Printed ISSN 1330–0016
Online ISSN 1333–9133
CD ISSN 1333–8390
CODEN FIZBE7

STUDY OF AN ABELINIZATION TRANSITION IN SU(2) GLUODYNAMICS AT FINITE TEMPERATURE

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Received 23 October 2007; Accepted XX June 2008
Online 20 June 2008

We discuss the problem of an effective description of the phase transition phenomena in the pure gluodynamics in $SU(2)$ symmetric QCD. The calculation method is choosen following the conjecture that the infrared sector of the theory possesses the same confinement characteristic as the full theory. It is shown that the analytic description of this phenomena is beyond the Gaussian method of evaluations of functional integrals. We point to a non-perturbative method of the evaluation of functional integral for two dimensional Wiener integral for x^4 theory which could solve this problem.

PACS numbers: 11.15.-q, 11.15.Tk, 11.25.Db UDC 539.125
Keywords: path integral, gluodynamics, phase transitions

1. Introduction

The evaluations of the measurable quantities in quantum field theory (QFT) are well defined theoretically, but many obstacles appear in the calculations connected with the non-ability to realize the calculations corresponding to the theoretical definitions. The calculations in QFT are usually performed analytically using perturbative calculus, numerically by lattice gauge theory (LGT), or by creating models based on the results of previous methods of calculations. The perturbative methods give analytical results, but various problems in QFT are non-perturbative phenomena, therefore they are applicable for the narrow group of problems only. LGT, the best method of evaluations at this time, suffers from finiteness of the lattice, problems of definitions of mathematical objects on the lattice and the continuum limit problems. We adopt the method of analytical calculations in the continuum to study of the transitions in dense matter. After a short introduction into the problem in Section 2, we propose in Section 3 the non-perturbative solution of the problem of evaluation of the functional integral with fourth-order term in the action. We found the result in the form of an asymptotical series, and a method of solution of the problem is described in articles [1] and [2].

2. *Effective description of the confinement in continuum*

The Euclidean finite temperature theory is defined by the functional integral for the partition function

$$Z(\beta) = N \int [D\mathcal{A}_\mu^a] \, \exp\left(-S_E\right),$$

where the Euclidean action is defined by

$$S_E = \frac{1}{4} \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \, \mathcal{F}_{\mu\nu}^a \mathcal{F}_{\mu\nu}^a \, .$$

The Euclidean color field strength is defined as follows

$$\mathcal{F}_{\mu\nu}^a = \partial_\mu \mathcal{A}_\nu^a - \partial_\nu \mathcal{A}_\mu^a - g\varepsilon^{abc} \mathcal{A}_\mu^b \mathcal{A}_\nu^c \, .$$

We require the periodicity of color potentials in the direction of the imaginary time variable

$$\mathcal{A}_\mu^a(\tau + \beta, x) = \mathcal{A}_\mu^a(\tau, x) \, .$$

The theory with corresponding partition function looks like a partition function for the finite temperature theory in statistical physics. This "isomorphism" allows us to investigate the QCD problem by methods of statistical physics.

We are concerned with the phase transitions to the quark gluon plasma at high densities of matter, the one of the challenging forecasts of QCD. We investigate some problems of this phase transition, usually known as confinement/deconfinement phase transition, by a model-independent method described below. Such description of the confinement problem relies on the conjecture that the infrared sector of the finite temperature theory possesses the same phase structure as the complete theory. This is the assertion of Appelquist - Carrazone decoupling theorem [3], where it was proven that by integrating off-massive modes of the theory, one obtains terms in the potential controlling dynamics (in this case the infrared variables) of the theory. With the help of this theorem, we reduce the infinite number of variables of the original theory to a finite number in the infrared sector effectively describing the phenomena studied.

To integrate over the massive modes, we proceed as follows:

1) We separate the field variable to the sum of the "classical" part and the "quantum fluctuation"

$$\mathcal{A}_\mu^a(t,x) = A_\mu^a(t,x) + a_\mu^a(t,x) \, .$$

We use the static gauge for evaluations of the functional integrals over massive modes. The saddle points of the action in this gauge are fixed as

$$A_i^a = 0, \quad i, a = 1, 2, 3, \quad A_0^1 = A_0^2 = 0, \quad A_0^3 = const.$$

We choose finite A_0^3 in the color space for the simplicity.

