0	0	0	0	1	0	0	0
$q+19$	0	0	1	0	-1	1	0	-1	0	0
$q+20$	0	1	0	-1	0	0	-1	0	0	1
$q+21$	1	0	-1	0	1	-1	0	0	1	-1
$q+22$	0	-1	0	1	0	0	0	1	-1	0
$q+23$	-1	0	0	0	0	0	1	0	0	0
$q+24$	0	0	1	0	-1	1	0	-1	0	0
$q+25$	0	1	0	-1	0	0	-1	0	0	1
$q+26$	1	0	-1	0	1	-1	0	0	1	-1
$q+27$	0	-1	0	0	0	0	0	1	0	0
$q+28$	-1	0	0	1	0	0	1	0	-1	0
$q+29$	0	0	1	0	-1	1	0	-1	0	0
$q+30$	0	1	0	-1	0	0	-1	0	0	1
$q+31$	1	0	-1	0	1	-1	0	0	1	-1
$q+32$	0	-1	0	0	0	0	0	1	0	0

TABLE 6. $q \leq f(0) \leq q+32$

	$f(q) > f(q+1) > f(q+2) > f(q+3) > f(q+4) > f(0) > f(1) > f(2) > f(3) > f(4)$									
$f(0)$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
0	0	0	1	0	-1	1	0	-1	0	0
1	0	1	0	-1	0	0	-1	0	0	1
2	1	0	-1	0	0	-1	0	0	1	0
3	0	-1	0	0	1	0	0	1	0	-1
4	-1	0	0	1	0	0	1	0	-1	0

TABLE 7. $q+33 \leq f(0) \leq 2q-1$

	$f(q) > f(q+1) > f(q+2) > f(q+3) > f(q+4) > f(0) > f(1) > f(2) > f(3) > f(4)$									
$f(0)$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
$2q$	0	0	-1	0	0	0	0	0	1	0
$2q+1$	1	-1	0	0	1	-1	0	1	0	-1
$2q+2$	-1	0	0	1	0	0	1	0	-1	0
$2q+3$	0	0	1	0	-1	1	0	-1	0	0
$2q+4$	0	1	0	-1	0	0	-1	0	0	1
$2q+5$	0	0	-1	0	0	0	0	0	1	0
$2q+6$	1	-1	0	0	1	-1	0	1	0	-1
$2q+7$	-1	0	0	1	0	0	1	0	-1	0
$2q+8$	0	0	1	0	-1	1	0	-1	0	0
$2q+9$	0	0	0	-1	0	0	0	0	0	1
$2q+10$	0	1	-1	0	0	0	-1	0	1	0
$2q+11$	1	-1	0	0	1	-1	0	1	0	-1
$2q+12$	1	0	0	-1	0	0	-1	0	1	0
$2q+13$	0	0	1	0	-1	1	0	-1	0	0
$2q+14$	0	0	0	-1	0	0	0	0	0	1
$2q+15$	0	1	-1	0	0	0	-1	0	1	0
$2q+16$	1	-1	0	0	1	-1	0	1	0	-1
$2q+17$	-1	0	0	1	0	0	1	0	-1	0
$2q+18$	0	0	0	0	-1	1	0	0	0	0
$2q+19$	0	0	1	-1	0	0	0	-1	0	1
$2q+20$	0	1	-1	0	0	0	-1	0	1	0
$2q+21$	1	-1	0	0	1	-1	0	1	0	-1
$2q+22$	-1	0	0	1	0	0	1	0	-1	0
$2q+23$	0	0	0	0	-1	1	0	0	0	0
$2q+24$	0	0	1	-1	0	0	0	-1	0	1
$2q+25$	0	1	-1	0	0	0	-1	0	1	0
$2q+26$	1	-1	0	0	1	-1	0	1	0	-1
$2q+27$	-1	0	0	0	0	0	1	0	0	0
$2q+28$	0	0	0	1	-1	1	0	0	-1	0
$2q+29$	0	0	1	-1	0	0	0	-1	0	1
$2q+30$	0	1	-1	0	0	0	-1	0	1	0
$2q+31$	1	-1	0	0	1	-1	0	1	0	-1
$2q+32$	-1	0	0	0	0	0	1	0	0	0

TABLE 8. $2q \leq f(0) \leq 2q+32$

	$f(q) > f(q+1) > f(q+2) > f(q+3) > f(q+4) > f(0) > f(1) > f(2) > f(3) > f(4)$									
$f(0)$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
0	0	0	0	1	-1	1	0	0	-1	0
1	0	0	1	-1	0	0	0	-1	0	1
2	0	1	-1	0	0	0	-1	0	1	0
3	1	-1	0	0	0	-1	0	1	0	0
4	-1	0	0	0	1	0	1	0	0	-1

