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ON THE $D(4)$-PAIRS $\{a, ka\}$ WITH $k \in \{2, 3, 6\}$

Kouèssi Norbert Adédji, Marija Bliznac Trebješanin, Alan Filipin and Alain Togbé

Université d'Abomey-Calavi, Bénin, University of Split, Croatia, University of Zagreb, Croatia and University Northwest, USA

Abstract. Let a and $b = ka$ be positive integers with $k \in \{2, 3, 6\}$, such that $ab+4$ is a perfect square. In this paper, we study the extensibility of the $D(4)$-pairs $\{a, ka\}$. More precisely, we prove that by considering families of positive integers c depending on a, if $\{a, b, c, d\}$ is a set of positive integers which has the property that the product of any two of its elements increased by 4 is a perfect square, then d is given by

$$d = a + b + c + \frac{1}{2}\left(abc \pm \sqrt{(ab+4)(ac+4)(bc+4)}\right).$$

As a corollary, we prove that any $D(4)$-quadruple tht contains the pair $\{a, ka\}$ is regular.

1. Introduction

The study of Diophantine sets goes back to the third century and it was the ancient Greek mathematician Diophantus of Alexandria who was the first to study such sets. In the fourth part of his book Arithmetica [13], exercise no. 20 states: "*Find four numbers (for Diophantus, this meant positive rational numbers) such that the product of any two among them increased by 1 gives a square.*" Diophantus therefore described in his book a procedure to solve this exercise. Hence, he found the first example that we call nowadays a rational Diophantine quadruple

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}.$$

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However, the first example for an integral Diophantine quadruple $\{1, 3, 8, 120\}$ was found by Fermat. Later, the concept evolved and led to the generalization mentioned in the following definition.

DEFINITION 1.1. *Let $n \neq 0$ be an integer. We call a set of m distinct positive integers a $D(n)$-m-tuple or an m-tuple with the property $D(n)$, if the product of any two of its distinct elements increased by n is a perfect square.*

For a $D(4)$-triple $\{a, b, c\}$, $a < b < c$, we define

$$d_{\pm} = d_{\pm}(a, b, c) = a + b + c + \frac{1}{2}\left(abc \pm \sqrt{(ab+4)(ac+4)(bc+4)}\right).$$

It is straightforward to check that $\{a, b, c, d_+\}$ is a $D(4)$-quadruple, which we will call a regular quadruple. A quadruple which is not regular is called an irregular quadruple. If $d_- \neq 0$ then $\{a, b, c, d_-\}$ is also a regular $D(4)$-quadruple with $d_- < c$. It is conjectured that an irregular quadruple does not exist.

CONJECTURE 1.2. *Any $D(4)$-quadruple is regular.*

In this paper, we consider extensions of the $D(4)$-pairs $\{a, ka\}$, $k = 2, 3, 6$ to a $D(4)$-quadruple $\{a, ka, c, d\}$ by following the method in [2], where the extensions of Diophantine pairs $\{a, ka\}$, with $k = 3, 8$, were studied. We conjecture that $d = d_+(a, ka, c)$, i.e., that there is no irregular quadruple of this form. The validity of the conjecture is shown for some special pairs and triples. One of the results of interest for our case is the following lemma, which gives us a lower bound for the second element of the pair $b = ka$.

LEMMA 1.3 ([6, Lemma 2.2]). *Let $\{a, b, c, d\}$ be a $D(4)$-quadruple such that $a < b < c < d_+ < d$. Then $b > 10^5$.*

If $\{a, ka\}$ is a $D(4)$-pair, then there exists $r \in \mathbb{N}$ such that

$$ka^2 + 4 = r^2. \tag{1.1}$$

Rewriting (1.1) as a Pellian equation, yields

$$r^2 - ka^2 = 4. \tag{1.2}$$

The theory of Pellian equations guarantees that there is only one fundamental solution (r_1, a_1) of (1.2), for any $k = 2, 3, 6$, namely $(r_1, a_1) \in \{(6, 4), (4, 2), (10, 4)\}$ (in that order). All solutions (r_p, a_p) of the equation (1.2) are given by

$$\frac{r_p + a_p\sqrt{k}}{2} = \left(\frac{r_1 + a_1\sqrt{k}}{2}\right)^p, \quad p \in \mathbb{N}. \tag{1.3}$$

It is easy to see that $\gcd(r, a) = 2$ holds in every case. Since $b = ka > 10^5$ we can also deduce a lower bound for a. For $k = 2$, we have $a \geq a_7 = 161564$, which also gives us $b = 2a \geq 323128$. In the case $k = 3$, we have $a \geq a_9 =$

81090 and $b = 3a \geq 3a_9 \geq 243270$. Finally, for $k = 6$, the lower bounds $a \geq a_5 = 38804$, $b \geq 232824$ hold.

In general, if we extend a $D(4)$-pair $\{a, b\}$ to a $D(4)$-triple $\{a, b, c\}$ then there exist $s, t \in \mathbb{N}$ such that

$$\begin{aligned} ac + 4 &= s^2, \\ bc + 4 &= t^2. \end{aligned}$$

Combining these two equalities yields a Pellian equation

$$at^2 - bs^2 = 4(a - b). \tag{1.4}$$

Its solutions (t, s) are given by

$$(t_\nu + s_\nu\sqrt{k}) = (t_0 + s_0\sqrt{k})\left(\frac{r + \sqrt{ab}}{2}\right)^{\nu}, \quad \nu \geq 0,$$

where (t_0, s_0) is a fundamental solution of the equation (1.4) and ν is a non-negative integer.

In [6, Lemma 6.1], it has been shown that $(t_0, s_0) = (\pm 2, 2)$ are the only fundamental solutions when $b \leq 6.85a$, which is our case. We can represent the solutions (t_ν, s_ν) as pair of binary recurrence sequences

$$t_0 = \pm 2,\ t_1 = b \pm r,\ t_{\nu+2} = rt_{\nu+1} - t_\nu, \tag{1.5}$$

$$s_0 = 2,\ s_1 = r \pm a,\ s_{\nu+2} = rs_{\nu+1} - s_\nu,\ \nu \geq 0. \tag{1.6}$$

Since $c = \frac{s^2 - 4}{a}$, we give an explicit expression for the third element c in the terms of a and b by

$$\begin{aligned} c = c_\nu^{\pm} = \frac{4}{ab}\Bigg\{&\left(\frac{\sqrt{b} \pm \sqrt{a}}{2}\right)^2 \left(\frac{r + \sqrt{ab}}{2}\right)^{2\nu} \\ &+ \left(\frac{\sqrt{b} \mp \sqrt{a}}{2}\right)^2 \left(\frac{r - \sqrt{ab}}{2}\right)^{2\nu} - \frac{a + b}{2}\Bigg\}, \end{aligned} \tag{1.7}$$

where $\nu \geq 0$ is an integer. From [6, Proposition 1.8], if $a \geq 35$ then $c \geq c_4^-$ cannot hold, i.e., it remains to observe the cases $c \in \{c_1^\pm, c_2^\pm, c_3^\pm\}$. Let us list them in a more suitable form

$$\begin{aligned} c_1^\pm &= a + b \pm 2r, \\ c_2^\pm &= (ab + 4)(a + b \pm 2r) \mp 4r, \\ c_3^\pm &= (a^2b^2 + 6ab + 9)(a + b \pm 2r) \mp 4r(ab + 3). \end{aligned}$$

Let us mention some observations in the case c_1^-. Note that if $k \in \{2, 3\}$, then $c_1^- < a < b$ and in case $k = 6$ we have $a < c_1^- < b$. So, in these cases, we consider the $D(4)$-triple $\{a, b, c\}$ of the form $\{c_1^-, a, b\}$ or $\{a, c_1^-, b\}$.