2) The periodicity of color potentials in the imaginary time variable offers a chance to use Fourier transformation of the potentials in this variable. This operation automatically separates the Fourier transforms of the potentials to the infrared, zero frequency modes and massive modes, where nonzero Fourier frequency modes ($\equiv$ Matsubara frequencies) appear as mass terms. Integrating over these "massive modes" by the Gaussian method, we obtain their contribution to the effective potential (in the scope of this article we take $V_{\rm eff} = V_{\rm eff}(A_0^3, b_0^3, b_i^a),\ a, i = 1, 2, 3$) and contributions of the A_0^3 dependent mass terms for the perpendicular quantum fluctuation degrees of freedom $(b_i^1, b_i^2, i = 1, 2, 3)$ in the infrared sector of the theory. This process is known as dimensional reduction, described by Appelquist and Pisarski [4], and offers possible calculation scheme for finite temperature field theories. The zero-modes action in quantum fluctuations fields reads

$$S_{\rm zero} = \beta \int {\rm d}^3 x \left\{ \frac{1}{4} G_{ij}^a G_{ij}^a + \frac{1}{2} (g A_0^3)^2 [(b_i^1)^2 + (b_i^2)^2] + \frac{1}{2} (\partial_i b_0^3)^2 + V_{\rm eff}(A_0^3, b_0^3, b_i^a) \right\}, \quad (1)$$

with the field strength

$$G_{ij}^a = \partial_i b_j^a - \partial_j b_i^a - g \varepsilon^{abc} b_i^b b_j^c \,.$$

We have found two different effective systems, depending on the value of the static color potential A_0^3.

1. Let $A_0^3 = 0$. In this case, all quadratic terms in the zero frequency field variables in Eq. (1) disappear. The effective infrared system depends on variables $b_i^a, b_0^3,\ a, i = 1, 2, 3$. The color $SU(2)$ symmetry of the system is maintained. This system corresponds to the interaction of the effective Higgs field, represented by chromoelectric potential b_0^3, with the chromomagnetic fields represented by the chromomagnetic potentials b_i^a.

2. Let $A_0^3 \gg 0$, at least as the lowest Matsubara frequency. In this case, the modes b_i^1, b_i^2 in Eq. (1) must be treated as massive, and these modes are integrated off the zero mode system. Then the infrared system depends on variables $b_i^3, b_0^3,\ i = 1, 2, 3$, and the remanent color symmetry group is $Z(2)$. It is the well-known abelian projection observed also in the lattice calculations. In the earlier continuum theory studies [5 − 7], we find in this sector the confinement/deconfinement transition with correct behavior of the potential between heavy quarks calculated as the corellators of Polyakov's loops.

But what happens when A_0^3 grows from zero to a finite value and infrared system undergoes the transition $SU(2) \to Z(2)$? We named this phenomenon the "abelinization transition" [8]. To study it, the use of the Gaussian method for evaluations of functional integrals is not sufficient. Preliminary investigations indicate that this transition could be non-continuous, and consequently the lattice formalism may be not powerful, too. We conclude that one needs some new formalism for the evaluations of the functional integrals with higher-power terms than the quadratic in the potential. This formalism is well defined also in the case of the vanishing mass term of the potential.

As a first attempt, we applied the semi-analytical method to evaluate the Wiener functional integral with x^4 term in the action. The short description of this method is reported in the next section.