TABLE 9. $2q+33 \leq f(0) \leq 3q-1$

	$f(q) > f(q+1) > f(q+2) > f(q+3) > f(q+4) > f(0) > f(1) > f(2) > f(3) > f(4)$									
$f(0)$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
$3q$	0	-1	0	0	0	0	0	1	0	0
$3q+1$	0	0	0	0	1	-1	1	0	0	-1
$3q+2$	0	0	0	1	-1	1	0	0	-1	0
$3q+3$	0	0	1	-1	0	0	0	-1	0	1
$3q+4$	0	1	-1	0	0	0	-1	0	1	0
$3q+5$	0	-1	0	0	0	0	0	1	0	0
$3q+6$	0	0	0	0	1	-1	1	0	0	-1
$3q+7$	0	0	0	1	-1	1	0	0	-1	0
$3q+8$	0	0	1	-1	0	0	0	-1	0	1
$3q+9$	0	0	-1	0	0	0	0	0	1	0
$3q+10$	0	0	0	0	0	0	-1	1	0	0
$3q+11$	0	0	0	0	1	-1	1	0	0	-1
$3q+12$	0	0	0	1	-1	1	0	0	-1	0
$3q+13$	0	0	1	-1	0	0	0	-1	0	1
$3q+14$	0	0	-1	0	0	0	0	0	1	0
$3q+15$	0	0	0	0	0	0	-1	1	0	0
$3q+16$	0	0	0	0	1	-1	1	0	0	-1
$3q+17$	0	0	0	1	-1	1	0	0	-1	0
$3q+18$	0	0	0	-1	0	0	0	0	0	1
$3q+19$	0	0	0	0	0	0	0	-1	1	0
$3q+20$	0	0	0	0	0	0	-1	1	0	0
$3q+21$	0	0	0	0	1	-1	1	0	0	-1
$3q+22$	0	0	0	1	-1	1	0	0	-1	0
$3q+23$	0	0	0	-1	0	0	0	0	0	1
$3q+24$	0	0	0	0	0	0	0	-1	1	0
$3q+25$	0	0	0	0	0	0	0	-1	1	0
$3q+26$	0	0	0	0	1	-1	1	0	0	-1
$3q+27$	0	0	0	0	-1	1	0	0	0	0
$3q+28$	0	0	0	0	0	0	0	0	-1	1
$3q+29$	0	0	0	0	0	0	0	-1	1	0
$3q+30$	0	0	0	0	0	0	-1	1	0	0
$3q+31$	0	0	0	0	1	-1	1	0	0	-1
$3q+32$	0	0	0	0	-1	1	0	0	0	0

TABLE 10. $3q \leq f(0) \leq 3q+32$

	$f(q) > f(q+1) > f(q+2) > f(q+3) > f(q+4) > f(0) > f(1) > f(2) > f(3) > f(4)$									
$f(0)$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
0	0	0	0	0	0	1	0	0	0	-1
1	0	0	0	0	0	0	0	0	-1	1
2	0	0	0	0	0	0	0	-1	1	0
3	0	0	0	0	0	0	-1	1	0	0
4	0	0	0	0	0	-1	1	0	0	0

TABLE 11. $3q+33 \leq f(0) \leq 4q-1$

	$f(q+1) > f(q+2) > f(q+3) > f(q+4) > f(0) > f(1) > f(2) > f(3) > f(4) > f(q)$									
$f(0)$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$
$4q$	0	0	0	0	0	1	0	0	0	-1
$4q+1$	0	0	0	0	0	0	0	0	-1	1
$4q+2$	0	0	0	0	0	0	0	-1	1	0
$4q+3$	0	0	0	0	0	0	-1	1	0	0
$4q+4$	0	0	0	0	0	-1	1	0	0	0
$4q+5$	0	0	0	0	0	1	0	0	0	-1
$4q+6$	0	0	0	0	0	0	0	0	-1	1
$4q+7$	0	0	0	0	0	0	0	-1	1	0
$4q+8$	0	0	0	0	0	0	-1	1	0	0

TABLE 12. $4q \leq f(0) \leq 4q+8$

	$f(q+2) > f(q+3) > f(q+4) > f(0) > f(1) > f(2) > f(3) > f(4) > f(q) > f(q+1)$									
$f(0)$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$
$4q+9$	0	0	0	0	0	1	0	0	0	-1
$4q+10$	0	0	0	0	0	0	0	0	-1	1
$4q+11$	0	0	0	0	0	0	0	-1	1	0
$4q+12$	0	0	0	0	0	0	-1	1	0	0
$4q+13$	0	0	0	0	0	-1	1	0	0	0
$4q+14$	0	0	0	0	0	1	0	0	0	-1
$4q+15$	0	0	0	0	0	0	0	0	-1	1
$4q+16$	0	0	0	0	0	0	0	-1	1	0
$4q+17$	0	0	0	0	0	0	-1	1	0	0