The present paper deals with two closely related families, viz. those of $D(4)$-triples mentioned in this section. The outcome of our study is the theorem below, showing that each of the triples under scrutiny has a unique extension to quadruple, which supports Conjecture 1.2.

THEOREM 1.4. *Let k and ν be positive integers such that $k \in \{2, 3, 6\}$. If $\{a, b, c_\nu^{\pm}, d\}$ is a $D(4)$-quadruple with $b = ka$, then it is regular. In other words, we have $d = d_{\pm}$.*

Notice that the choice of the $D(4)$-pairs $\{a, ka\}$ with $k \in \{2, 3, 6\}$ is not random. In fact, the idea is to investigate irregular $D(4)$-quadruples containing the pairs $\{a, ka\}$, for any positive integer k. In the second author's recent work in [6], it was shown that $a < b \leq 6.85a$ implies that c must be of the form (1.7). Thus, if we choose to observe $b = ka$, we will have $k \in \{2, 3, 4, 5, 6\}$. For $k = 4$, it is easy to check from equation (1.1) that there is no $D(4)$-pair of the form $\{a, 4a\}$. On the other hand, for $k = 5$, we obtain an equation of the type

$$r^2 - 5a^2 = 4,$$

whose positive solutions are defined by $(r, a) = (L_{2n}, F_{2n})$, where F_n and L_n denote the n-th Fibonacci and Lucas numbers respectively. Here also the extension of the $D(4)$-pair $\{F_{2n}, 5F_{2n}\}$ should be skipped since it has already been studied (see [8, 11]). Thus, it remains to only study $k = 2, 3, 6$, which is the goal of this paper.

The organization of this paper is as follows. In Sections 2 and 3 of the paper, we will essentially prove useful results to achieve our main goal. We devote Section 4 to the proof of Theorem 1.4.

2. Pellian equations and linear forms in three logarithms

The goal of this section is to provide and prove the main technical tools used in our proof of Theorem 1.4. These tools are related to the search for the intersection of linear recurrent sequences and to linear forms in logarithms.

2.1. *System of simultaneous Pellian equations.* Let us observe an extension of a triple $\{a, b, c\}$ to a quadruple $\{a, b, c, d\}$:

$$\begin{aligned} ad + 4 &= x^2, \\ bd + 4 &= y^2, \\ cd + 4 &= z^2. \end{aligned}$$

By eliminating d from these equations, we get a system of generalized Pellian equations

$$az^2 - cx^2 = 4(a - c), \tag{2.1}$$

$$bz^2 - cy^2 = 4(b - c), \tag{2.2}$$

$$ay^2 - bx^2 = 4(a - b). \tag{2.3}$$

Its solutions (z, x), (z, y), and (y, x) satisfy

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})\left(\frac{s + \sqrt{ac}}{2}\right)^m, \tag{2.4}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{a} + y_1\sqrt{c})\left(\frac{t + \sqrt{bc}}{2}\right)^n, \tag{2.5}$$

$$y\sqrt{a} + x\sqrt{b} = (y_2\sqrt{a} + x_2\sqrt{b})\left(\frac{r + \sqrt{ab}}{2}\right)^l, \tag{2.6}$$

where m, n, l are nonnegative integers and (z_0, x_0), (z_1, y_1), and (y_2, x_2) are fundamental solutions of these equations.

Any solution to the system satisfies $z = v_m = w_n$, where v_m, and w_n are the recurrent sequences defined by

$$v_0 = z_0,\ v_1 = \frac{1}{2}(sz_0 + cx_0),\ v_{m+2} = sv_{m+1} - v_m,$$
$$w_0 = z_1,\ w_1 = \frac{1}{2}(tz_1 + cy_1),\ w_{n+2} = tw_{n+1} - w_n.$$

The initial terms of these sequences are described in the next theorem.

THEOREM 2.1 ([6, Theorem 1.3]). *Suppose that $\{a, b, c, d\}$ is a $D(4)$-quadruple with $a < b < c < d$ and that w_m and v_n are defined as before.*

i) If the equation $v_{2m} = w_{2n}$ has a solution, then $z_0 = z_1$ and $|z_0| = 2$ or $|z_0| = \frac{1}{2}(cr - st)$.
ii) If the equation $v_{2m+1} = w_{2n}$ has a solution, then $|z_0| = t$, $|z_1| = \frac{1}{2}(cr - st)$ and $z_0z_1 < 0$.
iii) If the equation $v_{2m} = w_{2n+1}$ has a solution, then $|z_1| = s$, $|z_0| = \frac{1}{2}(cr - st)$ and $z_0z_1 < 0$.
iv) If the equation $v_{2m+1} = w_{2n+1}$ has a solution, then $|z_0| = t$, $|z_1| = s$ and $z_0z_1 > 0$.

Moreover, if $d > d_+$, then the case ii) cannot occur.

Remark that because of the mentioned results by Fujita [12] and the authors [3] on the extensibility of Diophantine pairs $\{k - 1, k + 1\}$ and on the extensibility of $D(4)$-pair of the form $\{k - 2, k + 2\}$, if we prove that under the assumption of the following lemma, $D(4)$-triple $\{a, ka, c\}$ has only two extensions to a quadruple (with $d = d_-$ and $d = d_+$) it will imply the statement of Theorem 1.4.

LEMMA 2.2. *Assume that* $\{a, b, c, c'\}$ *is not a* $D(4)$*-quadruple for any* c' *with* $0 < c' < c^{\pm}_{\nu-1}$. *We have:*

i) if the equation $v_{2m} = w_{2n}$ *has a solution, then* $z_0 = z_1 = \pm 2$ *and* $x_0 = y_1 = 2$,

ii) if the equation $v_{2m+1} = w_{2n+1}$ *has a solution, then* $z_0 = \pm t$, $z_1 = \pm s$, $x_0 = y_1 = r$ *and* $z_0 z_1 > 0$.

PROOF. The proof of this lemma is based on Theorem 2.1 and is similar to the proofs [3, Lemma 3], [1, Lemma 2.3] and [12, Lemma 5]. □

REMARK 2.3. If $c = c^{\pm}_1 = a + b \pm 2r$, then it is enough to observe the case $v_{2m} = w_{2n}$.

Now, we observe the solutions of the system of equations (2.2) and (2.6). More precisely, we will determine the intersections $y = A_n = B_l$ of sequences $(A_n)_n$ and $(B_l)_l$ defined by

$$A_0 = y_1,\ A_1 = \frac{1}{2}(ty_1 + bz_1),\ A_{n+2} = tA_{n+1} - A_n, \tag{2.7}$$

$$B_0 = y_2,\ B_1 = \frac{1}{2}(ry_2 + bx_2),\ B_{l+2} = rB_{l+1} - B_l,\ n, j \geq 0. \tag{2.8}$$

The next lemma, which is a part of Lemma 2 in [8], gives us a description of the solutions of Pell equations.

LEMMA 2.4 ([8, Lemma 2]). *If* (X, Y) *is a positive integer solution to a generalized Pell equation*

$$aY^2 - bX^2 = 4(a - b),$$

with $ab + 4 = r^2$, *then we have*

$$Y\sqrt{a} + X\sqrt{b} = (y_0\sqrt{a} + x_0\sqrt{b})\left(\frac{r + \sqrt{ab}}{2}\right)^n,$$

where $n \geq 0$ *is an integer and* (x_0, y_0) *is an integer solution of the equation such that*

$$1 \leq x_0 \leq \sqrt{\frac{a(b-a)}{r-2}} \quad \text{and} \quad 1 \leq |y_0| \leq \sqrt{\frac{(r-2)(b-a)}{a}}.$$

The next lemma will be proved following the steps of [2, Lemma 4].

