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SEMI-PARALLEL HOPF REAL HYPERSURFACES IN THE COMPLEX QUADRIC

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Abstract. In this paper, we introduce the new notion of semi-parallel real hypersurface in the complex quadric Q^m. Moreover, we give a nonexistence theorem for semi-parallel Hopf real hypersurfaces in the complex quadric Q^m for $m \geq 3$.

1. Introduction

In [4], Deprez initiated the study of semi-parallel or semi-symmetric submanifolds. A submanifold M in a Riemannian manifold is said to be *semi-parallel* (or also called *semi-symmetric*) if the second fundamental form h satisfies

$$R \cdot h = 0 \tag{†}$$

i.e. $R(X, Y) \cdot h = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})h = 0$ for all tangent vector fields X and Y on M, where the curvature tensor R of the van der Waerden-Bortolotti connection ∇ of M acts as a derivation on h, that is,

$$R(X, Y)(h(Z, W)) = (R(X, Y)h)(Z, W) + h(R(X, Y)Z, W) + h(Z, R(X, Y)W)$$

for any tangent vector fields X, Y, Z and W on M. This notion is an extrinsic analogue for semi-symmetric spaces, i.e. Riemannian manifolds for which $R \cdot R = 0$, that is, $R(X, Y) \cdot R = 0$. Also, the notion of semi-parallel submanifolds is a generalization of parallel submanifolds, i.e. submanifolds for which $\nabla h = 0$. In [4], Deprez showed that a submanifold M in Euclidean space $\mathbb{E}^{m+1}$ is semi-parallel implies that (M, g) is semi-symmetric. For more details

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on semi-symmetric spaces, we refer the readers to [29, 30] and references therein.

Deprez mainly paid attention to the case of semi-parallel immersions in Euclidean space $\mathbb{E}^{m+1}$ (see [4, 5]). Later, Dillen [6] showed that a semi-parallel hypersurface in non-flat real space forms $\mathbb{R}^{m+1}(c)$, $c \neq 0$, are flat surfaces, hypersurfaces with parallel Weigarten endomorphism or rotation hypersurfaces of certain helices.

Niebergall and Ryan [18] studied real hypersurfaces in non-flat complex two-dimensional complex space forms $M^2(c)$, $c \neq 0$. As an extension of this result, Ortega [19] proved that there are no semi-parallel real hypersurfaces in non-flat complex space forms $M^m(c)$, $c \neq 0$ of complex dimension $m \geq 2$. In [26, 27], Romero gave some examples of indefinite complex Einstein hypersurfaces of the indefinite complex flat space, which are not locally symmetric. Wang [36] studied a similar problem for semi-symmetric almost coKähler 3-manifolds.

On the other hand, as a typical model space of complex Grassmann manifolds of rank 2, we can consider the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$, which is the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. It is the unique compact irreducible Riemannian symmetric space with both a Kähler structure J and a quaternionic Kähler structure $\mathcal{J}$ (see [17, 37, 38]). Semi-parallel real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ were studied by Hwang, Lee and Woo [8] and Loo [16], independently. By Loo's result, we obtain a non-existence theorem as follows.

Theorem A. *There does not exist a semi-parallel real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ for $m \geq 3$.*

Motivated by these results, in this paper we want to classify semi-parallel real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. The complex quadric Q^m which is a complex hypersurface in the complex projective space $\mathbb{C}P^{m+1}$ can be regarded as a kind of real Grassmann manifold of compact type with rank 2 (see [1, 2, 7, 10]). Moreover, Q^m admits two important geometric structures, so-called a real structure A and a complex structure J which anti-commute with each other, that is, $AJ = -JA$. By using the method of Lie algebra in [11], the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see also [7, 25]).

On the complex quadric there exists a remarkable geometric structure $\mathfrak{A}$ which is a parallel rank 2 vector bundle, which is given by the set of all complex conjugations defined on Q^m, that is, $\mathfrak{A}_{[z]} = \{A_{\lambda\bar{z}} \mid \lambda \in S^1 \subset \mathbb{C}\}$ for any point $[z]$ of Q^m. Then $\mathfrak{A}_{[z]}$ becomes a parallel rank 2-subbundle of End $T_{[z]}Q^m$, $[z] \in Q^m$. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m. Here the notion of parallel vector bundle $\mathfrak{A}$ means that $(\bar{\nabla}_X A)Y = q(X)JAY$ for any vector fields X and Y on Q^m, where $\bar{\nabla}$ and q denote a connection and a certain 1-form defined on $T_{[z]}Q^m$, $[z] \in Q^m$ respectively (see [28]).

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called *singular* if it is tangent to more than one maximal flat in Q^m. Since Q^m is a Hermitian symmetric space of rank 2, there are two types of singular tangent vectors for the complex quadric Q^m: Let $V(A) = \{X \in T_{[z]}Q^m \mid AX = X\}$ and $JV(A) = \{X \in T_{[z]}Q^m \mid AX = -X\}$ be the $(+1)$-eigenspace and (-1)-eigenspace for the involution A on $T_{[z]}Q^m$ for $[z] \in Q^m$.

(a) If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A) = \{X \in T_{[z]}Q^m | \, AX = X\}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-*principal*.
(b) If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors Z_1, $Z_2 \in V(A)$ such that $W/||W|| = (Z_1 + JZ_2)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-*isotropic*.

Let (ϕ, ξ, η, g) be the almost contact metric structure induced on M by the Kähler structure of Q^m. We say that M is a contact hypersurface of a Kähler manifold if there exists an everywhere nonzero smooth function κ such that $d\eta(X, Y) = 2\kappa g(\phi X, Y)$ holds on M. It can be easily verified that a real hypersurface M is contact if and only if there exists an everywhere nonzero constant function κ on M such that $S\phi + \phi S = 2\kappa\phi$, where S is the shape operator of M with respect to the normal vector field N that allows us to define $\xi = -JN$.

From this property, we naturally obtain that a contact real hypersurface M of a Kähler manifold is Hopf. The notion of *Hopf* means that the Reeb vector field ξ of M is principal by the shape operator S of M, that is, $S\xi = g(S\xi, \xi)\xi = \alpha\xi$. When the Reeb (curvature) function $\alpha = g(S\xi, \xi)$ identically vanishes on M, we say that M has vanishing geodesic Reeb flow. Otherwise, we say that M has non-vanishing geodesic Reeb flow.

A typical characterization of contact real hypersurfaces in the complex quadric Q^m was introduced in Berndt and Suh [2] as follows.

Theorem B. *Let M be a connected orientable real hypersurface with constant mean curvature in the complex quadric Q^m, $m \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of a tube around the m-dimensional sphere S^m which is embedded in Q^m as a real form of Q^m.*

Hereafter, we will call such a real hypersurface given in Theorem 1 *a tube of type* (B) and denote such a model space $(\mathcal{T}_B)$.

Related to the study of Hopf real hypersurfaces in Q^m, recently many characterizations have been investigated by several differential geometers from various viewpoints (see [2, 12, 13, 20, 21, 23, 31], etc.). In [14], Lee and Suh gave a characterization of Hopf real hypersurfaces in the complex quadric Q^m as follows.

Theorem C ([14]). *Let M be a Hopf real hypersurface in the complex quadric Q^m for $m \geq 3$. Then the unit normal vector field N of M is $\mathfrak{A}$-principal if and only if M is locally congruent to an open part of a tube around the m-dimensional sphere S^m which is totally real and totally geodesic in Q^m.*

Under these background and motivations, in this paper we want to classify semi-parallel Hopf real hypersurfaces in the complex quadric Q^m. In order to do this, we first prove the following result.

Theorem 1. *Let M be a semi-parallel Hopf real hypersurface in the complex quadric Q^m for $m \geq 3$. Then, the unit normal vector field N of M in Q^m is singular, that is, either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

Then we can assert a non-existence result of semi-parallel Hopf real hypersurfaces in Q^m, $m \geq 3$, as follows.

Theorem 2. *There does not exist any semi-parallel Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$.*

On the other hand, as mentioned above, the notion of semi-parallel hypersurfaces in Kähler manifolds is a natural generalization of parallel hypersurfaces. From such a viewpoint, we introduce the following result given by Suh as a corollary of Theorem 2.