TABLE 13. $4q+9 \leq f(0) \leq 4q+17$

	$f(q+3) > f(q+4) > f(0) > f(1) > f(2) > f(3) > f(4) > f(q) > f(q+1) > f(q+2)$									
$f(0)$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$
$4q+18$	0	0	0	0	0	1	0	0	0	-1
$4q+19$	0	0	0	0	0	0	0	0	-1	1
$4q+20$	0	0	0	0	0	0	0	-1	1	0
$4q+21$	0	0	0	0	0	0	-1	1	0	0
$4q+22$	0	0	0	0	0	-1	1	0	0	0
$4q+23$	0	0	0	0	0	1	0	0	0	-1
$4q+24$	0	0	0	0	0	0	0	0	-1	1
$4q+25$	0	0	0	0	0	0	0	-1	1	0
$4q+26$	0	0	0	0	0	0	-1	1	0	0

TABLE 14. $4q+18 \leq f(0) \leq 4q+26$

	$f(q+4) > f(0) > f(1) > f(2) > f(3) > f(4) > f(q) > f(q+1) > f(q+2) > f(q+3)$									
$f(0)$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$
$4q+27$	0	0	0	0	0	1	0	0	0	-1
$4q+28$	0	0	0	0	0	0	0	0	-1	1
$4q+29$	0	0	0	0	0	0	0	-1	1	0
$4q+30$	0	0	0	0	0	0	-1	1	0	0
$4q+31$	0	0	0	0	0	-1	1	0	0	0
$4q+32$	0	0	0	0	0	1	0	0	0	-1
$4q+33$	0	0	0	0	0	0	0	0	-1	1
$4q+34$	0	0	0	0	0	0	0	-1	1	0
$4q+35$	0	0	0	0	0	0	-1	1	0	0

TABLE 15. $4q+27 \leq f(0) \leq 4q+35$

	$f(0)>f(1)>f(2)>f(3)>f(4)>f(q)>f(q+1)>f(q+2)>f(q+3)>f(q+4)$									
$f(0)$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$
0	0	0	0	0	0	1	0	0	0	-1
1	0	0	0	0	0	0	0	0	-1	1
2	0	0	0	0	0	0	0	-1	1	0
3	0	0	0	0	0	0	1	-1	0	0
4	0	0	0	0	0	-1	1	0	0	0

TABLE 16. $4q+36 \le f(0) \le 5q-1$

Recall that n is a fixed integer in the range $0 \le n \le \phi(5qr)$ and we have, by Lemma 2.2 (2),

$$a(5qr,n)=\sum_{i=0}^{4} a^*(5q,f(i))+\sum_{i=q}^{q+4}\Big(-a^*\big(5q,f(i)\big)\Big),$$

where

$$a^*(5q,f(i))=\begin{cases} a_{f(i)} & \text{if } \ f(i)\le \frac{n}{r};\\ 0 & \text{otherwise.}\end{cases}$$

Let

$$I=\{0,1,2,3,4,q,q+1,q+2,q+3,q+4\}.$$

If $f(i)>\frac{n}{r}$ holds for all $i\in I$, then $a^*(pq,f(i))=0$. So $a(5qr,n)=0$.

If $f(i)\le\frac{n}{r}$ holds for all $i\in I$, then $a^*(pq,f(i))=a_{f(i)}$. So

$$a(5qr,n)=\sum_{i=0}^{4} a_{f(i)}+\sum_{i=q}^{q+4}(-a_{f(i)}).$$

Then it follows from Lemma 2.2 (1) that $a(5qr,n)=0$.

If otherwise, then there certainly exist two neighboring symbols $f(i_1)$ and $f(i_2)$ in the first row of the corresponding table such that

$$f(i_1)>\frac{n}{r}\ge f(i_2).$$

If $0\le i_2\le 4$ (or $q\le i_2\le q+4$), the value of $a(5qr,n)$ is given by computing the sum of values from $a_{f(i_2)}$ (or $-a_{f(i_2)}$) to the end of the relevant row.

It is a routine matter to see that the sums of values about $\pm a_{f(i)}$, from anywhere to the end in all tables, is equal to ±1 or 0. That is to say, $a(5qr,n)\in\{\pm1,0\}$ for $0\le n\le\phi(5qr)$.

Therefore, $A(5qr)=1$ if $q\ge 41$, $q\equiv 1 \pmod 5$, and $9r\equiv 1 \pmod{5q}$, as desired.