LEMMA 2.5. *Assume that* $\{a, b, c', c\}$ *is not a* $D(4)$*-quadruple for any* c' *with* $0 < c' < c^{\pm}_{\nu-1}$ *and* $b \geq 832824$. *Then,* $A_{2n} = B_{2l+1}$ *has no solution. Moreover, if* $A_{2n} = B_{2l}$ *then* $y_2 = 2$. *In other cases we have* $y_2 = \pm 2$.

PROOF. It is straightforward to check by induction that

$$A_{2n} \equiv y_1 \pmod{b}, \quad A_{2n+1} \equiv \frac{1}{2}(ty_1 + bz_1) \pmod{b},$$
$$B_{2l} \equiv y_2 \pmod{b}, \quad B_{2l+1} \equiv \frac{1}{2}(ry_2 + bx_2) \pmod{b}.$$

From Lemma 2.4, we have

$$|y_2| \le \sqrt{\frac{(r-2)(b-a)}{a}} = \sqrt{\frac{(r-2)(k-1)a}{a}} = \sqrt{(k-1)(r-2)},$$

where we have used that $b = ka$, $k \in \{2, 3, 6\}$. This implies

$$|y_2| \le \sqrt{(k-1)\sqrt{b^2/k + 4}} \le \begin{cases} 0.85\sqrt{b}, & k = 2, \\ 1.075\sqrt{b}, & k = 3, \\ 1.43\sqrt{b}, & k = 6. \end{cases}$$

CASE 1: If $A_{2n} = B_{2l}$, then $y_1 \equiv y_2 \pmod{b}$. From Lemma 2.2, we have $y_1 = 2$, so $y_2 \equiv 2 \pmod{b}$. On the other hand,

$$y_2 < 1.43\sqrt{b} < 0.5b,$$

for $b \ge 9$, so $y_2 = 2$.

CASE 2: If $A_{2n} = B_{2l+1}$, then we have $y_1 = 2$ and

$$2 \equiv \frac{1}{2}(ry_2 + bx_2) \pmod{b}. \tag{2.9}$$

Since $ab + 4 = r^2$, we know that $g = \gcd(b, r) \in \{1, 2, 4\}$. After multiplying congruence (2.9) by 2 and using the fact that $r^2 \equiv 4 \pmod{b}$, we can divide the final congruence by r and get

$$r \equiv y_2 \left(\bmod\ \frac{b}{g}\right). \tag{2.10}$$

We can rewrite the upper bound on $|y_2|$ and get

$$|y_2| \le \begin{cases} 0.002b, & k = 2, \\ 0.0022b, & k = 3, \\ 0.003b, & k = 6, \end{cases} \tag{2.11}$$

where we have used a lower bound on b for each value of k. Also,

$$r = \sqrt{ab + 4} = \sqrt{b^2/k + 4} = \sqrt{1/k + 4/b^2} \cdot b,$$

so we can use lower bounds on b to get those on r in the terms of b. More precisely,

$$R_0 \cdot b < r < (R_0 + 0.01) \cdot b,$$

where

$$R_0 = \begin{cases} 0.7, & k = 2, \\ 0.57, & k = 3, \\ 0.4, & k = 6. \end{cases}$$

If $g = \gcd(b, r) = 1$, for each k, then we have $||y_2| + r| < 0.712b < b$. This implies $y_2 = r > 0.4b$, which is a contradiction with (2.11). On the other hand, if $g = \gcd(b, r) \in \{2, 4\}$ we get $y_2 = r + \frac{b}{4} \cdot p$, for some $p \in \mathbb{Z}$. If $p \geq 0$, then we have $|y_2| \geq r$ and a contradiction as in the previous case. If $p \leq -4$, then we have $|y_2| > 0.29b$. For the remaining cases, $p = -1, -2, -3$, we have $|y_2| > 0.13b, 0.09b, 0.04b$, which is a contradiction in each case.

Other possibilities for the parities of the indices of the sequences (A_n) and (B_l) are solved similarly to [2, Lemma 4]. Thus, we omit the details. □

Therefore, the fundamental solutions of equation (2.3) are $(y_2, x_2) = (\pm 2, 2)$. Finally, we need to look at $x = Q_m = P_l$, for some non-negative integers m and l, where the sequences $(Q_m)_{m \geq 0}$ and $(P_l)_{l \geq 0}$ are obtained using (2.4) and (2.6) and are given by

$$P_0 = x_2,\ P_1 = \frac{1}{2}\left(rx_2 + ay_2\right),\ P_{l+2} = rP_{l+1} - P_l, \tag{2.12}$$

$$Q_0 = x_0,\ Q_1 = \frac{1}{2}\left(sx_0 + az_0\right),\ Q_{m+2} = sQ_{m+1} - Q_m. \tag{2.13}$$

From the above, for the equation $x = P_l = Q_m$, we conclude that only the following two possibilities exist:
TYPE 1: If $l \equiv m \equiv 0 \pmod 2$, then $z_0 = \pm 2$, $x_0 = y_2 = 2$ and $x_2 = 2$.
TYPE 2: If $m \equiv 1 \pmod 2$, then $z_0 = \pm t$, $x_0 = r$, $y_2 = \pm 2$ and $x_2 = 2$.

For the rest of this paper, we will carefully examine the following equation

$$x = Q_m = P_l, \tag{2.14}$$

while using the fundamental solutions of Types 1 and 2. As we mentioned in Remark 2.3, we only need to consider solutions in Type 1 if $c = c_1^{\pm}$ since $\frac{1}{2}(cr - st) = \frac{1}{2}\left((a + b \pm 2r)r - (r \pm a)(b \pm r)\right) = \pm 2$.

2.2. *A linear form in three logarithms.* Solving recurrences (2.12) and (2.13), we obtain

$$P_l = \frac{1}{2\sqrt{b}}\left((y_2\sqrt{a} + x_2\sqrt{b})\alpha^l - (y_2\sqrt{a} - x_2\sqrt{b})\alpha^{-l}\right),$$
$$Q_m = \frac{1}{2\sqrt{c}}\left((z_0\sqrt{a} + x_0\sqrt{c})\beta^m - (z_0\sqrt{a} - x_0\sqrt{c})\beta^{-m}\right),$$

where

$$\alpha = \frac{r + \sqrt{ab}}{2} \quad \text{and} \quad \beta = \frac{s + \sqrt{ac}}{2}. \tag{2.15}$$

Let us define

$$\gamma = \frac{\sqrt{c}(y_2\sqrt{a} + x_2\sqrt{b})}{\sqrt{b}(z_0\sqrt{a} + x_0\sqrt{c})} \quad \text{and} \quad \gamma' = \frac{\sqrt{b}(y_2\sqrt{a} + x_2\sqrt{c})}{\sqrt{c}(z_0\sqrt{a} + x_0\sqrt{b})}. \tag{2.16}$$

We follow the strategy used in [2] with some improvements and define the following linear forms in three logarithms

$$\Lambda = l\log\alpha - m\log\beta + \log\gamma \quad \text{for} \quad c > b, \tag{2.17}$$

and

$$\Lambda' = l\log\beta - m\log\alpha + \log\gamma' \quad \text{for} \quad c < b. \tag{2.18}$$

Lemma 2.6. *i) If the equation $P_l = Q_m$ has a solution (l, m) of Type 1 with $m \geq 1$, then*

$$0 < \Lambda < 11.7\beta^{-2m} \quad \text{and} \quad 0 < \Lambda' < 11.7\alpha^{-2m}.$$

ii) If the equation $P_l = Q_m$ has a solution (l, m) of Type 2 with $m \geq 1$, then

$$0 < \Lambda < 4.4a^2\beta^{-2m}.$$

Proof. For $c > b$, we define

$$E = \frac{y_2\sqrt{a} + x_2\sqrt{b}}{\sqrt{b}}\alpha^l \quad \text{and} \quad F = \frac{z_0\sqrt{a} + x_0\sqrt{c}}{\sqrt{c}}\beta^m.$$