3. *Evaluation of φ^4 Wiener functional integral*

The simplest non-Gaussian functional integral is the Wiener functional integral with the x^4 term in the action. In the Euclidean sector of the theory, we have to evaluate the continuum Wiener functional integral

$$\mathcal{Z} = \int [\mathcal{D}\varphi(x)] \exp(-\mathcal{S}) \,, \tag{2}$$

In this case, the action possesses the fourth-order term

$$\mathcal{S} = \int_0^\beta \mathrm{d}\tau \left[c/2 \left(\frac{\partial \varphi(\tau)}{\partial \tau} \right)^2 + b\varphi(\tau)^2 + a\varphi(\tau)^4 \right] .$$

The continuum Wiener functional integral is defined by a formal limit

$$\mathcal{Z} = \lim_{N\to\infty} \mathcal{Z}_N \,,$$

where the finite dimensional integral $\mathcal{Z}_N$ is defined by the time-slicing method

$$\mathcal{Z}_N = \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \left(\frac{\mathrm{d}\varphi_i}{\sqrt{2\pi\triangle/c}} \right) \exp\left\{ -\sum_{i=1}^{N} \triangle \left[c/2 \left(\frac{\varphi_i - \varphi_{i-1}}{\triangle} \right)^2 + b\varphi_i^2 + a\varphi_i^4 \right] \right\}, \tag{3}$$

where $\triangle = \beta/N$, and a, b, c are the parameters of the model. The quantity Z_N represents the unconditional propagation from $\varphi = \varphi_0 = 0$ to any $\varphi = \varphi_N$ (Eq. (3) contains an integration over φ_N).

3.1. *Finite dimensional integral*

An important task is to calculate the one-dimensional integral

$$I_1 = \int_{-\infty}^{+\infty} \mathrm{d}x \, \exp\{-(Ax^4 + Bx^2 + Cx)\} \,,$$

where $Re\ A > 0$. The standard perturbative procedure relies on the Taylor's decomposition of $\exp(-Ax^4)$ term with consecutive replacements of the integration and summation order. The integrals can be calculated, but the sum is divergent.

Instead, we propose the power expansion in C

$$I_1 = \sum_{n=0}^{\infty} \frac{(-C)^n}{n!} \int_{-\infty}^{+\infty} \mathrm{d}x\ x^n \exp\{-(Ax^4 + Bx^2)\} .$$

The integrals in the above relation can be expressed by the parabolic cylinder functions $D_{-\nu-1/2}(z)$. Then, for the integral I_1 we have

$$I_1 = \frac{\Gamma(1/2)}{\sqrt{B}} \sum_{m=0}^{\infty} \frac{\xi^m}{m!} \mathcal{D}_{-m-1/2}(z) \ , \quad \xi = \frac{C^2}{4B} \ , \quad z = \frac{B}{\sqrt{2A}} \ , \tag{4}$$

with the abbreviation

$$\mathcal{D}_{-m-1/2}(z) = z^{m+1/2}\ e^{z^2/4}\ D_{-m-1/2}(z) \ .$$

It was shown that sum in Eq. (4) is convergent and for finite values of z this sum converges uniformly in the variable ξ.

Applying this idea of integration to the N-dimensional integral (3), we proved [1] the exact formula for the N-dimensional integral (3)

$$\mathcal{Z}_N = \left[\prod_{i=0}^{N} 2(1 + b\triangle^2/c)\omega_i \right]^{-\frac{1}{2}} \mathcal{S}_N \ ,$$

with

$$\mathcal{S}_N = \sum_{k_1,\cdots,k_{N-1}=0}^{\infty} \prod_{i=0}^{N} \left[\frac{(\rho)^{2k_i}}{(2k_i)!} \Gamma(k_{i-1} + k_i + 1/2)\ \sqrt{\omega_i}\ \mathcal{D}_{-k_{i-1}-k_i-1/2}\ (z) \right] ,$$

where $k_0 = k_N = 0$, $\rho = (1+b\triangle^2/c)^{-1}$, $z = c(1+b\triangle^2/c)/\sqrt{2a\triangle^3}$, $\omega_i = 1-A^2/\omega_{i-1}$, $\omega_0 = 1/2 + Ab\triangle^2/c$, $A = 1/[2(1 + b\triangle^2/c)]$. We see that ρ is independent of the coupling constant. Only the argument z of the parabolic cylinder function depends on the coupling constant a.

