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LETTER TO THE EDITOR

ACCELERATED EXPANSION OF A CLOSED UNIVERSE

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A novel formulation of an accelerated expansion of a closed universe in which the cosmological constant decays like $\Lambda = (\beta/t)(\dot{a}/a)$, where β is a positive parameter and $a(t)$ is the scale factor, is presented and discussed in some details. The phenomenological decaying law is motivated from the fractional action-like variational approach (or fractionally differentiated Lagrangian function (FDLF)) recently introduced by the author.

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1. Introduction

It is widely believed today that our universe is asymptotically flat, matter dominated and undergoes a phase of an accelerated expansion due to the presence of a mysterious dark energy. This fact is based on the recent available astronomical observations of the dynamics of galaxies, clusters of galaxies, of luminosity distances of the Type Ia supernovae (SNIa) as a function of redshift with redshifts $z > 0.35$, the first acoustic peak of the CMB temperature fluctuations or anisotropies and the recent findings of BOOMERANG experiments [1−4]. One of the main features of the dark energy is the violation of the strong energy condition $\rho + 3p \geq 0$; ρ and p are the density and pressure of the perfect fluid. Questions still linger about the nature of dark matter, especially its distribution in central region of clusters and galaxies. In summary, the assumption of the flat model with a positive cosmological constant is in good agreement with the observations.

A large number of theoretical and phenomenological competitive models, ranging from a positive cosmological constant to scalar field theories, have been presented to account for the gravitational effects of the dark energy including the cosmological constant Λ with the equation of state $p_\Lambda = -\rho_\Lambda = -\Lambda$, which is considered to be a strictly constant energy density inherent in empty spacetime [5]. Most of these theories are accompanied with problems and many difficulties, and we still ignore which of these models are the most viable or more realistic. We are grappled with deep cosmological enigmas and many unsolved problems. In this work, a modified Friedmann-Robertson-Walker (FRW) cosmology is discussed, in which the cosmological constant decays with the phenomenological law $\Lambda = (\beta/t)(\dot{a}/a)$, where β is a positive parameter and $a(t)$ is the scale factor of the universe. The chosen phenomenological law is in fact based on the fraction action-like variational approach (FALVA) of fractionally differentiated Lagrangian function (FDLF), introduced in 2005 by the author to model nonconservative and weak decaying classical and quantum dynamical systems. The fractional time integral introduces only one fractional parameter, $\alpha > 0$, while in other models an arbitrary number of fractional parameters (orders of derivatives) appear [6 – 12]. The standard functional action S is replaced by a fractional functional integral action $S_{\alpha>0}$ revealing interesting features. It is in fact defined as follows

$$S_{\alpha>0}\left[q\right] = \frac{1}{\Gamma\left(\alpha\right)} \int\limits_{t_0}^{t} L\left(\dot{q}\left(\tau\right), q\left(\tau\right), \tau\right)\left(t-\tau\right)^{\alpha-1} \mathrm{d}\tau$$

$$= \int\limits_{t_0=0}^{t} L\left(\dot{q}, q, \tau\right)\tau \mathrm{d}g_t\left(\tau\right), \quad [t_0, t_1] \in \mathcal{R},$$

where $L\left(\dot{q}, q, \tau\right)$ is the Lagrangian weighted with $\left(t-\tau\right)^{\alpha-1}/\Gamma(\alpha)$ and $\Gamma(1+\alpha)g_t(\tau) = t^\alpha - (t-\tau)^\alpha$, with the scaling properties $g_{\mu t}(\mu\tau) = \mu^\alpha g_t(\tau)$, $\mu > 0$. In reality, we consider a smooth action integral (a time smeared measure $dg_t(\tau)$ on the time interval $[0, t] \in \mathcal{R}^+$), which can be rewritten as the strictly singular Riemann-Liouville type fractional derivative Lagrangian

