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EQUIVALENT TOPOLOGIES ON THE CONTRACTING BOUNDARY

Vivian He
University of Toronto, Canada

Abstract. The contracting boundary of a proper geodesic metric space generalizes the Gromov boundary of a hyperbolic space. It consists of contracting geodesics up to bounded Hausdorff distances. Another generalization of the Gromov boundary is the κ–Morse boundary with a sublinear function κ. The two generalizations model the Gromov boundary based on different characteristics of geodesics in Gromov hyperbolic spaces. It was suspected that the κ–Morse boundary contains the contracting boundary. We will prove this conjecture: when $\kappa = 1$ is the constant function, the 1-Morse boundary and the contracting boundary are equivalent as topological spaces.

1. Introduction

There have been many attempts to construct a boundary on a proper geodesic metric space in order to generalize the Gromov boundary on a hyperbolic space. This paper studies the relation between two of such constructions: the (weakly) contracting boundary with the Cashen-Mackay topology [3] and the sublinearly Morse boundary [7, 8].

The *contracting boundary* on a proper geodesic metric space consists of *(weakly) contracting* geodesics up to bounded Hausdorff distances [4, 1]. The (weakly) contracting property was introduced to imitate the behaviour of geodesics in hyperbolic spaces. The space is sometimes referred to as the Morse boundary [5], as the Morse condition is equivalent to the (weakly) contracting condition [1]. Cashen and Mackay [3] defined a topology on this contracting boundary that is quasi-isometry invariant and metrizable when the space is the Cayley graph of a finitely generated group.

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The *sublinearly Morse boundary* is a modification of the Morse boundary by allowing the neighbourhoods to grow sublinearly [7, 8]. The sublinearly Morse boundary is the union of κ–Morse boundaries over all sublinear functions κ. Given a sublinear function κ, the topology on the κ–Morse boundary is quasi-isometry invariant and metrizable.

When $\kappa = 1$ is the constant function, the 1–Morse property on a quasi-geodesic is equivalent to the (weakly) contracting property [3, Theorem 2.2]. We will show in Lemma 2.7 that the equivalence relations defining the 1–Morse boundary and the contracting boundary are also equivalent, proving that the two boundaries are equal as sets. Moreover, we will show that the topologies are also equivalent.

THEOREM 1.1. *For a proper geodesic metric space X, the 1-Morse boundary $\partial_1 X$ and the contracting boundary $\partial_c X$ are equivalent as topological spaces.*

The κ–Morse boundary is metrizable for any proper geodesic metric space X and sublinear function κ [8, Theorem 4.10]. So we now have this immediate result.

COROLLARY 1.2. *The contracting boundary on a proper geodesic metric space is metrizable.*

This strengthens the result from [3] which shows the metrizability of the contracting boundary when X is the Cayley graph of a finitely generated group.

History. The Gromov boundary is a special case of the visual boundary on CAT(0) spaces. However, the quasi-isometry invariance property of the Gromov boundary fails on the visual boundary: Croke and Kleiner [2] constructed an example of two quasi-isometric CAT(0) spaces whose visual boundaries are not homeomorphic. This poses an issue in prescribing a boundary to a group, as different Cayley graphs of the group may have non-homeomorphic boundaries.

To fix this, Charney and Sultan [4] defined the contracting boundary of a complete CAT(0) space to be the subset of the visual boundary consisting of only (strongly) contracting geodesics. They defined a geodesic to be (strongly) contracting if the projection of any ball disjoint from the geodesic is uniformly bounded. Charney and Sultan showed that this boundary equipped with the direct limit topology is invariant under quasi-isometries.

On CAT(0) spaces, an equivalent formulation of the (strongly) contracting condition is the Morse condition [4]: a geodesic γ is Morse if any quasi-geodesic with endpoints on γ is within a bounded neighbourhood of γ. Using the Morse condition, Cordes [5] constructed the Morse boundary on proper geodesic metric spaces. Just as the contracting boundary, the Morse boundary with the direct limit topology is quasi-isomety invariant.