To evaluate S_N, we must solve the problem how to sum up the product of two parabolic cylinder functions. The parabolic cylinder functions are related to the representation of the group of the upper triangular matrices, so we implicitly expect a simplification of their product. This problem is not solved completely yet. We adopt less complex method of summation, namely we use the asymptotic expansion of one of them, then, exchanging the order of summations we can sum over k_i. Surely, the result is degraded to an asymptotic expansion only, but still we have an analytical solution of the problem. This procedure was widely discussed in detail in the article [1]. Here we show the result

$$S_N = \sum_{\mu=0}^{\mathcal{J}} \frac{(-1)^\mu}{\mu!\,(2z^2\triangle^3)^\mu}\,\triangle^{3\mu}\left\{C^{2\mu}(N)\right\}_{2\mu,2\mu}\,. \tag{5}$$

The evaluation of the symbols $\left\{C^{2\mu}(N)\right\}_{2\mu,2\mu}$ is described in the article [2]. We explicitly present the first nontrivial contribution in the continuum limit ($\mu = 1$, $\triangle \to 0,\ \triangle.N = \beta$)

$$\left\{C^2(a,b,c,\tau)\right\}_{2,2} = \frac{3}{8\gamma^3}\left[3\gamma\tau\tanh^2(\gamma\tau) + \tanh(\gamma\tau) - \gamma\tau\right], \tag{6}$$

where $\gamma = \sqrt{2b/c}$. The dependence of $S(a,b,c,\tau) = \lim_{N\to\infty} S_N$ on the variable b, the mass-squared of anharmonic oscillator, is shown in Fig. 1.

The parameters a, c, τ are fixed constants, and we took $\mathcal{J} = 3$ in Eq. (5). For $b < 0$, we see the singularities for $|\gamma\tau| = (n + 1/2)\pi$. This divergent behavior corresponds to the powers of $\tan(|\gamma\tau|)$.

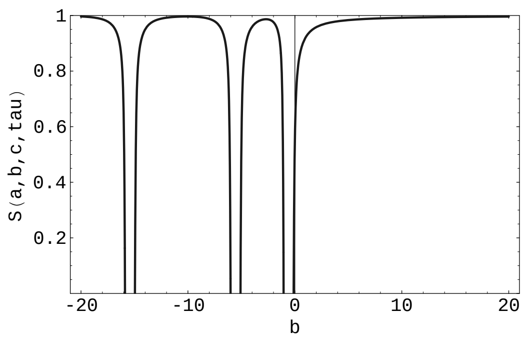

Fig. 1. b dependence of the continuum function $S(a,b,c,\tau)$ for fixed values $a = 0.1, c = 0.5, \tau = 1$. The first three nontrivial terms of the asymptotic series (5) were used.

3.2. Gelfand–Yaglom equation

Gelfand and Yaglom proved in Ref. [9] for the harmonic oscillator that nontrivial continuum limit of the finite dimensional integral approximation to the functional integral should be evaluated from N-dimensional integral results by a recurrent procedure. For the harmonic oscillator, Gelfand and Yaglom derived for the unconditional measure integral $Z(\beta)$ the differential equation

$$\frac{\partial^2}{\partial\tau^2}y(\tau) = \frac{2b}{c}\,y(\tau)\,, \tag{7}$$

where $Z(\beta) = \lim_{N\to\infty} \mathcal{Z}_N = y(\beta)^{-1/2}$

Following the idea of Gelfand and Yaglom, we found [1] for an-harmonic oscillator the generalized Gelfand–Yaglom equation. We define the unconditional measure functional integral $Z(\beta)$ by the relation

$$Z(\beta) = \lim_{N\to\infty} \mathcal{Z}_N = \frac{1}{\sqrt{F(\beta)}} \ .$$

The function $S(\tau)$ is given as the continuum limit of Eq. (5)

$$S(\tau) = \lim_{N\to\infty} \mathcal{S}_N \ .$$

The generalized Gelfand–Yaglom equation reads

$$\frac{\partial^2}{\partial\tau^2}F(\tau)+4\left(\frac{\partial}{\partial\tau}F(\tau)\right)\left(\frac{\partial}{\partial\tau}\ln\mathcal{S}(\tau)\right)=F(\tau)\left[\frac{2b}{c}-2\frac{\partial^2}{\partial\tau^2}\ln S(\tau)-4\left(\frac{\partial}{\partial\tau}\ln S(\tau)\right)^2\right], \tag{8}$$

accompanied by the initial conditions $F(0) = 1$, and $(\partial F(\tau)/\partial\tau)\big|_{\tau=0} = 0$.