$$S_{\beta\in(0,1)}\left[q\right] = D_t^{-1+\beta} L\left(\dot{q}\left(t\right), q\left(t\right), t\right)$$

$$= \int\limits_{0}^{t} L\left(\dot{q}\left(t\right), q\left(t\right), t\right) \frac{\mathrm{d}\tau}{\left(t-\tau\right)^\beta} \overset{\beta\to 0}{\to} \int\limits_{0}^{t} L\left(\dot{q}\left(t\right), q\left(t\right), t\right) \mathrm{d}\tau,$$

and thereby retrieved the standard action integral or functional integral. In this work, we have $\beta = 1 - \alpha$, $\alpha \in (0,1)$. Such type of functionals is known in mathematical economy, describing, for instance, the so called "discounting" economical dynamics. The resulting modified Euler-Lagrange equation and fractional Hamiltonian canonical equations were derived, and it was proved to have important cosmological consequences in agreement with recent astronomical observations [13 − 14]. Within the framework of the FDLF formalism, the gravity is perturbed and the dynamical equations result in a decaying cosmological constant (considered as a time-decaying friction), depending on the fractional parameter α, and decays like $\Lambda \propto (1/t)(\dot{a}/a)$. Motivated by the results obtained in our previous work, we will discuss in this paper the Friedmann-Robertson-Walker cosmology with normal action considered, i.e., $\alpha = 1$, where the cosmological constant is replaced by the phenomenological friction-like law $\Lambda(t) = (\beta/t)(\dot{a}/a)$, and β is a positive parameter.

2. *Modified FRW cosmology with friction lambda*

We consider the standard FRW metric modelled by the metric

$$ds^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right], \tag{1}$$

where $k = 0, \pm 1$ is the curvature parameter of the spatial sections (CPSS) in the standard case and r, θ and ϕ are dimensionless commoving coordinates. In the presence of the friction cosmological constant, the Friedmann dynamical equations are written as follows [15]

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3} + \frac{\beta}{3t}\,\frac{\dot{a}}{a}, \tag{2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho\,(1+3\gamma)}{3} + \frac{\beta}{3t}\,\frac{\dot{a}}{a}, \tag{3}$$

where the equation of state $p = \gamma\rho$ (γ is a real parameter) is assumed. Moreover, we will assume that the fluid density decays like $\rho = 3\delta/(8\pi G a^2)$, where δ is a positive parameter [16 − 25]. Here G is the gravitational constant. Consequently, Eqs. (2) and (3) give

$$\frac{\dot{a}^2}{a^2} + \frac{k-\delta}{a^2} = \frac{\beta}{3t}\,\frac{\dot{a}}{a}, \tag{4}$$

$$\frac{\ddot{a}}{a} = -\frac{\delta\,(1+3\gamma)}{2a^2} + \frac{\beta}{3t}\,\frac{\dot{a}}{a}, \tag{5}$$

An important consequence arises from Eq. (4) if $k = \delta$, the scale factor evolves as $a \propto t^{\beta/3}$ (the solution a = constant is not acceptable). Replacing into Eq. (5) gives

$$\frac{\beta}{3t^2} = \frac{\delta(1+3\gamma)}{2t^{2\beta/3}}, \tag{6}$$

yielding $\beta = 3$ ($a \propto t$) and $\delta\,(1+3\gamma) = 1$. For $\delta = 1$, $\gamma = 0$ (matter era), while for $\delta = 1/3$, $\gamma = 1/2$ (radiation era). This is interesting because it gives a universe expanding linearly with time in a closed spacetime in which the cosmological constant decays and the fluid density decays as $\propto t^{-2}$.