One can easily check that E, $F > 1$ if $l, m \geq 1$. Then, the equation $P_l = Q_m$ becomes

$$E + 4\left(\frac{b-a}{b}\right)E^{-1} = F + 4\left(\frac{c-a}{c}\right)F^{-1}. \tag{2.19}$$

Because $c > b > 10^5$, we have $\frac{c-a}{c} > \frac{b-a}{b}$. It follows that

$$E + 4\left(\frac{b-a}{b}\right)E^{-1} > F + 4\left(\frac{b-a}{b}\right)F^{-1} \tag{2.20}$$

and hence

$$(E - F)\left(EF - 4\left(\frac{b-a}{b}\right)\right) > 0.$$

Therefore, $E > F$. Moreover, by (2.19) we have

$$0 < E - F < 4\left(\frac{c-a}{c}\right)F^{-1} < 4F^{-1}.$$

It follows that $\Lambda > 0$, with

$$\Lambda = \log\frac{E}{F} = \log\left(1 + \frac{E-F}{F}\right) < \frac{E-F}{F} < 4F^{-2}.$$

TYPE 1: One has

$$\begin{aligned}\Lambda &< 4\frac{c}{(\pm 2\sqrt{a}+2\sqrt{c})^2}\beta^{-2m} = \frac{c}{(\pm\sqrt{a}+\sqrt{c})^2}\beta^{-2m}\\ &< \frac{k}{(\sqrt{k}-1)^2}\beta^{-2m} < 11.7\beta^{-2m},\ \text{for } c > b = ka > 10^5.\end{aligned}$$

Note that in the case $b > c$, we consider the $D(4)$-triple of the form $\{a, c_1^-, b\}$. So in this case, we similarly get $0 < \Lambda' < 11.7\alpha^{-2m}$, which is valid for $k = 2, 3, 6$.

TYPE 2:

- The case $z_0 = t$. We have

$$\Lambda < 4\frac{c}{(t\sqrt{a}+r\sqrt{c})^2}\beta^{-2m} < \frac{4}{r^2}\frac{c}{(\sqrt{a}+\sqrt{c})^2}\beta^{-2m} < \beta^{-2m},$$

for $c > b > 10^5$ and $r > 100$.

- The case $z_0 = -t$. From (2.19), we get

$$\begin{aligned}F = E + 4\left(\frac{b-a}{b}\right)E^{-1} - 4\left(\frac{c-a}{c}\right)F^{-1} &> E - 4\left(\frac{c-a}{c}\right)F^{-1}\\ &> E - 4\left(\frac{c-a}{c}\right) > 0.\end{aligned}$$

The above inequality comes from the fact that $F > 1$. Thus, we obtain

$$F^{-1} < \left(E - 4\left(\frac{c-a}{c}\right)\right)^{-1}.$$

Therefore, we have

$$\begin{aligned}E - F &= 4\left(\frac{c-a}{c}\right)F^{-1} - 4\left(\frac{b-a}{b}\right)E^{-1}\\ &< 4\left(\frac{c-a}{c}\right)\left(E - 4\left(\frac{c-a}{c}\right)\right)^{-1} - 4\left(\frac{b-a}{b}\right)E^{-1}.\end{aligned} \tag{2.21}$$

Moreover, in Type 2 and for $m \geq 3$, we have

$$\begin{aligned}F &\geq \frac{r\sqrt{c}-t\sqrt{a}}{\sqrt{c}}\beta^3 = \frac{4(c-a)}{\sqrt{c}(r\sqrt{c}+t\sqrt{a})}\cdot\left(\frac{s+\sqrt{ac}}{2}\right)^3\\ &> \frac{4(c-a)}{\sqrt{c}\cdot 2r\sqrt{c}}\cdot(\sqrt{ac})^3 > 4(c-a),\end{aligned} \tag{2.22}$$

which implies $E > F > 4(c-a)$ and then

$$\frac{c-a}{c}\left(E - 4\left(\frac{c-a}{c}\right)\right)^{-1} < E^{-1}. \tag{2.23}$$

When $m = 1$, we can easily see that $P_l \neq Q_1$ by following what is done in the proof of [1, Lemma 3.3]. So, combining (2.21) and (2.23), we obtain

$$E - F < 4E^{-1} - 4\left(\frac{b-a}{b}\right)E^{-1} = \frac{4a}{b}E^{-1} < \frac{4}{k}F^{-1}.$$

Therefore, one can see that

$$\Lambda = \log\frac{E}{F} = \log\left(1 + \frac{E-F}{F}\right) < \frac{E-F}{F} < \frac{4}{k}F^{-2}$$

and

$$\frac{4}{k}F^{-2} = \frac{4}{k}\cdot\frac{c}{(r\sqrt{c} - t\sqrt{a})^2}\beta^{-2m} < \frac{c^2r^2}{k(c-a)^2}\beta^{-2m}.$$

Using the fact that $c > b = ka > 10^5$, we get

$$r^2 = ka^2 + 4 < 1.1ka^2 \quad \text{and} \quad \frac{c^2}{k(c-a)^2} < \frac{k}{(k-1)^2}.$$

Hence, $\Lambda = \log\dfrac{E}{F} < 4.4a^2\beta^{-2m}$. Considering all cases in Type 2, we have $\Lambda < 4.4a^2\beta^{-2m}$. This completes the proof of Lemma 2.6. □

We now have the next result whose proof is similar to that of part 2) in [2, Lemma 8].

LEMMA 2.7. *If the equation $x = Q_m = P_l$ has a solution (l, m) with $m \geq 3$, then $m \leq l$.*

For any nonzero algebraic number α of degree d over $\mathbb{Q}$ whose minimal polynomial over $\mathbb{Z}$ is $a_0\prod_{j=1}^{d}(X - \alpha^{(j)})$, we denote by

$$h(\alpha) = \frac{1}{d}\left(\log|a_0| + \sum_{j=1}^{d}\log\max\left(1, \left|\alpha^{(j)}\right|\right)\right)$$

its absolute logarithmic height. We recall the following result due to Matveev [14].

LEMMA 2.8. *Denote by $\alpha_1, \ldots, \alpha_j$ algebraic numbers, not 0 or 1, by $\log\alpha_1, \ldots, \log\alpha_j$ determinations of their logarithms, by D the degree over $\mathbb{Q}$ of the number field $\mathbb{K} = \mathbb{Q}(\alpha_1, \ldots, \alpha_j)$, and by $b_1, \ldots, b_j$ integers. Define $B = \max\{|b_1|, \ldots, |b_j|\}$ and*

$$A_i = \max\{Dh(\alpha_i), |\log\alpha_i|, 0.16\} \ \ (1 \leq i \leq j),$$

where $h(\alpha)$ denotes the absolute logarithmic Weil height of α. Assume that the number

$$\Lambda = b_1\log\alpha_1 + \cdots + b_n\log\alpha_j$$

does not vanish. Then

$$|\Lambda| \geq \exp\{-C(j,\chi)D^2 A_1 \cdots A_j \log(eD)\log(eB)\},$$

where $\chi = 1$ *if* $\mathbb{K} \subset \mathbb{R}$ *and* $\chi = 2$ *otherwise and*

$$C(j,\chi) = \min\left\{\frac{1}{\chi}\left(\frac{1}{2}ej\right)^{\chi} 30^{j+3} j^{3.5}, 2^{6j+20}\right\}.$$