However, the direct limit topology on the Morse boundary is generally not first countable; this was shown by Murray [6]. On the same boundary, Cashen and Mackay [3] defined a different topology which is first countable, Hausdorff, and regular. Moreover, they showed that when the space is the Cayley graph of a finitely generated group, the boundary is metrizable. Cashen and Mackay's topology relies on a more general notion of contracting quasi-geodesics (see Definition 2.1). This notion was introduced by Arzhantseva, Cashen, Gruber, and Hume [1] as weakly contracting, which they showed to be equivalent to the Morse condition on proper geodesic metric spaces. Since Cashen and Mackay used the weakly contracting condition to define the topology, they referred to the Morse boundary as the contracting boundary, and weakly contracting sets as contracting sets in [3]. We will follow this notation.

Qing, Rafi, and Tiozzo [7] further generalized the contracting boundary to the sublinearly Morse boundary. They did this by allowing the neighbourhoods to grow sublinearly with respect to the distance from the origin. While preserving all the topological properties of the contracting boundary, this relaxation also encapsulates the asymptotic behaviour of random walks on spaces [7].

2. The contracting boundary and the κ–Morse boundary

In this section, we will define the contracting boundary and the κ–Morse boundary as introduced in [3, 7, 8]. We will then check that the contracting boundary is equal to the 1-Morse boundary as a set.

Let X be a proper geodesic metric space with base point $\mathfrak{o}$ and metric d_X. We say a function $\rho : [0,\infty) \to [1,\infty)$ is sublinear if $\lim_{x\to\infty} \frac{\rho(x)}{x} = 0$. For simplicity, we will also require ρ to be increasing and concave. The following definitions are equivalent.

Definition 2.1. *Let Z be a closed subset of X and $\pi_Z : X \to 2^Z$ be the closest point projection to Z. We say that Z is contracting if there is a sublinear function ρ_Z such that for all x and y in X,*

$$d(x,y) \le d(x,Z) \implies \operatorname{diam}(\pi_Z(x) \cup \pi_Z(y)) \le \rho(d(x,Z)).$$

Definition 2.2. *Let Z be a closed subset of X, we say Z is Morse if there exists a proper function $m_Z : [1,\infty) \times [0,\infty) \to \mathbb{R}$ for every $q \ge 1$ and every $Q \ge 0$, every (q,Q)–quasi-geodesic with endpoints in Z is contained in the $m_Z(q,Q)$-neighbourhood of Z.*

The equivalence of contracting sets and Morse sets is proven in [3]. The notion of Morse is generalized to κ–Morse, which is then used to define the κ–Morse boundary. For the definition of the contracting boundary, we continue working with the contracting definition.

Definition 2.3. *The contracting boundary of X, $\partial_c X$, is defined to be the set of equivalence classes of contracting quasi-geodesic rays based at $\mathfrak{o}$.*

Two contracting quasi-geodesics are equivalent if they are a bounded Hausdorff distance apart.

We now define the κ–Morse boundary. For any point $p \in X$, we use $||p||$ to denote $d_X(\mathfrak{o}, p)$. Given a quasi-geodesic ray α starting at $\mathfrak{o}$, let t_r be the first time that $||\alpha(t_r)|| = r$. We use α_r to denote $\alpha(t_r)$, and $\alpha|_r$ to denote $\alpha([0, t_r])$.

Given a sublinear function κ, the κ–Morse boundary introduced in [7] and [8] is attained by relaxing the Morse set. We first loosen the definition of a neighbourhood to allow a κ multiplicative error:

$$\mathcal{N}_\kappa(Z, m) := \{x \in X : d_X(x, Z) \leq m \cdot \kappa(||x||)\}.$$

When $\kappa = 1$ this is just the usual m–neighbourhood.

Definition 2.4. *We say Z is κ–Morse if there exists a proper function $m_Z : [1, \infty) \times [0, \infty) \to \mathbb{R}$ such that for every $q \geq 1$ and every $Q \geq 0$, every (q, Q)–quasi-geodesic $\beta : [s, t] \to X$ with endpoints on Z satisfies*

$$\beta[s, t] \subset \mathcal{N}_\kappa(Z, m_Z(q, Q)).$$

When $\kappa = 1$, this is equivalent to Definition 2.2. We will also work with the following definition of κ–Morse (sometimes called strongly Morse). When Z is a quasi-geodesic, the two definitions of κ–Morse are equivalent [8].