In addition, we have more matter in the matter epoch than in the radiation one. For $k \neq \delta$, Eqs. (4) and (5) give

$$\frac{\ddot{a}}{a} - \frac{\delta\,(1+3\gamma)}{2\,(k-\delta)}\,\frac{\dot{a}^2}{a^2} = -\left(\frac{\delta\,(1+3\gamma)}{2\,(k-\delta)} - 1\right)\frac{\beta}{3t}\,\frac{\dot{a}}{a}\,. \tag{7}$$

If $\delta\,(1+3\gamma) = 2\,(k-\delta)$, then the solution of Eq. (7) is $a = e^{Ht}$, where $H \equiv \dot{a}/a$ is the Hubble parameter. This results in rapid decays of the energy densities and the cosmological constant, leaving no time for galaxy formation. If, in contrast, $\gamma = -1/3$ ($p = -\rho/3$), then $k = \delta > 0$ and consequently, the scale factor evolves as $a \propto t^m, m = 1 + (\beta/3) > 1$ and this corresponds to an accelerated expansion in a closed spacetime. The cosmological constant is positive and decays like $\Lambda \propto t^{-2}$, while the fluid energy density decays like $\rho \propto t^{-2m}$ and, consequently, the fluid pressure tends to zero. In other words, the universe at the end of time is empty but closed. More generally, we will consider the following three solutions.

a)-Matter-dominated epoch

This corresponds to $\gamma = 0$. Equation (7) is then written like

$$\frac{\ddot{a}}{a} - \frac{\delta}{2\,(k-\delta)}\frac{\dot{a}^2}{a^2} = -\left(\frac{\delta}{2\,(k-\delta)} - 1\right)\frac{\beta}{3t}\,\frac{\dot{a}}{a}\,, \tag{8}$$

which can be solved to give $a \propto t^r$, where

$$r = \frac{3\,(2\,(k-\delta)) - \beta\,(3\delta - 2k)}{3\,(2k - 3\delta)}\,. \tag{9}$$

Note that for $k = 0$ (flat spacetime), $r = (2+\beta)/3 > 1$ if $\beta > 1$. Consequently, the cosmological constant is positive and decays like $\Lambda \propto t^{-2}$ ($\Lambda = \beta H^2/r = 3\beta H^2/(2+\beta)$), while the fluid density decays like $\rho \propto t^{-2r}$. The vacuum energy density is then given by

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G} = \frac{\beta r}{8\pi G t^2} = \frac{\beta\,(2+\beta)}{24\pi G t^2}\,. \tag{10}$$

As for the density parameter of the universe and the density parameter due to the vacuum contribution, they are defined, respectively, as

$$\Omega^{\mathrm{matter}} \equiv \frac{\rho}{\rho_c} = \frac{\delta}{t^{2r} H^2} = \frac{\delta}{p^2 t^{2(r-1)}}\,, \tag{11}$$

$$\Omega^{\Lambda} \equiv \frac{\Lambda}{3H^2} = \frac{\beta}{2+\beta}\,, \tag{12}$$

where $\rho_c = 3H^2/(8\pi G)$ is the critical energy density of the universe. Note that for $r > 1$, which corresponds to an accelerated expansion, $\Omega^{\mathrm{matter}} \to 0$, while for $r = 1$, $\Omega^{\mathrm{matter}} \to \delta/p^2$. That is, for $r > 1$, the accelerated expansion is dominated by the vacuum energy. Thus for $\beta = 3r$, $\Omega^{\mathrm{total}} = \Omega^{\mathrm{matter}} + \Omega^{\Lambda} \to 1$, as favoured by the inflationary scenario. In order to calculate the rate of particle creation (annihilation), we define

$$n_p = \frac{1}{a^3}\left.\frac{\mathrm{d}\left(\rho a^3\right)}{\mathrm{d}t}\right|_p = \frac{1}{8\pi G}\frac{\beta r}{t^3} = \frac{\beta}{\delta t^{2(1-\beta)/3}}\rho_p H_p\,, \tag{13}$$

where we have used the continuity equation $\mathrm{d}(\rho a^3)/\mathrm{d}t = -(a^3/(8\pi G))(d\Lambda/dt)$ for a constant G. Note that for $r > 1$, n_p increases with time, and for $p < 1$, n_p decreases with time. For $r = 1$ and $\beta = \delta = 1$, $n_p = \rho_p H_p$, less than that of the steady state model $(= 3n_p H_p)$ [15].