Corollary A ([31]). *There does not exist any parallel Hopf real hypersurface in the complex quadric Q^m for $m \geq 3$.*

The present paper is organized as follows: in Section 2 we review the geometric structure of complex quadric Q^m including its Riemannian curvature tensor $\bar{R}$. In Section 3, by using the properties of complex structure J and real structure $A \in \mathfrak{A}$ given on Q^m, the equations of Gauss and Codazzi could be derived from the curvature tensor $\bar{R}$ of Q^m. Moreover, in this section we introduce some important results for a Hopf real hypersurface with singular unit normal vector field in Q^m. In Section 4, we study semi-parallel Hopf real hypersurfaces in Q^m. Moreover, we show that such real hypersurfaces have a singular unit normal vector field, as mentioned in Theorem 1. By means of this result, in Section 5 we give a complete proof of Theorem 2.

2. The complex quadric

For more background to this section we refer to [9, 11, 13, 22, 24, 25, 32, 33, 34]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $z_1, \ldots, z_{m+2}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric. For a nonzero vector $z \in \mathbb{C}^{m+2}$ we denote by $[z]$ the complex span of z, that is, $[z] = \mathbb{C}z = \{\lambda z \,|\, \lambda \in S^1 \subset \mathbb{C}\}$. Note that by definition $[z]$ is a point in $\mathbb{C}P^{m+1}$. For each $[z] \in Q^m \subset \mathbb{C}P^{m+1}$ we identify $T_{[z]}\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus \mathbb{C}z$ of $\mathbb{C}z$ in $\mathbb{C}^{m+2}$ (see Kobayashi and Nomizu [11]). The tangent space $T_{[z]}Q^m$ can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus (\mathbb{C}z \oplus \mathbb{C}\rho)$ of $\mathbb{C}z \oplus \mathbb{C}\rho$ in $\mathbb{C}^{m+2}$, where $\rho \in \nu_{[z]}Q^m$ is a normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point $[z]$.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0, \dots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

For a unit normal vector ρ of Q^m at a point $[z] \in Q^m$ we denote by $A = A_\rho$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to ρ. The shape operator is an involution on the tangent space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_\rho) \oplus JV(A_\rho),$$

where $V(A_\rho)$ is the $(+1)$-eigenspace and $JV(A_\rho)$ is the (-1)-eigenspace of A_ρ. Geometrically this means that the shape operator A_ρ defines a real structure on the complex vector space $T_{[z]}Q^m$, or equivalently, is a complex conjugation on $T_{[z]}Q^m$. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations. There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $[z] \in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m. These real forms are congruent to each other under action of the center SO_2 of the isotropy subgroup of SO_{m+2} at $[z]$. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. In this way the family $\mathfrak{A}$ of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms S^m of Q^m containing $[z]$, and the subspaces $V(A)$ in $T_{[z]}Q^m$ correspond to the tangent spaces $T_{[z]}S^m$ of the real forms S^m of Q^m.

The Gauss equation for Q^m in $\mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J

and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned}\bar{R}(U,V)W = {} & g(V,W)U - g(U,W)V + g(JV,W)JU - g(JU,W)JV \\ & - 2g(JU,V)JW + g(AV,W)AU \\ & - g(AU,W)AV + g(JAV,W)JAU - g(JAU,W)JAV\end{aligned} \tag{2.1}$$

for any tangent vector fields U, V, and W on Q^m. It is well known that for every unit tangent vector $U \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors Z_1, $Z_2 \in V(A)$ such that

$$U = \cos(t)Z_1 + \sin(t)JZ_2$$

for some $t \in [0, \pi/4]$ (see [25]). The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $0 < t < \pi/4$ then the unique maximal flat containing U is $\mathbb{R}Z_1 \oplus \mathbb{R}JZ_2$.

3. Real hypersurfaces in Q^m

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure (see [3]). By using the Gauss and Weingarten formulas the left-hand side of (2.1) becomes, for any tangent vector fields X, Y, and Z on M

$$\begin{aligned}\bar{R}(X,Y)Z = {} & R(X,Y)Z - g(SY,Z)SX + g(SX,Z)SY \\ & + \big\{g((\nabla_X S)Y,Z) - g((\nabla_Y S)X,Z)\big\}N,\end{aligned}$$

where R and S denote the Riemannian curvature tensor and the shape operator of M in Q^m, respectively.

Note that $JX = \phi X + \eta(X)N$ and $JN = -\xi$, where ϕX is the tangential component of JX and N is a (local) unit normal vector field of M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker \eta$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$. Moreover, since the complex quadric Q^m has also a real structure A, we decompose AX into its tangential and normal components for a fixed $A \in \mathfrak{A}_{[z]}$ and $X \in T_{[z]}M$:

$$AX = BX + \delta(X)N$$

where BX denotes the tangential component of AX and $\delta(X) = g(AX,N) = g(X,AN)$.

As mentioned in Section 2, since the normal vector N belongs to $T_{[z]}Q^m$, $[z] \in M$, we can choose $A \in \mathfrak{A}_{[z]}$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors Z_1, $Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$ (see Proposition 3 in [25]). Note that t is a function on M. If $t = 0$, then $N = Z_1 \in V(A)$, therefore we see that N becomes an $\mathfrak{A}$-principal tangent vector. On the other hand, if $t = \frac{\pi}{4}$, then $N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$. That is, N is an $\mathfrak{A}$-isotropic tangent vector of Q^m. In addition, since $\xi = -JN$, we have

$$\begin{cases} \xi &= -JN = \sin(t)Z_2 - \cos(t)JZ_1, \\ AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1 \end{cases} \tag{3.1}$$

for orthonormal vectors Z_1 and Z_2 in $V(A)$. This implies

$$\delta(\xi) = g(A\xi, N) = g(\xi, AN) = 0. \tag{3.2}$$

Here we calculate it in detail

$$g(A\xi, N) = g(\sin(t)Z_2 + \cos(t)JZ_1, \cos(t)Z_1 + \sin(t)JZ_2) = 0,$$

where we have used that Z_1 and Z_2 are orthonormal vectors in $V(A)$ such that $g(Z_1, Z_2) = 0$ and J the Kähler structure satisfying

$$g(Z_1, JZ_1) = g(Z_2, JZ_2) = g(JZ_1, JZ_2) = 0.$$

Accordingly, we can assert that the vector field $A\xi$ is tangent to M, regardless of singular normal vector field N (see [2, 35]). From this fact and the anti-commuting property $JA = -AJ$, together with $JN = -\xi$, we get

$$AN = AJ\xi = -JA\xi = -\phi A\xi - g(A\xi, \xi)N,$$

which implies

$$\delta(X) = g(AX, N) = g(AN, X) = -g(\phi A\xi, X) \tag{3.3}$$

for any tangent vector field X on M. By using this formula and $A\xi = B\xi$, we obtain

$$\begin{aligned} JAX &= J(BX + g(AX, N)N) \\ &= \phi BX + g(BX, \xi)N - g(X, AN)\xi \\ &= \phi BX + g(\phi A\xi, X)\xi + g(A\xi, X)N \end{aligned}$$

for all $X \in TM$. In addition, from (3.1) we also obtain that

$$g(A\xi, \xi) = -g(AN, N) = -\cos(2t) \quad \left(0 \leq t \leq \frac{\pi}{4}\right)$$

on M. Using the formulas mentioned above and taking the tangential and normal components of (2.1) yields

$$\begin{aligned} &R(X,Y)Z - g(SY, Z)SX + g(SX, Z)SY \\ &\quad = g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &\qquad - 2g(\phi X, Y)\phi Z + g(BY, Z)BX - g(BX, Z)BY \\ &\qquad + g(\phi BY, Z)\phi BX - g(\phi BX, Z)\phi BY \\ &\qquad + g(\phi A\xi, Y)\eta(Z)\phi BX - g(\phi A\xi, X)\eta(Z)\phi BY \\ &\qquad + g(\phi BY, Z)g(\phi A\xi, X)\xi - g(\phi BX, Z)g(\phi A\xi, Y)\xi \end{aligned} \tag{3.4}$$

and

$$(\nabla_X S)Y - (\nabla_Y S)X$$
$$\begin{aligned}
&= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi - g(\phi A\xi, X)BY \\
&\quad + g(\phi A\xi, Y)BX + g(A\xi, X)\phi BY + g(A\xi, X)g(\phi A\xi, Y)\xi \\
&\quad - g(A\xi, Y)\phi BX - g(A\xi, Y)g(\phi A\xi, X)\xi,
\end{aligned} \tag{3.5}$$

which are called the equations of Gauss and Codazzi, respectively.