Definition 2.5. *Let κ be a concave sublinear function. We say Z is κ–Morse if there is a proper function $m_Z : \mathbb{R}^2 \to \mathbb{R}$ such that for any sublinear function κ' and any $r > 0$, there exists R such that for any (q, Q)–quasi-geodesic ray β with $m_Z(q, Q) \leq \frac{r}{2\kappa(r)}$,*

$$d_X(\beta_R, Z) \leq \kappa'(R) \implies \beta|_r \subset N_\kappa(Z, m_Z(q, Q)).$$

We call $m_Z(q, Q)$ the Morse gauge function. We will assume that $m_Z(q, Q) \geq \max(q, Q)$.

Definition 2.6. *The κ–Morse boundary, $\partial_\kappa X$, is defined to be the set of all equivalence classes of κ–Morse quasi-geodesic rays based at $\mathfrak{o}$. Two κ–Morse quasi-geodesic rays α and β are equivalent if they sublinearly fellow travel each other:*

$$\lim_{r \to \infty} \frac{d_X(\alpha_r, \beta_r)}{r} = 0.$$

From the definition we immediately have the following lemma.

Lemma 2.7. *When $\kappa = 1$, $\partial_1 X = \partial_c X$ as sets.*

Proof. When $\kappa = 1$, the Definition 2.4 of 1-Morse and the Definition 2.2 of Morse coincide. It remains to be shown that the equivalence relations are the same. Let α and β be 1-Morse quasi-geodesics with quasi-geodesic constants (q_1, Q_1) and (q_2, Q_2). If they are a bounded Hausdorff distance away, then

$$\lim_{r\to\infty} \frac{d_X(\alpha_r, \beta_r)}{r} = 0.$$

Conversely, if α and β fellow travel each other, then the function $\kappa'(R) := d_X(\alpha_X, \beta_X)$ is sublinear. Then for all $R > 0$,

$$d_X(\beta_R, \alpha) \leq \kappa'(R).$$

By Definition 2.5 for any $r \geq 2m_\alpha(q_2, Q_2)$,

$$\beta|_r \subset N_1(\alpha, m_\alpha(q_2, Q_2)).$$

Similarly, for any $r \geq 2m_\beta(q_1, Q_1)$,

$$\alpha|_r \subset N_1(\beta, m_\beta(q_1, Q_1)).$$

Since r can be arbitrarily large, the Hausdorff distance between α and β is bounded by the maximum of $m_\alpha(q_2, Q_2)$ and $m_\beta(q_1, Q_1))$. This proves the equivalence. ∎

3. Topologies

We now discuss the topologies on the contracting boundary and the κ–Morse boundary introduced in [3, 7, 8]. We will show that they are equivalent in the next section.

We first define the topology on the contracting boundary. Given a sublinear function ρ and constants $q \geq 1$, $Q \geq 0$, define

$$\kappa(\rho, q, Q) = \max\left\{3q, 3Q^2, 1 + \inf\left\{R > 0 \mid \forall r \geq R, 3q^2\rho(r) < r\right\}\right\}.$$

The constant κ is defined so that it satisfies the property that for $r \geq \kappa(\rho, L, A)$,

$$r - L^2\rho(r) - A \geq \frac{1}{3}r \geq L^2\rho(r).$$

This inequality proves the following theorem ([3, Theorem 4.2]), which we will use in our proof of equivalent topologies.

Theorem 3.1 (Quasi-geodesic image theorem). *Let Z be ρ-contracting. Let $\alpha : [0, T] \to X$ be a continuous (q, Q)–quasi-geodesics segment. If $d(\alpha, Z) \geq \kappa(\rho, q, Q)$ then*

$$\begin{aligned}&\operatorname{diam} \pi(\alpha(0)) \cup \pi(\alpha(T)) \\ &\qquad \leq \frac{q^2+1}{q^2}(Q + d(\alpha(T), Z)) + \frac{q^2-1}{q^2} d(\alpha(0), Z) + 2\rho(d(\alpha(0), Z)).\end{aligned}$$

Let $\mathbf{b} \in \partial_c X$ and let $b \in \mathbf{b}$ be a geodesic in the equivalence class (the existence of such a geodesic is due to [3, Lemma 5.2]). Let ρ_b be a sublinear function such that b is ρ_b-contracting. The topology is defined by the following open neighbourhood $U(\mathbf{b}, r)$.