b)Radiation-dominated epoch

This is characterized by the equation of state $p = \rho/3$, or $\gamma = 1/3$. The solution of Eq. (7) is then $a \propto t^q$, where

$$q = \frac{3\,(k-\delta) + \beta\,(k-2\delta)}{3\,(k-2\delta)}\,. \tag{14}$$

Note that for $k = 0$, $q = (3+2\beta)/6 > 1$ for $\beta > 3/2$ and consequently, for this value of β, the expansion of the universe accelerates faster in the matter epoch than in the radiation one. Consequently, the cosmological constant is positive and decays like $\Lambda \propto t^{-2}$ ($\Lambda = \beta H^2/q = 6\beta H^2/(3+2\beta)$), while the fluid density decays like $\rho \propto t^{-2q}$. The vacuum energy density decays like

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G} = \frac{\beta q}{8\pi G t^2} = \frac{\beta\,(3+2\beta)}{48\pi G t^2}\,. \tag{15}$$

c)-Inflationary era

This epoch corresponds to $p = -\rho$, or $\gamma = -1$. This yields the following differential equation

$$\frac{\ddot{a}}{a} + \frac{\delta}{k-\delta}\frac{\dot{a}^2}{a^2} = \left(\frac{k}{k-\delta}\right)\frac{\beta}{3t}\,\frac{\dot{a}}{a}\,. \tag{16}$$

A possible solution is given by $a \propto t^n$, where

$$n = \frac{k\beta + 3\,(k-\delta)}{3k}\,, \tag{17}$$

with $n > 1$ for $k\beta > 3\delta$. As the time grows up, the RHS of Eq. (16) tends to zero and $n \simeq (k - \delta)/k < 1$. That is, the Universe expansion in the closed spacetime decreases with time. Note that for $k = 0$, the solution is given by $a \propto e^{Ht}$, which corresponds to the inflation case.

3. *Time-varying gravitational constant*

We now discuss the case when the gravitational constant varies. The continuity equations are $\dot{\rho} + 3H(\gamma + 1)\rho = 0$ and $\dot{\Lambda} + 8\pi\dot{G}\rho = 0$ [16]. The first differential equation gives

$$(3\gamma + 1)\frac{\dot{a}}{a} = \frac{\dot{G}}{G}\,, \tag{18}$$

while the second differential equation gives

$$\beta\left(-\frac{1}{t^2}\frac{\dot{a}}{a} + \frac{1}{t}\left(\frac{2\,(k-\delta) - \delta\,(1+3\gamma)}{2a^2}\right)\right) + \frac{3\delta}{a^2}\frac{\dot{G}}{G} = 0\,. \tag{19}$$