At each point $[z] \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of $T_{[z]}M$, $[z] \in M$ as follows:

$$\mathcal{Q}_{[z]} = \{X \in T_{[z]}M \mid AX \in T_{[z]}M \text{ for all } A \in \mathfrak{A}_{[z]}\}.$$

It is known that if $N_{[z]}$ is $\mathfrak{A}$-principal, then $\mathcal{Q}_{[z]} = \mathcal{C}_{[z]}$ (see [31]).

We now assume that M is a Hopf hypersurface in the complex quadric Q^m. Then the shape operator S of M in Q^m satisfies $S\xi = \alpha\xi$ with the Reeb curvature function $\alpha = g(S\xi, \xi)$ on M. By Codazzi equation (3.5), we obtain the following lemma.

Lemma 3.1 ([2]). *Let M be a Hopf hypersurface in Q^m for $m \geq 3$. Then we obtain*

$$\begin{aligned}
X\alpha &= (\xi\alpha)\eta(X) - 2g(A\xi, \xi)g(\phi A\xi, X) \\
&= (\xi\alpha)\eta(X) + 2g(A\xi, \xi)g(X, AN)
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&2S\phi SX - \alpha\phi SX - \alpha S\phi X - 2\phi X - 2g(X, \phi A\xi)A\xi \\
&\quad + 2g(X, A\xi)\phi A\xi + 2g(X, \phi A\xi)g(\xi, A\xi)\xi - 2g(\xi, A\xi)\eta(X)\phi A\xi = 0
\end{aligned} \tag{3.7}$$

for any tangent vector fields X and Y on M.

Remark 3.2. From (3.6) we know that if M has vanishing geodesic Reeb flow (or constant Reeb curvature), then the normal vector field N is singular. In fact, under this assumption (3.6) becomes $g(A\xi, \xi)g(X, AN) = 0$ for any tangent vector field X on M. Since $g(A\xi, \xi) = -\cos(2t)$, the case of $g(A\xi, \xi) = 0$ implies that N is $\mathfrak{A}$-isotropic. Besides, if $g(A\xi, \xi) \neq 0$, that is, $g(AN, X) = 0$ for all $X \in TM$, then

$$AN = \sum_{i=1}^{2m} g(AN, e_i)e_i + g(AN, N)N = g(AN, N)N$$

for any basis $\{e_1, e_2, \ldots, e_{2m-1}, e_{2m} := N\}$ of $T_{[z]}Q^m$, $[z] \in Q^m$. Applying the real structure A to the above formula and using the property of the involution $A^2 = I$, we get $N = A^2N = g(AN, N)AN$. Taking the inner product of the above equation with the unit normal N, it follows that $g(AN, N) = \pm 1$. Since $g(AN, N) = \cos(2t)$ where $t \in [0, \frac{\pi}{4})$, we obtain $AN = N$. Hence N should be $\mathfrak{A}$-principal.

LEMMA 3.3 ([31]). *Let M be a Hopf hypersurface in Q^m such that the normal vector field N is $\mathfrak{A}$-principal everywhere. Then the Reeb function α is constant. Moreover, if $X \in \mathcal{C}$ is a principal curvature vector of M with principal curvature λ, then $2\lambda \neq \alpha$ and its corresponding vector ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

Moreover, if the normal vector N is $\mathfrak{A}$-isotropic, the tangent vector space $T_{[z]}M$, $[z] \in M$, is decomposed as

$$T_{[z]}M = [\xi] \oplus \mathrm{Span}\{A\xi, AN\} \oplus \mathcal{Q},$$

where $\mathcal{C} \ominus \mathcal{Q} = \mathcal{Q}^{\perp} = \mathrm{Span}\{A\xi, AN\}$.

LEMMA 3.4 ([15]). *Let M be a Hopf hypersurface in Q^m, $m \geq 3$, such that the normal vector field N is $\mathfrak{A}$-isotropic everywhere. Then the following statements hold.*

(a) The Reeb function α is constant.

(b) The unit tangent vector fields $A\xi$ and $AN = -\phi A\xi$ are principal for the shape operator and their principal curvature is zero, that is, $SA\xi = SAN = S\phi A\xi = 0$.

(c) If $X \in \mathcal{Q}$ is a principal curvature vector of M with principal curvature λ, then $2\lambda \neq \alpha$ and its corresponding vector ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.

On the other hand, from the property of $\delta(\xi) = g(A\xi, N) = 0$ in (3.2) for a real hypersurface M in Q^m we see that the non-zero vector field $A\xi$ is tangent to M. Hence by Gauss formula, $\bar{\nabla}_X Y = \nabla_X Y + g(SX, Y)N$ and $(\bar{\nabla}_X A)Y = q(X)JAY$ for any X, $Y \in TM$, it yields

$$\begin{aligned}
\nabla_X(A\xi) &= \bar{\nabla}_X(A\xi) - g(SX, A\xi)N \\
&= (\bar{\nabla}_X A)\xi + A(\bar{\nabla}_X \xi) - g(SX, A\xi)N \\
&= q(X)JA\xi + A(\nabla_X \xi) + g(SX, \xi)AN - g(SX, A\xi)N
\end{aligned}$$

for any $X \in TM$. By using $AN = AJ\xi = -JA\xi$ and $JA\xi = \phi A\xi + \eta(A\xi)N$, the tangential part and normal part of this formula give us, respectively,

$$\nabla_X(A\xi) = q(X)\phi A\xi + B\phi SX - g(SX, \xi)\phi A\xi$$

and

$$\begin{aligned}
q(X)g(A\xi, \xi) &= -g(AN, \nabla_X \xi) + g(SX, \xi)g(A\xi, \xi) + g(SX, A\xi) \\
&= 2g(SX, A\xi).
\end{aligned} \tag{3.8}$$

In particular, if M is Hopf, then (3.8) becomes

$$q(\xi)g(A\xi, \xi) = 2\alpha g(A\xi, \xi).$$

Now, if a real hypersurface M has $\mathfrak{A}$-principal normal vector field N in Q^m, then $A\xi = -\xi$ and $AN = N$. Therefore the following lemma holds.

Lemma 3.5 ([15]). *Let M be a real hypersurface with $\mathfrak{A}$-principal normal vector field N in the complex quadric Q^m, $m \geq 3$. Then we obtain:*

(a) $AX = BX$,
(b) $A\phi X = -\phi AX$,
(c) $A\phi SX = -\phi SX$ *and* $q(X) = 2g(SX,\xi)$,
(d) $ASX = SX - 2g(SX,\xi)\xi$ *and* $SAX = SX - 2\eta(X)S\xi$

for all $X \in T_{[z]}M$, $[z] \in M$.

Finally, we introduce one lemma derived from the Hessian tensor of the Reeb curvature function $\alpha = g(S\xi,\xi)$. Indeed, it is defined by

$$(\operatorname{Hess}\alpha)(X,Y) = g(\nabla_X \operatorname{grad}\alpha, Y)$$

for any X and Y tangent to M. Then, this tensor satisfies $(\operatorname{Hess}\alpha)(X,Y) = (\operatorname{Hess}\alpha)(Y,X)$, that is, $g(\nabla_X \operatorname{grad}\alpha, Y) = g(\nabla_Y \operatorname{grad}\alpha, X)$. From this property we obtain the following lemma which plays a key role in the proof of our Theorem 1.

Lemma 3.6 ([15]). *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. Then we obtain:*

$$X(\xi\alpha) = -2\beta g(SA\xi,X) + \xi(\xi\alpha)\eta(X) + 2\alpha\beta g(A\xi,X) \tag{3.9}$$

and

$$X\beta = -2g(S\phi A\xi,X), \tag{3.10}$$

where two smooth functions α and β are defined by $\alpha = g(S\xi,\xi)$ and $\beta = g(A\xi,\xi)$, respectively. Furthermore, by using (3.9) *and* (3.10) *we get*

$$\begin{aligned}
&-2\beta g(SA\xi,X)\eta(Y) + 2\alpha\beta g(A\xi,X)\eta(Y) + (\xi\alpha)g(\phi SX,Y)\\
&\quad + 4g(S\phi A\xi,X)g(\phi A\xi,Y) + 4g(SA\xi,X)g(A\xi,Y) - 2\beta g(BSX,Y)\\
&= -2\beta g(SA\xi,Y)\eta(X) + 2\alpha\beta g(A\xi,Y)\eta(X) + (\xi\alpha)g(\phi SY,X)\\
&\quad + 4g(S\phi A\xi,Y)g(\phi A\xi,X) + 4g(SA\xi,Y)g(A\xi,X) - 2\beta g(BSY,X)
\end{aligned} \tag{3.11}$$

for any tangent vector fields X and Y on M.