PROPOSITION 2.9. *Assume that* $c \in \{c_1^+, c_2^\pm\}$. *If* $Q_m = P_l$, *then*

$$\frac{l}{\log(el)} < 3.36 \cdot 10^{13} \cdot \log^2(5.9c^2), \quad \textit{with the solutions of Type } 1,$$

$$\frac{l}{\log(el)} < 6.44 \cdot 10^{13} \cdot \log^2(5.9c^2), \quad \textit{with the solutions of Type } 2.$$

If $Q_m = P_l$, *with* $c = c_1^-$, *then we get*

$$\frac{l}{\log(el)} < 8.5 \cdot 10^{13} \cdot \log^2(26.3a), \quad \textit{for } k = 2,3,6.$$

PROOF. We apply Lemma 2.8 with $j = 3$ and $\chi = 1$ to the linear form in logarithms (2.17). Here, we take

$$D = 4, b_1 = l, b_2 = -m, b_3 = 1, \alpha_1 = \alpha, \alpha_2 = \beta, \text{ and } \alpha_3 = \gamma.$$

Since $m \leq l$, we can take $B = l$. Also, we have

$$h(\alpha_1) = \frac{1}{2}\log\alpha \quad \text{and} \quad h(\alpha_2) = \frac{1}{2}\log\beta.$$

• In the case $c \in \{c_1^+, c_2^\pm\}$ with $b = ka$, we have $c - a > \left(1 - \frac{1}{k}\right)c$ and $r < \frac{1}{3}c$. Moreover, the conjugates of α_3 are

$$\frac{\sqrt{c}(y_2\sqrt{a} \pm x_2\sqrt{b})}{\sqrt{b}(z_0\sqrt{a} \pm x_0\sqrt{c})},$$

and the leading coefficient of the minimal polynomial of α_3 divides the number $16k^2(c-a)^2$. We proceed with the following estimates

$$\begin{aligned}
h(\alpha_3) &\leq \frac{1}{4}\left[\log(16k^2(c-a)^2) + 4\log\frac{\max\{|\sqrt{c}(y_2\sqrt{a} \pm x_2\sqrt{b})|\}}{\min\{|\sqrt{b}(z_0\sqrt{a} \pm x_0\sqrt{c})|\}}\right] \\
&= \frac{1}{4}\left[\log(16k^2(c-a)^2) + 4\log\frac{2\sqrt{c}(1+\sqrt{k})}{\sqrt{k}(-t\sqrt{a} + r\sqrt{c})}\right] \\
&< \frac{1}{4}\log\left[\frac{2^4 r^4 c^4 (1+\sqrt{k})^4}{(c-a)^2}\right] < \frac{1}{4}\log\left(\frac{k^2(1+\sqrt{k})^2 \cdot c^6}{(\sqrt{k}-1)^2}\right) \\
&< \frac{1}{4}\log(204c^6) < \frac{3}{4}\log(5.9c^2).
\end{aligned}$$

Thus, we can take

$$A_1 = 2\log\alpha, \quad A_2 = 2\log\beta, \quad A_3 = 3\log(5.9c^2).$$

Applying Lemma 2.8, we get

$$\log|\Lambda| > -1.3901 \cdot 10^{11} \cdot 16 \cdot 12 \cdot \log\alpha \cdot \log\beta \cdot \log(5.9c^2) \cdot \log(4e) \cdot \log(el).$$

From each inequality of Lemma 2.6, we have $l\log\alpha < 2m\log\beta$ and also

$$\log|\Lambda| < -1.9m\log\beta \quad \text{in Type 1, with } m \geq 1 \tag{2.24}$$

and

$$\log|\Lambda| < -0.99m\log\beta \quad \text{in Type 2, with } m \geq 2. \tag{2.25}$$

Moreover,

$$\log\beta < \frac{1}{2}\log(5.9c^2). \tag{2.26}$$

Combining (2.24), (2.24), (2.25), and (2.26), we respectively get according to Type 1 and Type 2 the following inequalities

$$\frac{l}{\log(el)} < 3.36 \cdot 10^{13} \cdot \log^2(5.9c^2) \text{ and } \frac{l}{\log(el)} < 6.44 \cdot 10^{13} \cdot \log^2(5.9c^2).$$

• In the case $c = c_1^-$ with the solutions of Type 1, we similarly get the following inequality

$$\frac{l}{\log(el)} < 8.5 \cdot 10^{13} \cdot \log^2(26.3a), \quad \text{for } k = 2, 3, 6,$$

which comes from the combination of Lemm 2.8 and the upper bound on Λ'. This ends the proof. □

3. Lower bounds for m and l in terms of a

In this section, we will apply congruence relations to obtain some lower bounds for the indices m and l satisfying the equation

$$x = Q_m = P_l.$$

Lemma 3.1. *If a is odd, then*

$$Q_{2m} \equiv x_0 + \frac{1}{2}a(cx_0m^2 + sz_0m) \pmod{a^2}, \tag{3.1}$$

$$P_{2l} \equiv x_2 + \frac{1}{2}a(bx_2l^2 + ry_2l) \pmod{a^2}. \tag{3.2}$$

If a is even, then

$$Q_{2m} \equiv x_0 + \frac{1}{2}a(cx_0m^2 + sz_0m) \pmod{\frac{1}{2}a^2},$$

$$P_{2l} \equiv x_2 + \frac{1}{2}a(bx_2l^2 + ry_2l) \pmod{\frac{1}{2}a^2}.$$

Proof. The proof is similar to that of [2, Lemma 15]. We give the proof of (3.1) by induction. The proofs of the other congruences can be done in the same way. We have $Q_0 = x_0$, $Q_2 = x_0 + \frac{1}{2}a(cx_0 + sz_0)$ and

$$Q_4 = \frac{1}{2}x_0a^2c^2 + \frac{1}{2}a^2z_0cs + \frac{1}{2}a(4cx_0 + 2sz_0) + x_0.$$

Assume that the assertion is valid for $m-1$ and m. Note also that, the sequence $(Q_{2m})_{m\geq 0}$ satisfies the recurrence relation

$$Q_{2m+2} = (ac+2)Q_{2m} - Q_{2m-2}.$$

Thus, we obtain

$$\begin{aligned} Q_{2m+2} &\equiv acx_0 + x_0 + \frac{1}{2}a[2cx_0m^2 + 2sz_0m - cx_0(m-1)^2 - sz_0(m-1)] \\ &\equiv x_0 + \frac{1}{2}a[cx_0(m+1)^2 + sz_0(m+1)] \pmod{a^2}. \end{aligned}$$

This completes the proof. □

We now consider the following result.

Lemma 3.2. *If the equation $P_l = Q_m$ has a solution (l,m) of Type 1, then we have*

$$l \geq \frac{1}{12}\left(-2 + \sqrt{4 + 3\sqrt{a}}\right).$$

Proof. Let $\nu \in \{1,2,3\}$, we have $c = c_\nu^\pm \equiv \pm 2\nu r \pmod{a}$. From (1.6), we see that $s \equiv 2, r \pmod{a}$. We also get that $b = ak \equiv 0 \pmod{a}$. Using Lemma 3.1, we have in all cases with the solutions in Type 1

$$cx_0m^2 + sz_0m \equiv bx_2l^2 + ry_2l \pmod{a}.$$

Hence, we get

$$\pm 4\nu rm^2 \pm 4m \equiv 2rl \pmod{a}, \quad \text{if} \quad s \equiv 2 \pmod{a}$$

and

$$\pm 4\nu m^2 \pm 2m - 2l \equiv 0 \pmod{\frac{a}{\gcd(a,r)}}, \quad \text{if} \quad s \equiv r \pmod{a}.$$