3.2. *The case when* $p=7$. For primes $7<q<r$ with $q\equiv 1 \pmod 7$ and $9r\equiv 1 \pmod{7q}$, our goal here is to use Lemma 2.2 to show that

$$a(7qr,2qr+22r+q+2)=2.$$

Let $n=2qr+22r+q+2$. By substituting the value of n into congruence (2.1), we infer that $f(i)\equiv 11q+40-9i \pmod{7q}$. On noting $0\le f(i)\le 7q-1$, we obtain that $f(i) = 11q + 40 - 9i$ for $0 \le i \le 6$ and $q \le i \le q+6$. Then $rf(i) \le n$ if $i \in \{q+2, q+3, q+4, q+5, q+6\}$ and $rf(i) > n$ if $i \in \{0, 1, \cdots, 6\} \cup \{q, q+1\}$. From (2.2), we have

$$a^*(7q, f(i)) = \begin{cases} a(7q, f(i)) & \text{if } i \in \{q+2, q+3, q+4, q+5, q+6\}; \\ 0 & \text{if } i \in \{0, 1, \cdots, 6\} \cup \{q, q+1\} \end{cases}$$

It follows from Lemma 2.2 that

$$a(7qr, n) = -\sum_{i=q+2}^{q+6} a(7q, f(i)). \tag{3.3}$$

Rewriting $f(q+2)$, $f(q+6)$ as $f(q+2) = 3*7 + 2q + 1$ and $f(q+6) = \frac{q-15}{7} \cdot 7 + q + 1$, we deduce, by Lemma 2.1, $a(pq, f(q+2)) = a(pq, f(q+6)) = -1$. Applying this to (3.3) yields

$$a(7qr, n) = 2 - \sum_{i=q+3}^{q+5} a(7q, f(i)).$$

Let $q+3 \le i \le q+5$. On one hand, it follows from $f(i) \not\equiv 0 \pmod 7$ and Lemma 2.1 that $a(7q, f(i)) \ne 1$; on the other hand, it follows from $f(i) \ne 7u + vp + 14$ with $0 \le u \le \frac{q-8}{7}$, $0 \le v \le 5$ that $a(7q, f(i)) \ne -1$. Hence $a(7q, f(i)) = 0$, and thus $a(7qr, n) = 2$.

3.3. *The case when $p > 7$.* In this part, we will prove the following proposition to complete the proof of Theorem 1.1.

PROPOSITION 3.1. *Let $7 < p < q < r$ be odd primes such that $q = kp + 1$ and $9r \equiv 1 \pmod{pq}$.*

(1) If $p \equiv 1 \pmod 9$, then

$$2 \le \begin{cases} a(pqr, pr + 10qr + q + r + \frac{2p-11}{9}) & \text{if } k = 4; \\ a(pqr, 3pr + 9qr + q + r + \frac{p-10}{9}) & \text{if } k \ge 6. \end{cases}$$

(2) If $p \equiv 2 \pmod 9$, then

$$2 \begin{cases} = A(pqr) & \text{if } k = 2 \text{ and } p = 11; \\ \le a(pqr, 10qr + q + r + \frac{2p-13}{9}) & \text{if } k = 2 \text{ and } p > 11; \\ \le a(pqr, qr + q + r + \frac{5p-10}{9}) & \text{if } k \ge 6. \end{cases}$$

(3) If $p \equiv 4 \pmod 9$, then $a(pqr, 3qr + r + \frac{7p-10}{9}) \le -2$.

(4) If $p \equiv 5 \pmod 9$, then

$$2 \le \begin{cases} a(pqr, -pr + 10qr + q + \frac{4p-11}{9}) & \text{if } k = 2; \\ a(pqr, 5pr + 9qr + r + q + \frac{2p-10}{9}) & \text{if } k \ge 6. \end{cases}$$

(5) If $p \equiv 7 \pmod 9$, then $a(pqr, 3pr + 9qr + q + r + \frac{4p-10}{9}) \ge 2$.

(6) If $p \equiv 8 \pmod 9$, then $a(pqr, 3qr + q + r + \frac{8p-10}{9}) \ge 2$.

Proof of Proposition 3.1. (1) Case 1. $k=4$.

Noting that $p\equiv 1 \pmod 9$ and $q=4p+1$, we get $p\geq 37$. Let $n=pr+10qr+q+r+\frac{2p-11}{9}$. In order to use Lemma 2.2, we first obtain, by substituting the value of l into congruence $rf(i)+i\equiv l \pmod{pq}$,

$$f(i)\equiv 12p+17q-8-9i \pmod{pq}.$$

Since $p\geq 37$ and $0\leq f(i)\leq pq-1$, we have

$$f(i)=12p+17q-8-9i, \tag{3.4}$$

where $0\leq i\leq p-1$ or $q\leq i\leq q+p-1$. It follows that $rf(i)>n$ whenever $i\in\{0,1,\cdots,p-1\}\cup\{q,q+1,\cdots,q+\frac{2p-20}{9}\}$ and $rf(i)\leq n$ whenever $i\in\{q+\frac{2p-11}{9},q+\frac{2p-2}{9},\cdots,q+p-1\}$. Thus, by (2.2),

$$a^*(pq,f(i))=\begin{cases}0 & \text{if } i\in[0,p-1]\cup[q,q+\frac{2p-20}{9}];\\ a(pq,f(i)) & \text{if } i\in[q+\frac{2p-11}{9},q+p-1].\end{cases}$$