4. Proof of Theorem 1

Now in this section we want to get some basic equations for semi-parallel shape operator from the equation of Gauss, and to show that the unit normal vector field N of a semi-parallel Hopf real hypersurface in Q^m is singular.

Let M be a semi-parallel Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. By submanifold theory the second fundamental form h of M satisfies $h(Z,W) = g(SZ,W)N$ for any tangent vector fields Z and W on M, where S denotes the shape operator of M. By such relation the condition (†) can be written as follows:

$$(R(X,Y)S)Z = 0 \tag{$*$}$$

for any tangent vector fields X, Y and Z on M. In addition, from $(R(X,Y)S)Z = R(X,Y)(SZ) - S(R(X,Y)Z)$ the condition $(*)$ is equivalent to

$$R(X,Y)(SZ) = S(R(X,Y)Z) \tag{$**$}$$

for any tangent vector fields X, Y and Z on M. Hence, from (3.3) and (3.4), $(**)$ becomes

$$\begin{aligned} g(Y,SZ)X &- g(X,SZ)Y + g(\phi Y,SZ)\phi X - g(\phi X,SZ)\phi Y \\ &- 2g(\phi X,Y)\phi SZ + g(BY,SZ)BX - g(BX,SZ)BY \\ &+ g(\phi BY,SZ)\phi BX + \alpha g(\phi A\xi,Y)\eta(Z)\phi BX \\ &+ g(\phi BY,SZ)g(\phi A\xi,X)\xi - g(\phi BX,SZ)\phi BY \\ &- \alpha g(\phi A\xi,X)\eta(Z)\phi BY - g(\phi BX,SZ)g(\phi A\xi,Y)\xi \\ &+ g(SY,SZ)SX - g(SX,SZ)SY \\ =\, & g(Y,Z)SX - g(X,Z)SY + g(\phi Y,Z)S\phi X - g(\phi X,Z)S\phi Y \\ &- 2g(\phi X,Y)S\phi Z + g(BY,Z)SBX - g(BX,Z)SBY \\ &+ g(\phi BY,Z)S\phi BX + g(\phi A\xi,Y)\eta(Z)S\phi BX \\ &+ \alpha g(\phi BY,Z)g(\phi A\xi,X)\xi - g(\phi BX,Z)S\phi BY \\ &- g(\phi A\xi,X)\eta(Z)S\phi BY - \alpha g(\phi BX,Z)g(\phi A\xi,Y)\xi \\ &+ g(SY,Z)S^2X - g(SX,Z)S^2Y \end{aligned} \tag{4.1}$$

for any vector fields X, Y and Z tangent to M.

Now, we want to prove that the unit normal vector field N of M in Q^m is singular. By Remark 3.2, we see that the unit normal vector field N becomes singular when M has vanishing geodesic Reeb flow, that is, the Reeb function $\alpha = g(S\xi,\xi)$ identically vanishes on M. So, in the remaining part of this section, we only consider the case that M has non-vanishing geodesic Reeb flow.

LEMMA 4.1. *Let M be a semi-parallel Hopf real hypersurface with non-vanishing geodesic Reeb flow in the complex quadric Q^m, $m \geq 3$. Then $S^2A\xi = \alpha SA\xi$. Moreover, we obtain*

$$\alpha\beta S^2X = \alpha\beta X - \alpha\eta(X)A\xi + \alpha BX + \alpha^2\beta SX - \beta SX + \eta(X)SA\xi - SBX \tag{4.2}$$

for any vector field X tangent to M.

PROOF OF LEMMA 4.1. If we put $Z = \xi$ in (4.1), it yields

$$\begin{aligned} \alpha\eta(Y)X &- \alpha\eta(X)Y + \alpha g(A\xi,Y)BX - \alpha g(A\xi,X)BY \\ &+ \alpha g(\phi A\xi,Y)\phi BX - \alpha g(\phi A\xi,X)\phi BY + \alpha^2\eta(Y)SX - \alpha^2\eta(X)SY \\ =\, & \eta(Y)SX - \eta(X)SY + g(A\xi,Y)SBX - g(A\xi,X)SBY \\ &+ g(\phi A\xi,Y)S\phi BX - g(\phi A\xi,X)S\phi BY + \alpha\eta(Y)S^2X - \alpha\eta(X)S^2Y \end{aligned} \tag{4.3}$$

for any $X, Y \in TM$.

Putting $Y = A\xi$ in (4.3) and using $BA\xi = A^2\xi - g(A^2\xi, N)N = \xi$, we have

$$\begin{aligned} &\alpha\beta X - \alpha\eta(X)A\xi + \alpha BX - \alpha g(A\xi, X)\xi + \alpha^2\beta SX - \alpha^2\eta(X)SA\xi \\ &\quad = \beta SX - \eta(X)SA\xi + SBX - \alpha g(A\xi, X)\xi + \alpha\beta S^2X - \alpha\eta(X)S^2A\xi, \end{aligned} \tag{4.4}$$

where β denotes the smooth function $g(A\xi, \xi)$, that is, $\beta := g(A\xi, \xi)$.

Moreover, putting $X = \xi$ in (4.4) provides

$$\alpha S^2A\xi = \alpha^2 SA\xi,$$

where we have used $B\xi = A\xi$ and $S\xi = \alpha\xi$. Since M has non-vanishing geodesic Reeb flow, this gives us

$$S^2A\xi = \alpha SA\xi. \tag{4.5}$$

If we substitute (4.5) into (4.4), it becomes

$$\alpha\beta X - \alpha\eta(X)A\xi + \alpha BX + \alpha^2\beta SX = \beta SX - \eta(X)SA\xi + SBX + \alpha\beta S^2X,$$

that is,

$$\alpha\beta S^2X = \alpha\beta X - \alpha\eta(X)A\xi + \alpha BX + \alpha^2\beta SX - \beta SX + \eta(X)SA\xi - SBX$$

for any $X \in TM$. So, we have finished the proof. $\square$

On the other hand, if the smooth function $\beta = g(A\xi, \xi)$ identically vanishes on M, it implies that the normal vector field N of M in Q^m becomes $\mathfrak{A}$-isotropic. In fact, from (3.1) we obtain that $\beta = g(A\xi, \xi) = \sin^2(t) - \cos^2(t) = -\cot(2t)$ for $t \in [0, \frac{\pi}{4}]$. So, $\beta = 0$ implies $t = \frac{\pi}{4}$. That is, the unit normal vector field N of M in Q^m can be expressed by

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for some orthonormal vector fields Z_1, $Z_2 \in V(A)$ (see Section 3). By the definition of $\mathfrak{A}$-isotropic tangent vector field of Q^m, it means that the unit vector field N is singular. Thus, hereafter unless otherwise stated, let us assume that the smooth function β satisfies $\beta = g(A\xi, \xi) \neq 0$.

Now, for our convenience sake, let us denote by

$$\mathcal{P}_X = g(SA\xi, X)A\xi + g(S\phi A\xi, X)\phi A\xi - g(A\xi, X)SA\xi - g(\phi A\xi, X)S\phi A\xi \tag{$\sharp$}$$

for any vector field X on M.

Lemma 4.2. *Let M be a semi-parallel Hopf real hypersurface with non-vanishing geodesic Reeb flow in the complex quadric Q^m, $m \geq 3$. If $\beta = g(A\xi, \xi) \neq 0$, then $\mathcal{P}_X$ becomes*

$$\begin{aligned} \mathcal{P}_X = {} & \alpha\beta\eta(X)A\xi - 2\beta^2 g(\phi A\xi, X)S\phi A\xi - \beta\eta(X)SA\xi \\ & - \alpha\beta g(A\xi, X)\xi + 2\beta^2 g(S\phi A\xi, X)\phi A\xi + \beta g(SA\xi, X)\xi \end{aligned} \tag{4.6}$$

and therefore

$$\mathcal{P}_X = \beta BSX - \beta SBX \tag{4.7}$$

for any tangent vector field X on M.