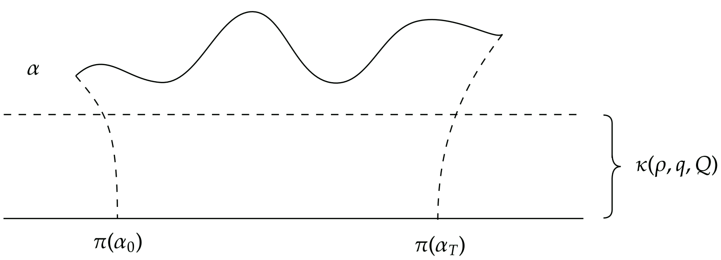

Figure 1. Quasi-geodesic image theorem

Definition 3.2. *Let $r > 0$, $U(\mathbf{b}, r)$ is defined to be the set of points $\mathbf{a} \in \partial X$ such that for all $Q \geq 1$ and $q \geq 0$ and every continuous (q, Q)–quasi-geodesic ray $\alpha \in \mathbf{a}$, we have*

$$d(\alpha, b \cap \mathcal{N}_r^c \mathfrak{o}) \leq \kappa(\rho_b, q, Q).$$

Here, $\mathcal{N}_r \mathfrak{o}$ stands for the r neighbourhood of $\mathfrak{o}$.

We now introduce the open neighbourhood in $\partial_\kappa X$ that defines the topology in [8].

Definition 3.3. *Let $r > 0$ and $\mathbf{b} \in \partial_\kappa X$. Let $b \in \mathbf{b}$ be a κ–Morse quasi-geodesic ray. Define $U_\kappa(\mathbf{b}, r)$ to be the set of points $\mathbf{a} \in \partial_\kappa X$ such that for any (q, Q)–quasi-geodesic ray $\alpha \in \mathbf{a}$,*

$$m_b(q, Q) \leq \frac{r}{2\kappa(r)} \implies \alpha|_r \subset N_\kappa(b, m_b(q, Q)).$$

4. Proof of equivalent topologies

We now show that U_1 and U define equivalent topologies on $\partial_1 X = \partial_c X$. We will prove both directions of containment for the open neighbourhoods. Let $\mathbf{b} \in \partial_c X$ and $r > 0$.

Proposition 4.1. *Given $U(\mathbf{b}, r)$, there is $R > 0$ such that $U(\mathbf{b}, R) \subset U_1(\mathbf{b}, r)$.*

Proof. Let b be a geodesic in the class of $\mathbf{b}$. Let

$$K = \sup_{m_b(q,Q) \leq r/2} \kappa(\rho_b, q, Q),$$

where ρ_b is the sublinear function corresponding to the geodesic b as in Definition 2.1. K is well defined since $\kappa(\rho_b, q, Q)$ is bounded when $\max\{q, Q\} \leq m_b(q, Q) \leq r/2$. In particular K is sublinear with respect to r.

Since b is 1-Morse, there exists R such that for any (q, Q)–quasi-geodesic ray α with $m_b(q, Q) \leq r/2$, if $d_X(\alpha|_R, b) \leq K$ then $\alpha|_r \subset \mathcal{N}(b, m_b(q, Q))$. So $U(\mathbf{b}, R) \subset U_1(\mathbf{b}, r)$. ∎

For the other direction, we will make use of Theorem 3.1.

PROPOSITION 4.2. *Given* $\mathbf{b} \in \partial X, r > 0$*, there exists* $R > 0$ *such that* $U_1(\mathbf{b}, R) \subset U(\mathbf{b}, r)$*.*

PROOF. Choose R sufficiently large so that

$$R > \max(r, 1), \qquad R - 4\sqrt{R} > r, \qquad \rho_b(\sqrt{R}/2) < \sqrt{R}/2,$$

and

$$R > 4 \max_{q' \leq \sqrt{2r/3},\ Q' \leq 2r/3} m_b(q', Q')^2.$$

The third inequality holds for sufficiently large R because ρ_b is sublinear.