Hence

$$a(pqr,n)=-\sum_{j=\frac{2p-11}{9}}^{p-1}a(pq,f(q+j)). \tag{3.5}$$

Noticing that $f(q+\frac{2p-11}{9})=p+10q+1$, $f(q+p-1)=3p+8q+1$, we deduce from Lemma 2.1 that $a(pq,f(q+\frac{2p-11}{9}))=a(pq,f(q+p-1))=-1$. Applying this to (3.5) gives

$$a(pqr,n)=2-\sum_{j=\frac{2p-2}{9}}^{p-2}a(pq,f(q+j)).$$

Let $\frac{2p-2}{9}\leq j\leq p-2$. It is clear that the binary coefficient $a(pq,f(q+j))$ takes on one of three values: -1, 0, or 1. To show $a(pqr,n)\geq 2$, it suffices to prove that $a(pq,f(q+j))\neq 1$. If the assertion did not hold, then, by Lemma 2.1, we would obtain

$$f(q+j)\equiv 0 \pmod p.$$

Combining with (3.4) yields $9j\equiv 0 \pmod p$. This leads to a contradiction and yields $a(pqr,n)\geq 2$.

Case 2. $k\geq 6$.

Let $n=3pr+9qr+q+r+\frac{p-10}{9}$. Then by using the congruence (2.1) we have $f(i)=4p+18q-9-9i$ for $0\leq i\leq p-1$ and $q\leq i\leq q+p-1$. Thus, $f(i)>\frac{n}{r}$ if $i\in[0,p-1]\cup[q,q+\frac{p-19}{9}]$, and $f(i)\leq\frac{n}{r}$ if $i\in[q+\frac{p-10}{9},q+p-1]$. Invoking Lemma 2.2, we have

$$a^*(pq,f(i))=\begin{cases}0 & \text{if } i\in[0,p-1]\cup[q,q+\frac{p-19}{9}];\\ a(pq,f(i)) & \text{if } i\in[q+\frac{p-10}{9},q+p-1].\end{cases}$$

Hence

$$a(pqr, n) = -\sum_{j=\frac{p-10}{9}}^{p-1} a(pq, f(q+j)). \tag{3.6}$$

Noting that $f(q+\frac{p-10}{9}) = 3p+9q+1$, $f(q+p-1) = (k-5)p+8q+1$, and $k \geq 6$, we deduce from Lemma 2.1 that $a(pq, f(q+\frac{p-10}{9})) = a(pq, f(q+p-1)) = -1$. Combing this with (3.6) gives

$$a(pqr, n) = 2 - \sum_{j=\frac{p-1}{9}}^{p-2} a(pq, f(q+j)).$$

Next, we claim that $a(pq, f(q+j)) \neq 1$ for $\frac{p-1}{9} \leq j \leq p-2$. If the assertion did not hold, then, by Lemma 2.1, we would obtain

$$f(q+j) = 4p + 9q - 9 - 9j \equiv 0 \pmod{p}.$$

Thus, $p|9j$, a contradiction to $\frac{p-1}{9} \leq j \leq p-2$.

Hence we infer that $a(pq, f(q+j)) \in \{-1, 0\}$ for $\frac{p-1}{9} \leq j \leq p-2$, and then $a(pqr, n) \geq 2$.

(2) Case 1. $k = 2$ and $p = 11$.

By using PARI/GP, we have $A(11 \cdot 23 \cdot 1237) = 2$. Then the claim follows from $9 \cdot 1237 \equiv 1 \pmod{11 \cdot 23}$ and Lemma 2.3 that $A(pqr) = 2$ in the case $p = 11$, $q = 23$, and $9r \equiv 1 \pmod{pq}$.

Case 2. $k = 2$ and $p > 11$.

Let $n = 10qr + q + r + \frac{2p-13}{9}$. By substituting the value of n into congruence $rf(i) + i \equiv n \pmod{pq}$ and noting that $0 \leq f(i) \leq pq-1$, we have

$$f(i) = 2p + 19q - 12 - 9i, \tag{3.7}$$

where $0 \leq i \leq p-1$ or $q \leq i \leq q+p-1$. Therefore,

$$\frac{n}{r} \begin{cases} > f(i) & \text{if } q + \frac{2p-13}{9} \leq i \leq q+p-1; \\ < f(i) & \text{if } 0 \leq i \leq p-1 \text{ or } q \leq i \leq q + \frac{2p-22}{9}. \end{cases}$$