Proof of Lemma 4.2. Putting $Z = \xi$ and $Y = \xi$ in (4.1) implies

$$\begin{aligned} \alpha S^2 X = \alpha X + \alpha\beta BX - \alpha g(A\xi, X)A\xi - \alpha g(\phi A\xi, X)\phi A\xi \\ + \alpha^2 SX - SX - \beta SBX + g(A\xi, X)SA\xi + g(\phi A\xi, X)S\phi A\xi \end{aligned} \tag{4.8}$$

for all $X \in TM$. Since $\beta \neq 0$, (4.8) becomes

$$\begin{aligned} \alpha\beta S^2 X = \alpha\beta X + \alpha\beta^2 BX - \alpha\beta g(A\xi, X)A\xi - \alpha\beta g(\phi A\xi, X)\phi A\xi \\ + \alpha^2\beta SX - \beta SX - \beta^2 SBX \\ + \beta g(A\xi, X)SA\xi + \beta g(\phi A\xi, X)S\phi A\xi \end{aligned} \tag{4.9}$$

for any tangent vector field X on M. From (4.2) and (4.9) we obtain

$$\begin{aligned} &- \alpha\eta(X)A\xi + \alpha BX + \eta(X)SA\xi - SBX \\ &\quad = \alpha\beta^2 BX - \alpha\beta g(A\xi, X)A\xi - \alpha\beta g(\phi A\xi, X)\phi A\xi - \beta^2 SBX \\ &\qquad + \beta g(A\xi, X)SA\xi + \beta g(\phi A\xi, X)S\phi A\xi, \end{aligned}$$

that is,

$$\begin{aligned} &- \alpha\eta(X)A\xi + \alpha BX + \eta(X)SA\xi - SBX - \alpha\beta^2 BX \\ &\quad + \alpha\beta g(A\xi, X)A\xi + \alpha\beta g(\phi A\xi, X)\phi A\xi + \beta^2 SBX \\ &\quad - \beta g(A\xi, X)SA\xi - \beta g(\phi A\xi, X)S\phi A\xi = 0 \end{aligned} \tag{4.10}$$

for any tangent vector field X on M. If we take $X = BX$ in (4.10), it follows

$$\begin{aligned} &- \alpha g(A\xi, X)A\xi + \alpha B^2 X + g(A\xi, X)SA\xi - SB^2 X \\ &\quad - \alpha\beta^2 B^2 X + \alpha\beta g(A\xi, BX)A\xi + \alpha\beta g(\phi A\xi, BX)\phi A\xi \\ &\quad + \beta^2 SB^2 X - \beta g(A\xi, BX)SA\xi - \beta g(\phi A\xi, BX)S\phi A\xi = 0, \end{aligned} \tag{4.11}$$

where we have used $\eta(BX) = g(BX, \xi) = g(X, A\xi)$ for any $X \in TM$.

On the other hand, from $JA = -AJ$, $A^2 = I$, $JN = -\xi$ and $A\xi \in TM$, we obtain

$$AN = AJ\xi = -JA\xi = -\phi A\xi - g(A\xi, \xi)N = -\phi A\xi - \beta N, \tag{4.12}$$

$$BA\xi = A^2\xi - g(A^2\xi, N)N = \xi \tag{4.13}$$

and

$$\phi BX + g(X, \phi A\xi)\xi = -B\phi X + \eta(X)\phi A\xi \tag{4.14}$$

for any vector field X tangent to M. Putting $X = A\xi$ in (4.14) and using (4.13) provides

$$B\phi A\xi = -\phi BA\xi + \beta\phi A\xi = \beta\phi A\xi. \tag{4.15}$$

Moreover, from $A^2 = I$, together with (4.12) and (4.15), we get

$$\begin{aligned} X = A^2X &= A(BX + g(AX, N)N) \\ &= B^2X + g(ABX, N)N + g(AN, X)AN \\ &= B^2X - g(BX, \phi A\xi)N + g(\phi A\xi, X)\phi A\xi + \beta g(\phi A\xi, X)N \\ &= B^2X - \beta g(\phi A\xi, X)N + g(\phi A\xi, X)\phi A\xi + \beta g(\phi A\xi, X)N \\ &= B^2X + g(\phi A\xi, X)\phi A\xi, \end{aligned}$$

that is,

$$B^2X = X - g(\phi A\xi, X)\phi A\xi, \qquad \forall X \in TM. \tag{4.16}$$

By using (4.13), (4.15) and (4.16), equation (4.11) can be rearranged as

$$\begin{aligned} &- \alpha g(A\xi, X)A\xi + \alpha X - \alpha g(\phi A\xi, X)\phi A\xi + g(A\xi, X)SA\xi - SX \\ &\quad + g(\phi A\xi, X)S\phi A\xi - \alpha\beta^2X + 2\alpha\beta^2 g(\phi A\xi, X)\phi A\xi + \alpha\beta\eta(X)A\xi \\ &\quad + \beta^2SX - 2\beta^2 g(\phi A\xi, X)S\phi A\xi - \beta\eta(X)SA\xi = 0, \qquad \forall X \in TM. \end{aligned} \tag{4.17}$$

In addition, taking the symmetric part of (4.17), it follows

$$\begin{aligned} &- \alpha g(A\xi, X)A\xi + \alpha X - \alpha g(\phi A\xi, X)\phi A\xi + g(SA\xi, X)A\xi - SX \\ &\quad + g(S\phi A\xi, X)\phi A\xi - \alpha\beta^2X + 2\alpha\beta^2 g(\phi A\xi, X)\phi A\xi + \alpha\beta g(A\xi, X)\xi \\ &\quad + \beta^2SX - 2\beta^2 g(S\phi A\xi, X)\phi A\xi - \beta g(SA\xi, X)\xi = 0, \qquad \forall X \in TM. \end{aligned} \tag{4.18}$$

Subtracting (4.18) from (4.17) yields

$$\begin{aligned} &g(A\xi, X)SA\xi + g(\phi A\xi, X)S\phi A\xi + \alpha\beta\eta(X)A\xi - 2\beta^2 g(\phi A\xi, X)S\phi A\xi \\ &\quad - \beta\eta(X)SA\xi - g(SA\xi, X)A\xi - g(S\phi A\xi, X)\phi A\xi - \alpha\beta g(A\xi, X)\xi \\ &\quad + 2\beta^2 g(S\phi A\xi, X)\phi A\xi + \beta g(SA\xi, X)\xi = 0, \end{aligned}$$

that is,

$$\begin{aligned} &g(A\xi, X)SA\xi + g(\phi A\xi, X)S\phi A\xi - g(SA\xi, X)A\xi - g(S\phi A\xi, X)\phi A\xi \\ &\quad = -\alpha\beta\eta(X)A\xi + 2\beta^2 g(\phi A\xi, X)S\phi A\xi + \beta\eta(X)SA\xi \\ &\qquad + \alpha\beta g(A\xi, X)\xi - 2\beta^2 g(S\phi A\xi, X)\phi A\xi - \beta g(SA\xi, X)\xi \end{aligned} \tag{4.19}$$

for any tangent vector field X on M. From (4.19), we obtain (4.6) in Lemma 4.2.

The symmetric part of (4.8) yields

$$\begin{aligned} \alpha S^2X &= \alpha X + \alpha\beta BX - \alpha g(A\xi, X)A\xi - \alpha g(\phi A\xi, X)\phi A\xi \\ &\quad + \alpha^2SX - SX - \beta BSX + g(SA\xi, X)A\xi + g(S\phi A\xi, X)\phi A\xi \end{aligned} \tag{4.20}$$

for any $X \in TM$. Subtracting (4.20) from (4.8) follows

$$\begin{aligned} 0 &= -\beta SBX + g(A\xi, X)SA\xi + g(\phi A\xi, X)S\phi A\xi \\ &\quad + \beta BSX - g(SA\xi, X)A\xi - g(S\phi A\xi, X)\phi A\xi, \end{aligned}$$

which implies that

$$\begin{aligned}\beta SBX - \beta BSX = g(A\xi, X)SA\xi + g(\phi A\xi, X)S\phi A\xi \\ - g(SA\xi, X)A\xi - g(S\phi A\xi, X)\phi A\xi.\end{aligned} \tag{4.21}$$

Consequently, (4.21) implies (4.7) in Lemma 4.2. ∎

In order to give our Theorem 1, the following remark is necessary.