The case $s \equiv r \pmod{a}$. Recall that in our case $\gcd(a,r) = 2$. Thus,

$$\left|\pm 4\nu m^2 \pm 2m - 2l\right| \geq \frac{a}{\gcd(a,r)} = \frac{a}{2}.$$

Note also that $m \leq l$ and $\nu \leq 3$. Thus, we get $12l^2 + 4l \geq \frac{a}{2}$, which implies

$$l \geq \frac{1}{12}\left(-2 + \sqrt{4 + 6a}\right). \tag{3.3}$$

THE CASE $s \equiv 2 \pmod{a}$. In this case, by multiplying the congruence obtained by r and using the fact that $r^2 \equiv 4 \pmod{a}$, we get

$$\pm 16\nu m^2 \pm 4mr - 8l \equiv 0 \pmod{a}. \tag{3.4}$$

Because $r^2 \equiv 4 \pmod{a}$, we conclude that $r \equiv \pm 2 \pmod{a'}$ for some a' which is a divisor of a and $a' \geq \sqrt{a}$. It follows that

$$\pm 16\nu m^2 \pm 8m - 8l \equiv 0 \pmod{a'}.$$

Therefore, we deduce that

$$\left|\pm 16\nu m^2 \pm 8m - 8l\right| \geq a' \geq \sqrt{a}.$$

Using again $m \leq l$ and $\nu \leq 3$, we get

$$48l^2 + 16l \geq \sqrt{a},$$

which implies

$$l \geq \frac{1}{12}\left(-2 + \sqrt{4 + 3\sqrt{a}}\right). \tag{3.5}$$

Combining the inequalities (3.3) and (3.5), we obtain the desired inequality. This completes the proof. □

Let us denote

$$T_\tau + U_\tau\sqrt{ab} = \alpha^\tau = \left(\frac{r + \sqrt{ab}}{2}\right)^\tau, \tag{3.6}$$

where $(2T_\tau, 2U_\tau)$ is the τ-th positive integer solution to the Pell equation

$$T^2 - abU^2 = T^2 - (r^2 - 4)U^2 = 4.$$

It is easy to show by induction that

$$T_0 = 1,\ T_1 = \frac{r}{2},\ T_{\tau+2} = rT_{\tau+1} - T_\tau, \tag{3.7}$$

$$U_0 = 0,\ U_1 = \frac{1}{2},\ U_{\tau+2} = rU_{\tau+1} - U_\tau, \tag{3.8}$$

for $\tau \geq 0$. From (1.5), (1.6), (3.7), and (3.8), we have

$$s = s_\tau^{\pm} = 2T_\tau \pm 2aU_\tau \text{ and } t = t_\tau^{\pm} = \pm 2T_\tau + 2bU_\tau. \tag{3.9}$$

Considering congruences modulo r, we get

$$2T_\tau \equiv \pm 2 \pmod{r},\ 2U_\tau \equiv 0 \pmod{r} \quad \text{if} \quad \tau \equiv 0 \pmod{2},$$

and

$$2T_\tau \equiv 0 \pmod{r},\ 2U_\tau \equiv \pm 1 \pmod{r} \quad \text{if} \quad \tau \equiv 1 \pmod{2},$$

which implies that $s \equiv \pm 2, \pm a \pmod{r}$. The case $s \equiv \pm a \pmod{r}$ leads to a contradiction if $c = c_2^{\pm}$. Let us prove it. In this case, we have

$$s^2 = ac_2^{\pm} + 4 \equiv a^2 \pmod{r}.$$

Multiplying the above congruence by k and using the fact that $ka^2 = r^2 - 4 \equiv -4 \pmod{r}$, we get

$$kac_2^{\pm} \equiv -4k - 4 \pmod{r}. \tag{3.10}$$

Furthermore, from $c = c_2^{\pm}$ we have

$$kac_2^{\pm} \equiv 0 \pmod{r}. \tag{3.11}$$

Thus, combining (3.10) and (3.11), we get $4k + 4 \equiv 0 \pmod{r}$ and therefore

$$r \mid 4k + 4 \in \{12, 16, 28\},$$

which is not possible in our case since $r > 28$ by $b > 10^5$. In conclusion, we summarize what we have proved as the following result.

PROPOSITION 3.3. *Let* $c = c_2^{\pm} = (ab + 4)(a + b \pm 2r) \mp 4r$. *There is no* $D(4)$*-triple* $\{a, b, c\}$ *if* $s \equiv \pm a \pmod{r}$.

Therefore, for solutions of Type 2 with $s \equiv \pm 2 \pmod{r}$, we get the following key result.

LEMMA 3.4. *Assume that* $c = c_2^{\pm}$. *If the equation* $P_l = Q_m$ *has a solution* (l, m) *of Type* 2, *then we have*

$$m > \begin{cases} (3\sqrt{2} - 4)a/4, & k = 2, \\ (6 - 3\sqrt{3.1})a/4, & k = 3, \\ (5\sqrt{6} - 12)a/4, & k = 6. \end{cases}$$

PROOF. By induction, we obtain

$$P_l \equiv \pm 2, \pm a \pmod{r} \quad \text{and} \quad Q_m \equiv \frac{1}{2} amz_0 \pmod{r}.$$

Because $t = t_\tau^{\pm} = \pm 2T_\tau + 2bU_\tau \equiv \pm 2, \pm b \pmod{r}$ and $z_0 = \pm t$, we would get

$$Q_m \equiv \pm am, \pm 2m \pmod{r}.$$

• THE CASE $Q_m \equiv \pm 2m \pmod{r}$. Thus, first the equation $Q_m = P_l$ implies

$$\pm 2m \equiv \pm 2 \pmod{r},$$

which becomes $\pm m \pm 1 \equiv 0 \pmod{r/2}$ and then $m \geq r/2 - 1$. Secondly, we see that

$$\pm 2m \equiv \pm a \pmod{r},$$

which leads to $2m + a \geq r$. It follows that

$$m \geq \frac{1}{2}(r - a) > \frac{1}{2}(\sqrt{k} - 1)a, \quad \text{for } k \in \{2, 3, 6\}.$$

• THE CASE $Q_m \equiv \pm am \pmod{r}$. Combining this with $P_l \equiv \pm a \pmod{r}$, the equation $Q_m = P_l$ implies $am \equiv \pm a \pmod{r}$, which, using $\gcd(a, r) = 2$, gives $m \equiv \pm 1 \pmod{r/2}$ and $m \geq r/2 - 1$. Now, we use the fact that $P_l \equiv \pm 2 \pmod{r}$ to see that the equation $Q_m = P_l$ implies

$$ma \equiv \pm 2 \pmod{r}. \tag{3.12}$$

Multiplying the above congruence by ka and adding $4m$, we would get

$$m(ka^2 + 4) \equiv 4m \pm 2ka \pmod{r},$$

which gives

$$2ka \pm 4m \equiv 0 \pmod{r}. \tag{3.13}$$

Then, $m \geq \frac{1}{2}ka$ or the left hand side is positive and we get for $k = 2$ the possibilities:

$$4a + 4m = 3r, 4r, 5r, \ldots \tag{3.14}$$

$$4a - 4m = r, 2r. \tag{3.15}$$

From (3.14), we have $4a + 4m \geq 3r$ and using $r = \sqrt{2a^2 + 4} > a\sqrt{2}$ we get

$$m > \frac{1}{4}(3\sqrt{2} - 4)a. \tag{3.16}$$

In the case (3.15), we have $4a - 4m \leq 2r$. Since $r < a\sqrt{2.1}$, we obtain

$$m > \frac{1}{2}(2 - \sqrt{2.1})a. \tag{3.17}$$

We conclude that for $k = 2$, the relation (3.16) holds in all cases. The other cases ($k = 3, 6$) can be treated in the same way. So, we omit them. □

4. Proof of Theorem 1.4

In this section, we complete the proof of Theorem 1.4 in two subsections according to the values of c.