It follows from Lemma 2.2 that

$$a(pqr, n) = -\sum_{j=\frac{2p-13}{9}}^{p-1} a(pq, f(q+j)).$$

Noting that $f(q+\frac{2p-13}{9}) = 10q+1$ and $f(q+p-1) = p+6q+1$, we infer from Lemma 2.1 that $a(pq, f(q+\frac{2p-13}{9})) = a(pq, f(q+p-1)) = -1$, and thus

$$a(pqr, n) = 2 - \sum_{j=\frac{2p-4}{9}}^{p-2} a(pq, f(q+j)).$$

Let $\frac{2p-4}{9} \le j \le p-2$. Our task now is to show that $a(pq, f(q+j)) \ne 1$. Otherwise we have

$$f(q+j) \equiv 0 \pmod p$$

in view of Lemma 2.1. Applying (3.7) to the above congruence yields $9j+2 \equiv 0 \pmod p$. Hence we deduce that $9j+2 \in \{wp : 2 \le w \le 8\}$, a contradiction to $p \equiv 2 \pmod 9$. According to Lemma 2.1, we have $a(pq, f(q+j)) \in \{0, -1\}$, and thus $a(pqr, n) \ge 2$.

Case 3. $k \ge 6$.

Let $n = qr + q + r + \frac{5p-10}{9}$. Then we have $f(i) = 5p + 10q - 9 - 9i$ for $0 \le i \le p-1$ and $q \le i \le q+p-1$. Hence $\frac{n}{r} > f(i)$ whenever $i \in \{q+\frac{5p-10}{9}, q+\frac{5p-1}{9}, \cdots, q+p-1\}$ and $\frac{n}{r} < f(i)$ whenever $i \in \{0, 1, \cdots, p-1\} \cup \{q, q+1, \cdots, q+\frac{5p-19}{9}\}$. Thus, we infer

$$a(pqr, n) = -\sum_{j=\frac{5p-10}{9}}^{p-1} a(pq, f(q+j)). \tag{3.8}$$

Noting that $f(q+\frac{5p-10}{9}) = q+1$ and $f(q+p-1) = (k-4)p+1$, we deduce $a(pq, f(q+\frac{5p-10}{9})) = a(pq, f(q+p-1)) = -1$ from Lemma 2.1. Then (3.8) becomes

$$a(pqr, n) = 2 - \sum_{j=\frac{5p-1}{9}}^{p-2} a(pq, f(q+j)).$$

Recall that the coefficients of binary cyclotomic polynomials $\Phi_{pq}(x)$ can only take on one of three values: ± 1 or 0. For the purpose of proving $a(pqr, n) \ge 2$, it suffices to show $a(pq, f(q+j)) \ne 1$ for $\frac{5p-1}{9} \le j \le p-2$. In view of Lemma 2.1, if $a(pq, f(q+j)) = 1$, then $f(q+j) \equiv 0 \pmod p$. This yields

$$9j + 8 \equiv 0 \pmod p.$$

It follows from $5p+7 \le 9j+8 \le 9p-10$ that $9j+8 \in \{wp : 6 \le w \le 8\}$. This is contrary to $p \equiv 2 \pmod 9$, completing the proof.

(3) Let $n = 3qr + r + \frac{7p-10}{9}$. Then $f(i) = 7p + 3q - 9 - 9i$ for $0 \le i \le p-1$ and $f(q+j) = pq + 7p - 6q - 9j$ for $0 \le j \le p-1$. Therefore, $\frac{l}{r} > f(i)$ whenever $i \in [\frac{7p-10}{9}, p-1]$ and $\frac{l}{r} < f(i)$ whenever $i \in [0, \frac{7p-19}{9}] \cup [q, q+p-1]$. Thus,

$$a(pqr, n) = \sum_{j=\frac{7p-10}{9}}^{p-1} a(pq, f(j)). \tag{3.9}$$

Since $f(\frac{7p-10}{9}) = 3q+1$ and $f(p-1) = (k-2)p + 2q + 1$, then, by Lemma 2.1, $a(pq, f(\frac{7p-10}{9})) = a(pq, f(p-1)) = -1$. By substituting this into (3.9), we obtain

$$a(pqr, n) = -2 + \sum_{j=\frac{7p-1}{9}}^{p-2} a(pq, f(j)).$$

Our next destination is to show that $a(pq, f(j)) \neq 1$ for $\frac{7p-1}{9} < j < p-2$. If $a(pq, f(j)) = 1$, according to Lemma 2.1, then

$$f(j) = 7p + 3q - 9 - 9j \equiv 0 \pmod{p},$$

and thus $9j + 6 \equiv 0 \pmod{p}$. It follows from $7p + 5 \leq 9j + 6 \leq 9p - 12$ that $9j + 6 = 8p$, a contradiction to $p \equiv 4 \pmod{9}$. Taking Lemma 2.1 into consideration, we derive $a(pq, f(j)) \in \{-1, 0\}$, and thus $a(pqr, n) \leq -2$.

(4) Case 1. $k = 2$.