Remark 4.3. From (3.11), we note that $\mathcal{P}_X$ mentioned at (♯) can be given by

$$\begin{aligned}4\mathcal{P}_X = -2\beta\eta(X)SA\xi + 2\alpha\beta\eta(X)A\xi - (\xi\alpha)S\phi X - 2\beta SBX \\ + 2\beta g(SA\xi, X)\xi - 2\alpha\beta g(A\xi, X)\xi - (\xi\alpha)\phi SX + 2\beta BSX\end{aligned} \tag{4.22}$$

for any vector field X tangent to M.

Proposition 4.4. *Let M be a semi-parallel Hopf real hypersurface with non-vanishing geodesic Reeb flow in the complex quadric Q^m, $m \geq 3$. Then, the unit normal vector field N of M is singular.*

Proof of Proposition 4.4. As mentioned above, if $\beta = g(A\xi, \xi) = 0$, then the unit normal vector field N of M is $\mathfrak{A}$-isotropic. So, from now on let us consider the case $\beta \neq 0$.

From (4.7) in Lemma 4.2 and (4.22) in Remark 4.3, we get

$$\begin{aligned}4\beta BSX - 4\beta SBX &= 4\mathcal{P}_X \\ &= -2\beta\eta(X)SA\xi + 2\alpha\beta\eta(X)A\xi - (\xi\alpha)S\phi X \\ &\quad - 2\beta SBX + 2\beta g(SA\xi, X)\xi - 2\alpha\beta g(A\xi, X)\xi \\ &\quad - (\xi\alpha)\phi SX + 2\beta BSX,\end{aligned}$$

which implies

$$\begin{aligned}&- 2\beta\eta(X)SA\xi + 2\alpha\beta\eta(X)A\xi - (\xi\alpha)S\phi X + 2\beta SBX \\ &\quad + 2\beta g(SA\xi, X)\xi - 2\alpha\beta g(A\xi, X)\xi - (\xi\alpha)\phi SX - 2\beta BSX = 0.\end{aligned} \tag{4.23}$$

On the other hand, from (4.6) and (4.7) in Lemma 4.2, we have

$$\begin{aligned}\beta BSX - \beta SBX &= \mathcal{P}_X \\ &= \alpha\beta\eta(X)A\xi - 2\beta^2 g(\phi A\xi, X)S\phi A\xi - \beta\eta(X)SA\xi \\ &\quad - \alpha\beta g(A\xi, X)\xi + 2\beta^2 g(S\phi A\xi, X)\phi A\xi + \beta g(SA\xi, X)\xi.\end{aligned}$$

From this, equation (4.23) becomes

$$\begin{aligned}0 &= -2\beta\eta(X)SA\xi + 2\alpha\beta\eta(X)A\xi - (\xi\alpha)S\phi X \\ &\quad + 2\beta g(SA\xi, X)\xi - 2\alpha\beta g(A\xi, X)\xi - (\xi\alpha)\phi SX \\ &\quad - 2\big\{\alpha\beta\eta(X)A\xi - 2\beta^2 g(\phi A\xi, X)S\phi A\xi - \beta\eta(X)SA\xi \\ &\qquad - \alpha\beta g(A\xi, X)\xi + 2\beta^2 g(S\phi A\xi, X)\phi A\xi + \beta g(SA\xi, X)\xi\big\} \\ &= -(\xi\alpha)S\phi X - (\xi\alpha)\phi SX + 4\beta^2 g(\phi A\xi, X)S\phi A\xi - 4\beta^2 g(S\phi A\xi, X)\phi A\xi,\end{aligned}$$

that is,

$$(\xi\alpha)\big(S\phi+\phi S\big)X = 4\beta^2\big\{g(\phi A\xi, X)S\phi A\xi - g(S\phi A\xi, X)\phi A\xi\big\} \tag{4.24}$$

for any tangent vector field X on M. Introducing (4.24) in Remark 4.3 implies

$$\begin{aligned}4\mathcal{P}_X = &-2\beta\eta(X)SA\xi + 2\alpha\beta\eta(X)A\xi - 2\beta SBX\\ &+ 2\beta g(SA\xi, X)\xi - 2\alpha\beta g(A\xi, X)\xi + 2\beta BSX\\ &- 4\beta^2 g(\phi A\xi, X)S\phi A\xi + 4\beta^2 g(S\phi A\xi, X)\phi A\xi.\end{aligned}$$

Bearing in mind (4.6) in Lemma 4.2, this equation becomes

$$\begin{aligned}&4\alpha\beta\eta(X)A\xi - 8\beta^2 g(\phi A\xi, X)S\phi A\xi - 4\beta\eta(X)SA\xi\\ &\qquad - 4\alpha\beta g(A\xi, X)\xi + 8\beta^2 g(S\phi A\xi, X)\phi A\xi + 4\beta g(SA\xi, X)\xi\\ &\quad = 4\mathcal{P}_X\\ &\quad = -2\beta\eta(X)SA\xi + 2\alpha\beta\eta(X)A\xi - 2\beta SBX\\ &\qquad + 2\beta g(SA\xi, X)\xi - 2\alpha\beta g(A\xi, X)\xi + 2\beta BSX\\ &\qquad - 4\beta^2 g(\phi A\xi, X)S\phi A\xi + 4\beta^2 g(S\phi A\xi, X)\phi A\xi,\end{aligned}$$

which yields

$$\begin{aligned}&\beta BSX - \beta SBX\\ &\quad = \alpha\beta\eta(X)A\xi - 2\beta^2 g(\phi A\xi, X)S\phi A\xi - \beta\eta(X)SA\xi\\ &\qquad - \alpha\beta g(A\xi, X)\xi + 2\beta^2 g(S\phi A\xi, X)\phi A\xi + \beta g(SA\xi, X)\xi.\end{aligned} \tag{4.25}$$

By using (4.7) in Lemma 4.2, equation (4.25) gives

$$\begin{aligned}&g(SA\xi, X)A\xi + g(S\phi A\xi, X)\phi A\xi - g(A\xi, X)SA\xi - g(\phi A\xi, X)S\phi A\xi\\ &\quad = \alpha\beta\eta(X)A\xi - 2\beta^2 g(\phi A\xi, X)S\phi A\xi - \beta\eta(X)SA\xi\\ &\qquad - \alpha\beta g(A\xi, X)\xi + 2\beta^2 g(S\phi A\xi, X)\phi A\xi + \beta g(SA\xi, X)\xi\end{aligned} \tag{4.26}$$

for any vector field X tangent to M.

Taking the inner product of (4.26) with $A\xi$, we get

$$\begin{aligned}&g(SA\xi, X) - g(A\xi, X)g(SA\xi, A\xi) - g(\phi A\xi, X)g(S\phi A\xi, A\xi)\\ &\quad = \alpha\beta\eta(X) - 2\beta^2 g(\phi A\xi, X)g(S\phi A\xi, A\xi) - \beta\eta(X)g(SA\xi, A\xi)\\ &\qquad - \alpha\beta^2 g(A\xi, X) + \beta^2 g(SA\xi, X)\end{aligned} \tag{4.27}$$

for all $X \in TM$. Putting $X = \phi A\xi$ in (4.27) and using $g(\phi A\xi, \phi A\xi) = 1-\beta^2$, it becomes

$$\begin{aligned}&g(SA\xi, \phi A\xi) - (1-\beta^2)g(S\phi A\xi, A\xi)\\ &\quad = -2\beta^2(1-\beta^2)g(S\phi A\xi, A\xi) + \beta^2 g(SA\xi, \phi A\xi).\end{aligned}$$

That is, this implies

$$2\beta^2(1-\beta^2)g(S\phi A\xi, A\xi) = 0.$$

Since $\beta \neq 0$, it becomes

$$(1-\beta^2)g(S\phi A\xi, A\xi) = 0,$$

which gives the following two cases.