Assuming the power-law behaviour $a \propto t^y$ and $G \propto t^x$, Eqs. (18) and (19) give $(3\gamma + 1)y = x$ with $y = 1$ in order to obtain a consistent solution with $x = 3\gamma + 1$ and $(6 - \beta)\delta x = \beta(2 - 2(k - \delta))$. Thus for $\gamma = 0$ and $k = 0$, $x = 1$ and $(6 - \beta)\delta = \beta(2-2(k-\delta))$. From Eq. (4) we obtain the quadratic equation $\delta^2+\delta-1 = 0$ and thus $\delta_+ = (-1 + \sqrt{5})/2 \approx 0.6$, and consequently for $k = \delta$, $\beta \approx 1.38$. The fluid density then decays like $\rho \propto t^{-3}$, while the cosmological constant decays like $\Lambda \propto t^{-2}$. In conclusion, the flat universe expands linearly with time as a power-law and is filled with a fluid whose equation of state is given by $p = 0$, and where the gravitational constant increases linearly with time. The interesting feature of this model is that all its parameters are determined through the theory itself. Moreover, for $\gamma = 1/3$ (radiation-dominated epoch), $x = 2$, and thus the gravitational constant increases as $G \propto t^2$, while the fluid density decays like $\rho \propto t^{-4}$ or $\rho \propto a^{-4}$, as it is expected. Thus the gravitational constant passes from $G \propto t^2 \to G \propto t$, while the universe expands as $a \propto t$. It is worth mentioning that for $k = \delta$ and $\gamma = -1/3$, Eq. (19) gives $a = \pm t$ with $\beta = 3\delta$ for a non-constant scale factor a. Thus, an increasing gravitational constant with decaying vacuum and fluid densities may occur in a closed linearly expanding spacetime. Note that $a = -t$ is also a possible solution to the problem and such type of solution appears in pre-Big-Bang cosmology [26]. The presence of the bulk viscous stress term Π may also be taken into account when we deal with dissipative effects on the evolution of the Universe [27]. That is, the pressure term is modified and $p \to p + \Pi$. In this case, the first continuity equation is written as $\dot{\rho} + 3H(\gamma + 1)\rho = -3\Pi H$, and in order to have a consistent solution, one may choose $-8\pi\Pi \propto t^{-2y-x}$, so that $-\delta\,(x - 2y) + 3\delta y(\gamma + 1) = y$. For $\gamma = 0$ (matter epoch), $(5\delta-1)y = \delta x$, while for $\gamma = 1/3$ (radiation epoch), $(6\delta-1)y = \delta x$. From Eq. (19), $y = 1$; thus $5\delta - 1 = \delta x$ and $\beta(2k - 3\delta - 2) = -6\delta x$ for the matter epoch. Taking $2k = 3\delta$ yields $\beta = 3\delta x$, and consequently, $-8\pi\Pi \propto t^{-2-\beta/3\delta}$.

Finally, for $\gamma = 1/3$ (radiation epoch), $6\delta - 1 = \delta x$ and $\beta(k - 2\delta - 1) = -3\delta x$. Taking $k = 2\delta$, $\beta = \delta x$, we obtain again $-8\pi\Pi \propto t^{-2-\beta/3\delta}$. It is interesting to have a CPSS and, consequently, a varying topology of the universe evolving from the radiation dominated epoch to the radiation (? matter) dominated one. If in contrast $2k = 3\delta$, i.e., the CPSS is unchanged during the cosmic evolution, than $\beta(\delta + 2) = 6\delta x$ and $-8\pi\Pi \propto t^{-2-\beta(\delta+2)/6\delta}$ and thus Π decays more rapidly in the matter-dominated epoch than in the radiation one.

4. *Conclusions*

While a vast literature exists that addresses the observational fact of the current expansion and evolution of the universe, we are not aware of models similar to the one developed in this paper. We have shown in this work that the modified FRW cosmology with decaying friction-like cosmological constant, motivated from the fractionally differentiated Lagrangian function formalism, has important and interesting consequences in describing the evolution of the universe. One may attribute the accelerated expansion of the universe to the decay of the cosmological constant in its special law $\Lambda(t) = (\beta/t)(\dot{a}/a)$. The class of solutions obtained in this work, which corresponds to a closed spacetime, is totally appealing, since this type of solution appears in string theory [28]. This model is a radical alternative to the standard big-bang/inflationary theory. Further details and consequences are in progress.

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UBRZANO ŠIRENJE ZATVORENOG SVEMIRA

Predstavljamo i podrobno raspravljamo novu formulaciju ubrzanog širenja zatvorenog svemira, u kojoj kozmološka konstanta opada kao $\Lambda = (\beta/t)(\dot{a}/a)$, gdje je β pozitivan parametar a $a(t)$ mjerni množitelj. Ta fenomenološka postavka podstaknuta je nedavno uvedenim frakcijskim činidbenim varijacijskim pristupom (odn. frakcijski diferenciranom Lagrangeovom funkcijom (FDLF)).