4.1. *Proof of Theorem 1.4 for $c = c_1^{\pm}, c_2^{\pm}$.* We start with the following case:

- The case $b = 2a$. From (1.3), we easily get the following relation

$$a = a_p = \frac{1}{\sqrt{2}}\left((3 + 2\sqrt{2})^p - (3 - 2\sqrt{2})^p\right), \tag{4.1}$$

which gives using Proposition 2.9, Lemmas 3.2 and 3.4 the following result.

Lemma 4.1. *i) For a $D(4)$-triple $\{a, 2a, c_1^{\pm}\}$ with $a = a_p$ ($p \geq 1$), if the equation $P_l = Q_m$ has a solution (l, m) with $m \geq 3$, then $p \leq 111$ and $l \leq 2.53 \cdot 10^{20}$.*

ii) For a $D(4)$-triple $\{a, 2a, c_2^{\pm}\}$ with $a = a_p$ ($p \geq 1$), if the equation $P_l = Q_m$ has a solution (l, m) in Type 1 with $m \geq 3$, then $p \leq 116$ and $l \leq 2.57 \cdot 10^{21}$. If the equation $P_l = Q_m$ has a solution (l, m) in Type 2 with $m \geq 3$, then $p \leq 28$ and $l \leq 2.81 \cdot 10^{20}$.

• THE CASE $b = 3a$. Using (1.3), we get

$$a = a_p = \frac{1}{\sqrt{3}}\left((2+\sqrt{3})^p - (2-\sqrt{3})^p\right). \tag{4.2}$$

Combining this with Proposition 2.9, Lemmas 3.2 and 3.4, we obtain the following result.

LEMMA 4.2. *i) For a $D(4)$-triple $\{a, 3a, c_1^{\pm}\}$ with $a = a_p$ $(p \geq 1)$, if the equation $P_l = Q_m$ has a solution (l, m) with $m \geq 3$, then $p \leq 149$ and $l \leq 2.55 \cdot 10^{20}$.*

ii) For a $D(4)$-triple $\{a, 3a, c_2^{\pm}\}$ with $a = a_p$ $(p \geq 1)$, if the equation $P_l = Q_m$ has a solution (l, m) in Type 1 with $m \geq 3$, then $p \leq 156$ and $l \leq 2.6 \cdot 10^{21}$. If the equation $P_l = Q_m$ has a solution (l, m) in Type 2 with $m \geq 3$, then $p \leq 37$ and $l \leq 2.74 \cdot 10^{20}$.

• THE CASE $b = 6a$. By (1.3), we have

$$a = a_p = \frac{1}{\sqrt{6}}\left((5+2\sqrt{6})^p - (5-2\sqrt{6})^p\right). \tag{4.3}$$

Combining this with Proposition 2.9, Lemmas 3.2 and 3.4 we obtain the following result.

LEMMA 4.3. *i) For a $D(4)$-triple $\{a, 6a, c_1^{\pm}\}$ with $a = a_p$ $(p \geq 1)$, if the equation $P_l = Q_m$ has a solution (l, m) with $m \geq 3$, then $p \leq 85$ and $l \leq 2.52 \cdot 10^{20}$.*

ii) For a $D(4)$-triple $\{a, 6a, c_2^{\pm}\}$ with $a = a_p$ $(p \geq 1)$, if the equation $P_l = Q_m$ has a solution (l, m) in Type 1 with $m \geq 3$, then $p \leq 89$ and $l \leq 2.56 \cdot 10^{21}$. If the equation $P_l = Q_m$ has a solution (l, m) in Type 2 with $m \geq 3$, then $p \leq 22$ and $l \leq 3 \cdot 10^{20}$.

For the remaining cases, we will use the following lemma which is a slight modification of the original version of Baker-Davenport reduction method (see [7, Lemma 5a]).

LEMMA 4.4. *Assume that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of κ such that $q > 6M$ and let*

$$\eta = \|\mu q\| - M \cdot \|\kappa q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequality

$$0 < l\kappa - m + \mu < AB^{-l}$$

in integers l and m with

$$\frac{\log(Aq/\eta)}{\log(B)} \leq l \leq M.$$

THE CASE $c = c_1^-$. Dividing $0 < \Lambda' < 11.7\alpha^{-2m}$ by $\log\alpha$ and using the fact that $\alpha^{-2m} < \beta^{-l}$ we get

$$0 < l\kappa - m + \mu < AB^{-l}, \tag{4.4}$$

where

$$\kappa := \frac{\log\beta}{\log\alpha}, \quad \mu := \frac{\log\gamma'}{\log\alpha}, \quad A := \frac{11.7}{\log\alpha}, \quad B := \beta.$$

THE CASE $c \in \{c_1^+, c_2^\pm\}$. Dividing $0 < \Lambda < 11.7\beta^{-2m}$ and $0 < \Lambda < 4.4a^2\beta^{-2m}$ by $\log\beta$ and using the fact that we have $\beta^{-2m} < \alpha^{-l}$ leads to an inequality of the form

$$0 < l\kappa - m + \mu < AB^{-l}, \tag{4.5}$$

where we consider solutions of Type 1

$$\kappa := \frac{\log\alpha}{\log\beta}, \quad \mu := \frac{\log\gamma}{\log\beta}, \quad A := \frac{11.7}{\log\beta}, \quad B := \alpha,$$

and for solutions of Type 2

$$\kappa := \frac{\log\alpha}{\log\beta}, \quad \mu := \frac{\log\gamma}{\log\beta}, \quad A := \frac{4.4a^2}{\log\beta}, \quad B := \alpha.$$

For the remaining proof, we use Mathematica to apply Lemma 4.4. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\eta > 0$, then we use the next convergent until we find the one that satisfies the conditions. For $c = c_1^\pm$ we get $l \le 4$ in each case $b = ka$, $k = 1, 2, 3$, and for $c = c_2^\pm$ we get $l \le 8$ in each case.

Therefore, in all cases, we can conclude that

$$3 \le m \le l \le 8. \tag{4.6}$$

Combining this with Lemma 3.2 and the relations (4.1), (4.2), and (4.3), for solutions in Type 1, we get

$$p \le \begin{cases} 8, & k = 2, \\ 11, & k = 3, \\ 6, & k = 6, \end{cases}$$

and therefore the equation $Q_m = P_l$ has no solutions in this range. Combining now (4.6) with Lemma 3.4, in all cases, we get $a \le 131$, which contradicts the fact that $b = ka > 10^5$, with $k = 2, 3, 6$. Finally, we need to observe the cases $m \in \{0, 1, 2\}$. For $m = 0$, we get $x = Q_0 = P_0 = 2$, which gives $d = 0$. By following what is done in the proof of [1, Lemma 3.3], one can easily see that the only solution of equation (2.14) if $m \in \{1, 2\}$ is $(l, m) = (\tau, 1)$ for $z_0 = t$ i.e. $x = Q_1 = P_{\tau+1}$. In this case we have

$$x = Q_1 = P_{\tau+1} = r(T_\tau \pm aU_\tau) + a(bU_\tau \pm T_\tau),$$

which implies $d = (x^2 - 4)/a = d_\pm$.

4.2. *Proof of Theorem 1.4 with $c = c_3^{\pm}$.* In the case $c = c_3^{\pm}$, recall that the problem is more complicated to solve if $s \equiv \pm a \pmod{r}$ by considering the equation (2.14). The difficulty lies in the fact that it is not easy to find a lower bound of l and m in terms of a in the equation (2.14) if $s \equiv \pm a \pmod{r}$. To overcome this situation, we will deal with this case by examining the equation $z = v_m = w_n$ using Lemma 2.2. Now, we will give the lower bounds of the indices m and n in the equation $z = v_m = w_n$, for $2 < n < m < 2n$ (where the relationship between m and n follows from [10, Lemma 5] if m and n have the same parity). First, we have the following result.