Let $n = -pr + 10qr + q + \frac{4p-11}{9}$. Then $f(i) = 3p + 19q - 11 - 9i$ for $i \in [0, p-1] \cup [q, q+p-1]$. Consequently,

$$\frac{n}{r} \begin{cases} > f(i) & \text{if } q + \frac{4p-11}{9} \leq i \leq q + p - 1; \\ < f(i) & \text{if } 0 \leq i \leq p-1 \text{ or } q \leq i \leq q + \frac{4p-20}{9}. \end{cases}$$

Invoking Lemma 2.2, we have

$$a^*(pq, f(i)) = \begin{cases} a(pq, f(i)) & \text{if } i \in [q + \frac{4p-11}{9}, q + p - 1]; \\ 0 & \text{if } i \in [0, p-1] \cup [q, q + \frac{4p-20}{9}]. \end{cases}$$

In particular, we have $f(q + \frac{4p-11}{9}) = p + 9q + 1$ and $f(q + p - 1) = 7q + 1$. It follows from Lemma 2.1 that $a(pq, f(q + \frac{4p-11}{9})) = a(pq, f(q + p - 1)) = -1$, and thus

$$a(pqr, n) = 2 - \sum_{j=\frac{4p-2}{9}}^{p-2} a(pq, f(q + j)).$$

Next we will show that $a(pq, f(q+j)) \neq 1$ for $\frac{4p-2}{9} \leq j \leq p-2$. Otherwise, by Lemma 2.1, we have $f(q + j) = 3p + 10q - 11 - 9j \equiv 0 \pmod{p}$. This yields $9j + 1 \equiv 0 \pmod{p}$. Since $4p - 1 \leq 9j + 1 \leq 9p - 17$, we obtain that $9j + 1 \in \{wp : 4 \leq w \leq 8\}$, a contradiction to $p \equiv 5 \pmod{9}$. Thus, by Lemma 2.1, $a(pq, f(q + j)) = -1$ or 0, and then $a(pqr, n) \geq 2$.

Case 2. $k \geq 6$.

Proceeding as above, let $n = 5pr + 9qr + q + r + \frac{2p-10}{9}$. Then $f(i) = 7p + 18q - 9 - 9i$ for $i \in [0, p-1] \cup [q, q+p-1]$. It is easy to show that $f(i) \leq \frac{n}{r}$ if $q + \frac{2p-10}{9} \leq i \leq q + p - 1$ and $f(i) > \frac{n}{r}$ if $0 \leq i \leq p-1$ or $q \leq i \leq q + \frac{2p-19}{9}$.

Owing to Lemma 2.2, we have

$$a^*(pq, f(i)) = \begin{cases} a(pq, f(i)) & \text{if } i \in [q + \frac{2p-10}{9}, q + p - 1]; \\ 0 & \text{if } i \in [0, p-1] \cup [q, q + \frac{2p-19}{9}]. \end{cases}$$

Noting that $f(q+\frac{2p-10}{9}) = 5p+9q+1$ and $f(q+p-1) = (k-2)p+8q+1$, we infer from Lemma 2.1 that $a(pq, f(q+\frac{2p-10}{9})) = a(pq, f(q+p-1)) = -1$. According to Lemma 2.2, we then find that

$$a(pqr, n) = 2 - \sum_{j=\frac{2p-1}{9}}^{p-2} a(pq, f(q+j)).$$

In order to prove $a(pqr, n) \geq 2$, it remains to show that $a(pq, f(q+j)) \neq 1$ for $\frac{2p-1}{9} \leq j \leq p-2$. Otherwise, by Lemma 2.1, we have

$$f(q+j) = 7p+9q-9-9j \equiv 0 \pmod{p}.$$

This gives $9j \equiv 0 \pmod{p}$, a contradiction to $\frac{2p-1}{9} \leq j \leq p-2$.

(5) Applying $n = 3pr+9qr+q+r+\frac{4p-10}{9}$ to congruence (2.1), we infer that $f(i) = 7p+18q-9-9i$, where $0 \leq i \leq p-1$ and $q \leq i \leq q+p-1$. It is easy to see that $f(i) \leq \frac{n}{r}$ when $q+\frac{4p-10}{9} \leq i \leq q+p-1$, and $f(i) > \frac{n}{r}$ when $0 \leq i \leq p-1$ and $q \leq i \leq q+\frac{4p-19}{9}$. It follows from (2.2) and (2.3) that

$$a(pqr, n) = -\sum_{j=\frac{4p-10}{9}}^{p-1} a(pq, f(q+j)).$$

Noting $f(q+\frac{4p-10}{9}) = 3p+9q+1$ and $f(q+p-1) = (k-2)p+8q+1$, we obtain from Lemma 2.1 that $a(pq, f(q+\frac{4p-10}{9})) = a(pq, f(q+p-1)) = -1$. Thus,

$$a(pqr, n) = 2 - \sum_{j=\frac{4p-1}{9}}^{p-2} a(pq, f(q+j)).$$

Let $\frac{4p-1}{9} \leq j \leq p-2$. If $a(pq, f(q+j)) = 1$, then, by Lemma 2.1,

$$f(q+j) = 7p+9q-9-9j \equiv 0 \pmod{p}.$$

This yields $9j \equiv 0 \pmod{p}$, a contradiction to the range of j. Consequently, $a(pq, f(q+j)) = -1$ or 0. Therefore $a(pqr, n) \geq 2$.