Case I. $1-\beta^2 = 0$ (that is, $\beta^2 = 1$)

The assumption of $\beta^2 = 1$ implies $\beta = \pm 1$. Meanwhile, from (3.1) we see that the smooth function $\beta = g(A\xi, \xi)$ satisfies $\beta = -\cos(2t)$ for $t \in [0, \frac{\pi}{4})$. With these relations, $t = 0$ holds. This means that the unit normal vector field N satisfies $N = Z_1 \in V(A)$. Therefore, we claim that the unit normal vector field N is $\mathfrak{A}$-principal.

Case II. $1-\beta^2 \neq 0$ (that is, $g(S\phi A\xi, A\xi) = 0$)

From our assumption and putting $X = A\xi$ in (4.24), we get

$$(\xi\alpha)(S\phi A\xi + \phi SA\xi) = 0. \tag{4.28}$$

Subcase II-1. $\xi\alpha = 0$

Let us suppose that $\xi\alpha = 0$ on M. Then, (3.9) provides

$$SA\xi = \alpha A\xi. \tag{4.29}$$

Putting $X = A\xi$ in (3.7) and using (4.29) yields

$$\alpha S\phi A\xi = (\alpha^2 + 2\beta^2)\phi A\xi.$$

Since $\alpha \neq 0$, it implies that the vector field $\phi A\xi$ is principal with principal curvature $\lambda = \frac{\alpha^2+2\beta^2}{\alpha}$, that is,

$$S\phi A\xi = \lambda\phi A\xi, \quad \text{ where } \lambda = \frac{\alpha^2+2\beta^2}{\alpha}. \tag{4.30}$$

Putting $X = A\xi$ and $Z = A\xi$ in (4.1), together with (4.14), (4.16), (4.29) and (4.30), becomes

$$\begin{aligned} & -\alpha Y - 3\alpha g(\phi A\xi, Y)\phi A\xi - \alpha\beta BY - \alpha^2 SY \\ & = -SY - 3\lambda g(\phi A\xi, Y)\phi A\xi - \beta SBY - \alpha S^2 Y, \qquad \forall Y \in TM. \end{aligned} \tag{4.31}$$

Taking the inner product of (4.31) with $\phi A\xi$ and using $g(\phi A\xi, \phi A\xi) = 1-\beta^2$, together with (4.15) and (4.30), yields

$$\begin{aligned} & -\alpha g(Y, \phi A\xi) - 3\alpha(1-\beta^2)g(\phi A\xi, Y) - \alpha\beta^2 g(Y, \phi A\xi) - \alpha^2\lambda g(Y, \phi A\xi) \\ & = -\lambda g(Y, \phi A\xi) - 3\lambda(1-\beta^2)g(\phi A\xi, Y) - \beta^2\lambda g(Y, \phi A\xi) - \alpha\lambda^2 g(Y, \phi A\xi), \end{aligned}$$

that is,

$$(\alpha - \lambda)(2\beta^2 - \alpha\lambda - 4)g(\phi A\xi, Y) = 0$$

for any vector field Y tangent to M. Putting $Y = \phi A\xi$ gives

$$(1-\beta^2)(\alpha - \lambda)(2\beta^2 - \alpha\lambda - 4) = 0.$$

Now, as $\beta^2 \neq 1$ it follows

$$(\alpha - \lambda)(2\beta^2 - \alpha\lambda - 4) = 0. \tag{4.32}$$

From (4.30) we get $\alpha\lambda = \alpha^2 + 2\beta^2$. Hence $2\beta^2 - \alpha\lambda - 4 = -(\alpha^2 + 4)$ and it does not vanish on M, that is, $2\beta^2 - \alpha\lambda - 4 \neq 0$. So, (4.32) gives us $\alpha = \lambda$, which gives a contradiction. In fact, bearing in mind (4.30), the condition $\alpha = \lambda$ means that $2\beta^2 = 0$, that is, $\beta = 0$. But we consider the case of $\beta \neq 0$ on M.

Thus, the case of $\xi\alpha = 0$ does not occur in (4.28). Hence we obtain

$$\phi SA\xi = -S\phi A\xi. \tag{4.33}$$

Subcase II-2. $\phi SA\xi + S\phi A\xi = 0$

Putting $X = A\xi$ in (3.7) and using (4.33), we obtain

$$S^2\phi A\xi = -\beta^2\phi A\xi. \tag{4.34}$$

In addition, putting $X = \phi A\xi$ in (4.8) and using (4.15) yields

$$\begin{aligned}\alpha S^2\phi A\xi &= \alpha\phi A\xi + \alpha\beta^2\phi A\xi - \alpha(1-\beta^2)\phi A\xi\\ &\quad + \alpha^2 S\phi A\xi - S\phi A\xi - \beta^2 S\phi A\xi + (1-\beta^2)S\phi A\xi\\ &= 2\alpha\beta^2\phi A\xi + (\alpha^2 - 2\beta^2)S\phi A\xi,\end{aligned} \tag{4.35}$$

where we have used $g(\phi A\xi, \phi A\xi) = -g(\phi^2 A\xi, A\xi) = 1 - \beta^2$. Substituting (4.34) into (4.35) yields

$$3\alpha\beta^2\phi A\xi + (\alpha^2 - 2\beta^2)S\phi A\xi = 0. \tag{4.36}$$

Let us suppose that $\alpha^2 - 2\beta^2 = 0$, that is, $\beta^2 = \frac{\alpha^2}{2}$. Then, (4.34) gives

$$0 = 3\alpha\beta^2\phi A\xi = \frac{3}{2}\alpha^3\phi A\xi,$$

which implies $\phi A\xi = 0$. From its inner product with $\phi A\xi$, we obtain $\beta^2 = 1$. It makes a contradiction. That is, $\alpha^2 - 2\beta^2$ does not vanish on M. Hence, (4.36) implies

$$S\phi A\xi = \mu\phi A\xi, \qquad \text{where } \mu = -\frac{3\alpha\beta^2}{\alpha^2 - 2\beta^2}.$$

From this, we obtain

$$S^2\phi A\xi = \mu S\phi A\xi = \mu^2\phi A\xi.$$

But, bearing in mind (4.34), this equation gives $\mu^2 = -\beta^2$. It makes a contradiction. So, we assert that there does not exist a semi-parallel Hopf real hypersurface satisfying $\beta^2 \neq 1$. $\square$

Summing up Remark 3.2 and Proposition 4.4 we assert our Theorem 1.

5. Proof of Theorem 2

In Section 4, we have proved that the unit normal vector field N of a semi-parallel Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, is singular. According to the definition of singular tangent vector field on Q^m, it means that N is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. So, first we consider the case of a semi-parallel Hopf real hypersurface M with a $\mathfrak{A}$-isotropic unit normal vector field N in the complex quadric Q^m, $m \geq 3$. Then N can be expressed as

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for some orthonormal vector fields Z_1, $Z_2 \in V(A)$, where $V(A)$ denotes the $(+1)$-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2),\ AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2) \text{ and } JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

Then it gives that

$$g(\xi, A\xi) = g(JN, AJN) = 0, \quad g(\xi, AN) = 0 \quad \text{and} \quad g(AN, N) = 0,$$

which means that both vector fields $AN = -\phi A\xi$ and $A\xi$ are tangent to M. From these facts and Lemma 3.4, we obtain the following result.

Proposition 5.1. *There does not exist any semi-parallel Hopf real hypersurface M with $\mathfrak{A}$-isotropic unit normal vector field N in the complex quadric Q^m, $m \geq 3$.*

Proof of Proposition 5.1. Since the unit normal vector field N is $\mathfrak{A}$-isotropic, we see that $\beta = g(A\xi, \xi) = 0$. Bearing in mind Lemma 3.4, putting $Y = A\xi$ and $Z = \xi$ in (4.1) yields

$$-\alpha\eta(X)A\xi + \alpha BX - \alpha g(A\xi, X)\xi = SBX - \alpha g(A\xi, X)\xi,$$

that is, we obtain

$$SBX = -\alpha\eta(X)A\xi + \alpha BX, \qquad \forall X \in TM. \tag{5.1}$$

Taking $X = BX$ in (5.1) and using $B^2X = X - g(AN, X)AN$, together with $AN = -\phi A\xi$ and $SAN = 0$, we get

$$\begin{aligned} SX = SX - g(AN, X)SAN = SB^2X &\\ &= -\alpha\eta(BX)A\xi + \alpha B^2X \\ &= -\alpha g(A\xi, X)A\xi + \alpha X - \alpha g(AN, X)AN \\ &= -\alpha g(A\xi, X)A\xi + \alpha X - \alpha g(\phi A\xi, X)\phi A\xi, \end{aligned}$$

that is,

$$SX = \alpha X - \alpha g(A\xi, X)A\xi - \alpha g(\phi A\xi, X)\phi A\xi, \quad \forall X \in TM. \tag{5.2}$$