Let $b \in \mathbf{b}$ be a geodesic in the class. For any $\mathbf{a} \in U_1(\mathbf{b}, R)$, let $\alpha \in \mathbf{a}$ be a continuous (q, Q)–quasi-geodesic. Recall that we can assume $\max(q, Q) < m_b(q, Q)$.

First consider the case where $m_b(q, Q) \leq \sqrt{R}/2$. In particular,

$$m_b(q, Q) \leq \sqrt{R}/2 < R/2.$$

So by definition, $\alpha|_R \subset \mathcal{N}(b, m_b(q, Q))$. For contradiction, suppose that for all $r \leq t \leq R$,

$$d_X(\alpha(t), b) > \kappa(\rho_b, q, Q).$$

By Theorem 3.1,

$$\begin{aligned}
&\operatorname{diam} \pi(\alpha_r) \cup \pi(\alpha_R) \\
&\quad \leq \frac{q^2+1}{q^2}(Q + d(\alpha_R, Z)) + \frac{q^2-1}{q^2} d(\alpha_r, Z) + 2\rho_b(d(\alpha_r, Z)) \\
&\quad \leq \frac{q^2+1}{q^2}(Q + m_b(q, Q)) + \frac{q^2-1}{q^2} m_b(q, Q) + 2\rho_b(m_b(q, Q)) \\
&\quad = \frac{q^2+1}{q^2} Q + 2m_b(q, Q) + 2\rho_b(m_b(q, Q)) \\
&\quad \leq 2m_b(q, Q) + 2m_b(q, Q) + 2\rho_b(\sqrt{R}/2) \\
&\quad \leq 3\sqrt{R}.
\end{aligned}$$

On the other hand, the projection can be bounded below by

$$\operatorname{diam} \pi(\alpha_r) \cup \pi(\alpha_R) \geq (R - r) - d_X(\alpha_r, \pi(\alpha_r)) - d_X(\alpha_R, \pi(\alpha_R)) \geq R - r - \sqrt{R}.$$

Combining the two inequalities, we have that

$$3\sqrt{R} \geq R - r - \sqrt{R}.$$

But this contradicts the assumption that $R - 4\sqrt{R} > r$. We conclude that $d_X(\alpha_t, b) \leq \kappa(\rho_b, q, Q)$ for some $r \leq t \leq R$.

We are left with the case where $m_b(q, Q) > \sqrt{R}/2$. In this case,

$$m_b(q, Q) > \sqrt{R}/2 > \max_{q' \leq \sqrt{2r/3}, Q' \leq 2r/3} m_b(q', Q')^2.$$

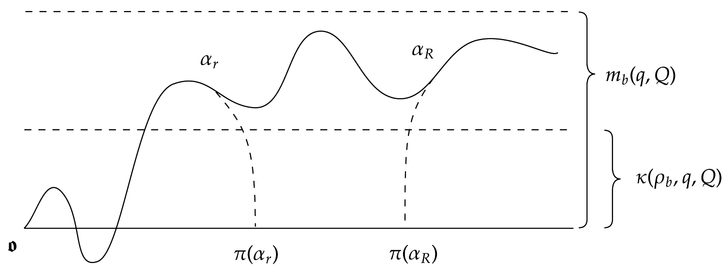

FIGURE 2. Proof of Proposition 4.2 in the case where $m_b(q, Q) \leq \sqrt{R}/2$.

Then either $q > \sqrt{2r/3}$ or $Q > 2r/3$, in which case

$$\kappa(\rho_b, q, Q) \geq \max\{3Q, 3q^2\} > 2r \geq d(\alpha, b \cap N_r^c o).$$

The last inequality holds because

$$d(\alpha, b \cap N_r^c o) \leq d(\alpha_r, o) + d(b_r, o) = 2r.$$

We conclude that for sufficiently large R, $d(\alpha, b \cap N_r^c o) \leq \kappa(\rho_b, q, Q)$. Hence $U_1(\mathbf{b}, R) \subset U(\mathbf{b}, r)$. ∎

We have proven Theorem 1.1 by combining Lemma 2.7, Proposition 4.1, and Proposition 4.2.

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V. He
Department of Mathematics
University of Toronto
Canada
E-mail: vivian.he@mail.utoronto.ca

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