LEMMA 4.5. *i) If the equation $z = v_{2m} = w_{2n}$ has a solution (m, n) with $n > 1$, then $m > 0.495b^{-0.5}c^{0.5}$.*

ii) If the equation $z = v_{2m+1} = w_{2n+1}$ has a solution (m, n) with $n > 1$, then $m^2 > 0.0625b^{-1}c^{0.5}$.

PROOF. i) The statement follows from the proof of [4, Proposition 2.3]. Here we only have to use that $b > 10^5 > 10^4$.

ii) Using Lemma 2.2 in the case of odd indices and from [10, Lemma 12], we get

$$\pm\frac{1}{2}astm(m+1) + r(2m+1) \equiv \pm\frac{1}{2}bstn(n+1) + r(2n+1) \pmod{c}. \tag{4.7}$$

Because $(st)^2 \equiv 16 \pmod{c}$, we conclude that $st \equiv \pm 4 \pmod{c'}$ for some divisor c' of c with $c' \geq \sqrt{c}$. Also the $\pm$ sign means that one of the congruences is true. Hence, we get

$$\pm 2am(m+1) + r(2m+1) \equiv \pm 2bn(n+1) + r(2n+1) \pmod{c'}. \tag{4.8}$$

Let us now assume the opposite i.e., $m^2 \leq 0.0625b^{-1}c^{0.5}$. Then, it is easy to see that both sides of the congruence relation (4.8) are less than c' and they have the same sign. More precisely, we have

$$\max\{2am(m+1), r(2m+1), 2bn(n+1), r(2n+1)\} \leq 2bm(m+1)$$

and

$$2bm(m+1) < 4bm^2 \leq \frac{c'}{4}.$$

Therefore, we get

$$|\pm 2am(m+1) + r(2m+1)| < \frac{c'}{2} \quad \text{and} \quad |\pm 2bn(n+1) + r(2n+1)| < \frac{c'}{2}.$$

Note that in the case of the sign "$-$", the two quantities $\pm 2am(m+1) + r(2m+1)$ and $\pm 2bn(n+1) + r(2n+1)$ are negative and in the case of the sign "+" they are positive. Thus, we actually have the equations

$$\pm 2am(m+1) + r(2m+1) = \pm 2bn(n+1) + r(2n+1) \tag{4.9}$$

instead of a congruence. Notice that we assume that $b = ka$ with $k = 2,3,6$. Now considering congruence modulo a, we get

$$r(2m+1) \equiv r(2n+1) \pmod{a},$$

which implies

$$2m \equiv 2n \pmod{\frac{a}{2}}. \tag{4.10}$$

Since $c_3^{\pm} < 519a^5$, we get

$$2n < 2m \leq 2 \cdot 0.0625^{0.5} \cdot b^{-0.5} \cdot \left(519a^5\right)^{0.25} < \frac{a}{4}.$$

Therefore, the congruence (4.10) gives an equation of the form $2m = 2n$. Therefore, from (4.9), we get $m = n = 0$, which is a contradiction. This completes the proof. □

Now, we will combine the lower bounds for indices m and n together with the result obtained using Baker's theory of linear forms in logarithms to prove the main Theorem for large values of p. Using the main result in [5] the third author proved in [9] that $z = v_m = w_n$, for $n > 2$, implies

$$\frac{m}{\log(m+1)} < 6.543 \cdot 10^{15} \log^2 c. \tag{4.11}$$

Combining this with Lemma 4.5, in the case of even indices, we get

$$\frac{2 \cdot 0.495 b^{-0.5} c^{0.5}}{\log(2 \cdot 0.495 b^{-0.5} c^{0.5} + 1)} < 6.543 \cdot 10^{15} \log^2 c, \tag{4.12}$$

and in the case of odd indices, we get the inequality

$$\frac{2 \cdot 0.0625^{0.5} b^{-0.5} c^{0.25} + 1}{\log(2 \cdot 0.0625^{0.5} b^{-0.5} c^{0.25} + 2)} < 6.543 \cdot 10^{15} \log^2 c. \tag{4.13}$$

Therefore, using Maple, the solutions obtained for inequalities (4.12) and (4.13) are summarized in the following lemma.

LEMMA 4.6. *i) For the $D(4)$-triples $\{a, 2a, c_3^{\pm}\}$ with $a = a_p$ $(p \geq 1)$ defined in (4.1), if $z = v_{2m} = w_{2n}$ has a solution (m,n), then $p \leq 14$ and $m \leq 2.6 \cdot 10^{21}$ but if $z = v_{2m+1} = w_{2n+1}$ has a solution (m,n), then $p \leq 40$ and $m \leq 2.1 \cdot 10^{22}$.*

ii) For the $D(4)$-triples $\{a, 3a, c_3^{\pm}\}$ with $a = a_p$ $(p \geq 1)$ defined in (4.2), if $z = v_{2m} = w_{2n}$ has a solution (m,n), then $p \leq 19$ and $m \leq 3 \cdot 10^{21}$ but if $z = v_{2m+1} = w_{2n+1}$ has a solution (m,n), then $p \leq 54$ and $m \leq 2.2 \cdot 10^{22}$.

iii) For the $D(4)$-triples $\{a, 6a, c_3^{\pm}\}$ with $a = a_p$ $(p \geq 1)$ defined in (4.3), if $z = v_{2m} = w_{2n}$ has a solution (m,n), then $p \leq 11$ and $m \leq 2.6 \cdot 10^{21}$ but if $z = v_{2m+1} = w_{2n+1}$ has a solution (m,n), then $p \leq 30$ and $m \leq 2.1 \cdot 10^{22}$.

Now, it remains to see what happens for small values of p by applying Lemma 4.5. For this, we also need the inequality, which follows from $v_m = w_n, n > 2$ (that is [11, Lemma 9]),

$$0 < m \log\left(\frac{s+\sqrt{ac}}{2}\right) - n \log\left(\frac{t+\sqrt{bc}}{2}\right) + \log \frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})}$$
$$< 2ac\left(\frac{s+\sqrt{ac}}{2}\right)^{-2m}.$$

Using Lemma 4.6, we apply Lemma 4.4 considering $c = c_3^{\pm}$. Note that in the case of even indices we have $z_0 = z_1 = \pm 2$, $x_0 = y_1 = 2$ and in the case of odd indices we have $x_0 = y_1 = r$, $z_0 = \pm t$, $z_1 = \pm s$ and $z_0 z_1 > 0$. We have done the reduction using Mathematica. In all cases according to $b = ka$ with $k = 2, 3, 6$, after at most 2 steps of reduction, we see that $z = v_m = w_n$ implies $n \leq m \leq 2$. In these small ranges, it is not difficult to check that all solutions of $z = v_m = w_n$ will give the extension of $D(4)$-triple $\{a, b, c\}$ to a quadruple with $d = d_-$ or $d = d_+$. This completes the proof of Theorem 1.4.

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K. N. Adédji
Institut de Mathématiques et de Sciences Physiques
Université d'Abomey-Calavi
Bénin
E-mail: adedjnorb1988@gmail.com

M. Bliznac Trebješanin
Faculty of Science
University of Split
21000 Split
Croatia
E-mail: marbli@pmfst.hr

A. Filipin
Faculty of Civil Engineering
University of Zagreb
10000 Zagreb
Croatia
E-mail: filipin@grad.hr

A. Togbé
Department of Mathematics and Statistics
Purdue University Northwest
1401 S, U.S. 421, Westville IN 46391
USA
E-mail: atogbe@pnw.edu

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