Let $\mathcal{Q}$ be the orthogonal complement of the 3-dimensional distribution $\mathcal{Q}^{\perp} := \text{span}\{\xi, A\xi, AN\}$ in the tangent bundle TM, that is, the tangent vector bundle TM is given by

$$TM = \text{span}\{\xi, A\xi, AN\} \oplus \mathcal{Q}.$$

Let X_0 be any unit tangent vector field of $\mathcal{Q}$. Then (5.2) tells us that X_0 is principal satisfying $SX_0 = \alpha X_0$. Then, by Lemma 3.4 we see that the corresponding unit vector field ϕX_0 becomes a principal curvature vector field of M with principal curvature $\mu := \frac{\alpha^2+2}{\alpha}$, that is,

$$S\phi X_0 = \mu \phi X_0 \quad \text{where} \quad \mu := \frac{\alpha^2+2}{\alpha} \tag{5.3}$$

for any $X_0 \in \mathcal{Q}$.

On the other hand, substituting X by ϕX in (5.2) we get

$$\begin{aligned} S\phi X &= \alpha\phi X - \alpha g(A\xi, \phi X)A\xi - \alpha g(\phi A\xi, \phi X)\phi A\xi \\ &= \alpha\phi X + \alpha g(\phi A\xi, X)A\xi - \alpha g(A\xi, X)\phi A\xi \end{aligned}$$

for any vector field X tangent to M. From this, we get

$$S\phi X_0 = \alpha \phi X_0 \tag{5.4}$$

for any $X_0 \in \mathcal{Q}$.

From (5.3) and (5.4) we have $\mu = \alpha$, which gives a contradiction. It completes the proof of our Proposition 5.1. □

By virtue of Theorem 1 and Proposition 5.1, we see that the unit normal vector field N of M becomes $\mathfrak{A}$-principal. From this result, together with Theorem C, we assert the following.

Proposition 5.2. *Let M be a semi-parallel Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. Then, M is locally congruent to an open part of a tube $(\mathcal{T}_B)$ of type* (B).

As mentioned in Section 1, the model space $(\mathcal{T}_B)$ means the tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around the m-dimensional sphere S^m which is embedded in Q^m as a real form of Q^m.

From now on let us check the converse statement of Proposition 5.2, that is,

Does the tube $(\mathcal{T}_B)$ of Type (B) in Q^m satisfy the assumption of semi-parallelism mentioned in Proposition 5.2?

In order to do this, we introduce the following proposition given in [31].

Proposition A. *Let $(\mathcal{T}_B)$ be a tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around the m-dimensional sphere S^m in Q^m. Then the following statements hold:*

(i) $(\mathcal{T}_B)$ is a Hopf hypersurface.

(ii) The normal bundle of $(\mathcal{T}_B)$ consists of $\mathfrak{A}$-principal vector fields.

(iii) $(\mathcal{T}_B)$ *has three distinct constant principal curvatures. The principal curvatures and corresponding principal curvature spaces of* $(\mathcal{T}_B)$ *are as in Table 1.*

principal curvature	eigenspace	multiplicity
$\alpha = -\sqrt{2}\cot(\sqrt{2}r)$	$T_\alpha = \mathbb{R}JN$	1
$\lambda = \sqrt{2}\tan(\sqrt{2}r)$	$T_\lambda = \{X \in \mathcal{C} \mid AX = X\}$	$m-1$
$\mu = 0$	$T_\mu = \{X \in \mathcal{C} \mid AX = -X\}$	$m-1$

TABLE 1

By (i) and (ii) in Proposition A, it follows that $(\mathcal{T}_B)$ is a Hopf real hypersurface with $\mathfrak{A}$-principal normal vector field N in the complex quadric Q^m, $m \geq 3$.

Now, let us check if a real hypersurface $(\mathcal{T}_B)$ is semi-parallel, that is, the shape operator S of $(\mathcal{T}_B)$ satisfies $(**)$ for any tangent vector fields X, Y and Z on $(\mathcal{T}_B)$. Indeed, by (3.4) the left and right sides of $(**)$ are respectively given by

$$\begin{aligned}\text{Left side} &= R(X,Y)(SZ)\\ &= g(Y,SZ)X - g(X,SZ)Y + g(\phi Y,SZ)\phi X - g(\phi X,SZ)\phi Y\\ &\quad - 2g(\phi X,Y)\phi SZ + g(AY,SZ)AX - g(AX,SZ)AY\\ &\quad + g(\phi AY,SZ)\phi AX - g(\phi AX,SZ)\phi AY\\ &\quad + g(SY,SZ)SX - g(SX,SZ)SY\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\text{Right side} &= S(R(X,Y)Z)\\ &= g(Y,Z)SX - g(X,Z)SY + g(\phi Y,Z)S\phi X\\ &\quad - g(\phi X,Z)S\phi Y - 2g(\phi X,Y)S\phi Z\\ &\quad + g(AY,Z)SAX - g(AX,Z)SAY\\ &\quad + g(\phi AY,Z)S\phi AX - g(\phi AX,Z)S\phi AY\\ &\quad + g(SY,Z)S^2X - g(SX,Z)S^2Y,\end{aligned} \tag{5.6}$$

where we have used $AN = N$, $A\xi = -\xi$ and Lemma 3.5.

Putting $Y = Z = \xi \in T_\alpha \subset T(\mathcal{T}_B)$ in (5.5) and (5.6) yields

$$\text{Left side} = \alpha X - 2\alpha\eta(X)\xi - \alpha AX + \alpha^2 SX - \alpha^3\eta(X)\xi \tag{5.7}$$

and

$$\text{Right side} = SX - 2\alpha\eta(X)\xi - SAX + \alpha S^2X - \alpha^3\eta(X)\xi \tag{5.8}$$

for any vector field X tangent to $(\mathcal{T}_B)$.

Suppose that $(\mathcal{T}_B)$ is a semi-parallel real hypersurface in Q^m. Then, the shape operator S satisfies $(**)$ for any vector fields X, Y and Z tangent to $(\mathcal{T}_B)$. Hence, when $Y = Z = \xi \in T_\alpha$, together with (5.7) and (5.8), this property provides

$$\alpha X - \alpha AX + \alpha^2 SX = SX - SAX + \alpha S^2 X.$$

It can be rearranged as

$$\alpha X - \alpha AX + \alpha^2 SX - SX + SAX - \alpha S^2 X = 0 \tag{5.9}$$

for any tangent vector field X on M. Bearing in mind Proposition 5, the left side of (5.9) becomes

$$\begin{aligned} &\alpha X - \alpha AX + \alpha^2 SX - SX + SAX - \alpha S^2 X \\ &= \begin{cases} 0 & \text{if } X \in T_\alpha \\ \alpha\lambda(\alpha-\lambda)X & \text{if } X \in T_\lambda \\ 2(\alpha-\mu)X & \text{if } X \in T_\mu. \end{cases} \end{aligned} \tag{5.10}$$

It gives us a contradiction with our assumption that a real hypersurface $(\mathcal{T}_B)$ is semi-parallel. In fact, when a real hypersurface $(\mathcal{T}_B)$ is semi-parallel, (5.10) yields

$$\begin{cases} \alpha - \lambda = 0 & \text{on } T_\lambda \\ \alpha = 0 & \text{on } T_\mu \end{cases}$$

by using $\alpha\lambda = (-\sqrt{2}\cot(\sqrt{2}r)) \cdot (\sqrt{2}\tan(\sqrt{2}r)) = -2$ and $\mu = 0$. But the principal curvature α is given by $\alpha = -\sqrt{2}\cot(\sqrt{2}r)$ for $r \in (0, \frac{\pi}{2\sqrt{2}})$, which does not vanish on T_μ. It gives us a contradiction. From this, we can assert that the shape operator S of $(\mathcal{T}_B)$ does not satisfy the assumption of semi-parallelism.

Consequently, this result and Proposition 5.2 give a complete proof of our Theorem 1 in the introduction. That is, we assert that there does not exist any semi-parallel Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$.

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