(6) By substituting $n = 3qr+q+r+\frac{8p-10}{9}$ into (2.1), we deduce that $f(i) = 8p+12q-9-9i$ for $i \in [0, p-1] \cup [q, q+p-1]$. It is straightforward to verify that $f(i) \leq \frac{n}{r}$ when $i \in [q+\frac{8p-10}{9}, q+p-1]$, and $f(i) > \frac{n}{r}$ when $i \in [0, p-1] \cup [q, q+\frac{8p-19}{9}]$. Then it follows from Lemma 2.2 that

$$a(pqr, n) = -\sum_{j=\frac{8p-10}{9}}^{p-1} a(pq, f(q+j)). \tag{3.10}$$

Noticing $f(q+\frac{8p-10}{9}) = 3q+1$ and $f(q+p-1) = (k-1)p+2q+1$, we obtain from Lemma 2.1 that $a(pq, f(q+\frac{8p-10}{9})) = a(pq, f(q+p-1)) = -1$.

Combining this with (3.10) gives

$$a(pqr, n) = 2 - \sum_{j=\frac{8p-1}{9}}^{p-2} a(pq, f(q+j)).$$

Our task now is to show that $a(pq, f(q+j)) \neq 1$ for $\frac{8p-1}{9} \leq j \leq p-2$. If the assertion did not hold, then, by Lemma 2.1, we would have

$$f(q+j) = 8p + 3q - 9 - 9j \equiv 0 \pmod{p}.$$

Consequently, we infer that $9j + 6 \equiv 0 \pmod{p}$. This contradicts the fact that $8p + 5 \leq 9j + 6 \leq 9p - 12$. Hence the binary coefficient $a(pq, f(q+j))$ takes on one of two values: -1 or 0. Finally, we have $a(pqr, n) \geq 2$. This completes the proof of Proposition 3.1. ∎

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References

[1] G. Bachman, *Flat cyclotomic polynomials of order three*, Bull. London Math. Soc. **38** (2006), 53–60.

[2] S. Elder, *Flat cyclotomic polynomials: a new approach*, arXiv:1207.5811v1 (2012).

[3] T. J. Flanagan, *On the coefficients of ternary cyclotomic polynomials*, MS Thesis, University of Nevada, Las Vegas, 2007.

[4] G. B. Imhoff, *On the coefficients of cyclotomic polynomials*, MS Thesis, California State University, Northridge, 2013.

[5] C. G. Ji, *A specific family of cyclotomic polynomials of order three*, Sci. China Math. **53** (2010), 2269–2274.

[6] N. Kaplan, *Flat cyclotomic polynomials of order three*, J. Number Theory **127** (2007), 118–126.

[7] N. Kaplan, *Flat cyclotomic polynomials of order four and higher*, Integers **10** (2010), 357–363.

[8] T. Y. Lam ands K. H. Leung, *On the cyclotomic polynomial* $\Phi_{pq}(X)$, Amer. Math. Monthly **103** (1996), 562–564.

[9] C. Sanna, *A survey on coefficients of cyclotomic polynomials*, Expo. Math. **40** (2022), 469–494.

[10] R. Thangadurai, *On the coefficients of cyclotomic polynomials*, in: Cyclotomic fields and related topics, eds. S. D. Adhikari, S. A. Katre, D. Thakur, Bhaskaracharya Pratishthana, Pune, 2000, 311–322.

[11] B. Zhang and Y. Zhou, *On a class of ternary cyclotomic polynomials*, Bull. Korean Math. Soc. **52** (2015), 1911–1924.

[12] B. Zhang, *Remarks on the flatness of ternary cyclotomic polynomials*, Int. J. Number Theory **13** (2017), 529–547.

[13] B. Zhang, *The flatness of a class of ternary cyclotomic polynomials*, Publ. Math. Debrecen **97** (2020), 201–216.

[14] B. Zhang, *The flatness of ternary cyclotomic polynomials*, Rend. Semin. Mat. Univ. Padova **145** (2021), 1–42.

[15] B. Zhang, *A remark on flat ternary cyclotomic polynomials*, Glas. Mat. Ser. III **56** (2021), 241–261.

[16] J. Zhao and X. K. Zhang, *Coefficients of ternary cyclotomic polynomials*, J. Number Theory **130** (2010), 2223–2237.

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