For $S(\tau)$, one can use a perturbative expansion in the coupling constant a and then solve Eq. (8). This procedure gives a non-perturbative approximation of the functional integral (2). For the harmonic oscillator limit, we have $S(\tau) \to 1$. In the case when function $S(\tau)$ is known exactly, the problem of the functional integral calculation is trivial. Problems arises in the situations, when $S(\tau)$ is known approximately, as the result of a perturbative approach. Then τ dependence of the $S(\tau)$ can be represented by a power expansion, where $S(\tau)$ is reasonably approximated in proximity of $\tau = 0$, but for $\tau = \beta$, the perturbative approach to $S(\beta)$ is not so exact. The development of the function $F(\tau)$ from $\tau = 0$ to $\tau = \beta$ is controlled by the differential equation (8), therefore the approximative knowledge of the function $F(\tau)$ in the proximity of $\tau = 0$ will lead to a more reliable result for $F(\beta)$. This philosophy of calculation corresponds to the idea of evaluation of the physical values by the renormalization group approach.

The dependence of the function $-2(\partial^2/\partial\tau^2)\ln S(\tau) - 4\left((\partial/\partial\tau)\ln S(\tau)\right)^2$ on τ for positive fixed values of parameters a, b, c is shown in Fig. 2.

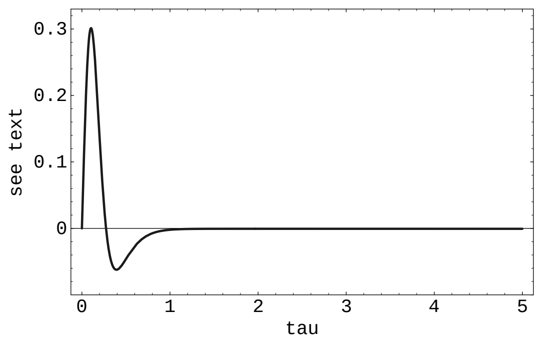

Fig. 2. τ dependence of the continuum function $-2\partial_\mu^2\ln(S(a,b,c,\tau))-4[\partial_\mu\ln(S(a,b,c,\tau))]^2$ for fixed $a = 0.1, b = 5, c = 0.5$. The first three nontrivial terms of the asymptotic series (5) were used.

4. *Conclusions*

We described some features of the confinement problem in the continuum theory, which can't be solved by the perturbative method of evaluations of the functional integral. At the same time, in the second part of the article, we show a new approach to the solution of such problems. We find for the functional integral of an anharmonic oscillator the non-perturbative equation (8). By solving this equation, we can find the analytical solution of the partition sum for an-harmonic oscillator problem.

The skeptic view of such solution of the problem may be expressed by the comment that we reformulate the problem to the language of differential equations. This is true. But our opinion is that the theory of differential equations is more elaborated and flexible than approaches based on the naive perturbative theory and it can give more reliable results than the perturbative theory.

Acknowledgements

This work was supported by VEGA projects No. 2/6074/26.

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PROUČAVANJE ABELIZACIJSKOG PRIJELAZA U SU(2) GLUODINAMICI NA KONAČNOJ TEMPERATURI

Raspravljamo problem učinkovitog opisa pojava faznih prijelaza u čistoj gluodinamici u simetričnom $SU(2)$ QCD. Odabrali smo računalni pristup slijedeći slutnju da infracrveni sektor teorije ima jednake značajke zarobljivanja kao potpuna teorija. Pokazuje se da analitički opis problema nije moguć primjenom Gaussove metode računanja funkcionalnih integrala. Ukazujemo na neperturbativan način izračunavanja funkcionalnih integrala za dvodimenzijski Wienerov integral za x^4 teoriju koji bi mogao riješiti problem.