Printed ISSN 1330–0016
Online ISSN 1333–9133
CD ISSN 1333–8390
CODEN FIZBE7

BLACK HOLE GROWTH AND ACCRETION ENERGY FROM FRACTIONAL ACTION-LIKE VARIATIONAL APPROACH

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Received 12 January 2007; Revised manuscript received 3 September 2008
Accepted 3 September 2008 Online 17 October 2008

Some interesting aspects and features of black hole growth and accretion energy rather than evaporation are discussed within the framework of fractional action-like variational approach recently introduced by the author.

PACS numbers: 04.70.-s, 45.10.Hj UDC 524.882

Keywords: fractional action-like variational approach, fractional geodesic equation, black hole growth, energy accretion, entropy

1. *Introduction*

Motivations and Fractional Action-Like Variational Approach (FALVA). The physics of supermassive black holes dissipative processes and their growth in galaxies pose very significant challenges in astrophysics. In Einsteins General Relativity theory they appear as classical solutions to the Einstein's vacuum equations and they represent matter that has collapsed down to a point with infinite density known as "singularity". From classical point of view, dealing with singularities is quite delicate, but in reality the real puzzles of black holes arise at the quantum level. Following Hawking theory, black holes have temperature and can consequently radiate [1]. In fact, for almost thirty years, since Hawking first proposed the idea of black holes as thermal objects, mathematicians and physicists have been highly interested in the contradiction to quantum mechanics that arises due to the nonunitarity of the evaporation and decaying processes. From physical point of view, the black hole mass decays as does the area of the horizon due to radiation process, but including the entropy of the emitted radiation, the total entropy increases as demanded by the second law of classical thermodynamics. From other sights, there exists some theoretical evidence for extracting energy from accretion into Kerr (rotating) black hole. In other others, Li and Paczyński proposed a new efficient process of converting rest mass to energy by alternating the following two processes: first the ordinary matter is extracted from a thin disk into a Kerr black hole, then energy is extracted magnetically from the black hole to the disk. These cycles can be repeated (in principle) for an indefinite period and consequently the black hole mass increase by 66% per cycle, and up to 43% of accreted mass are radiated away by the disk [2].

In fact, as suggested by current astrophysical observations, black holes with mass larger than $\approx 4 \times 10^{24}$ g would accrete a considerable amount of energy rather than evaporate in a Hawking process. In this context, all supermassive black holes masses would increase with time until they reach the critical mass of about 6×10^{55} g, comparable to the mass contained within the present day cosmological horizon [3]. As a result, one may ask if we really understood the black hole entropy principle. It has been a mystery for many years what the microscopic degrees of freedom are that give rise to this entropy and how we can formulate the dissipative processes occurring in the vicinity of the black hole. Many phenomenological models were proposed in particular those emerged from string theory [4] but none of them succeed to describe correctly the dissipative behavior. Since string theory is a quantum theory of gravitation, one would expect to get a possible result from it. But the major problem has been that due to the large mass of black holes, strong interaction occurred and string theory as it is known is defined perturbatively. So only after some difficult non-perturbative tools were obtainable, was this mystery addressed. From classical point of view, there exist some alternatives. Recently, in order to interpret black holes decay, Gupta and Sen proposed a novel geometrical formalism for analyzing the near-horizon conformal symmetry (NHCS) of Schwarzschild black holes using a scalar field probe [5]. The black hole equation is identified as the geodesic equation in the space of black hole mass. Their approach was based on conformal group structure in the near-horizon region and on the requirement of unitarity of the conformal theory and self-adjointness of the associated near-horizon Hamiltonian. Despite that a precise correction term to the usual expression for the decay rate of black holes of mass M and to the characteristic logarithmic correction to the Bekenstein-Hawking entropy was given, dissipative processes were ignored in their theory. Note that the behaviour of dissipative accreting matter close to a black hole provides important astrophysical features of galactic and extra-galactic black holes candidates [6]. In addition, we are interested on the one-way nature of black hole event horizons where more energy will enter the black hole throat than will be released and consequently, the black hole can gain mass and thereby, $\mathrm{d}M/\mathrm{d}t > 0$ and the entropy increases for non-adiabatic processes, in contrast to Gupta and Sen scenario.

We are totally aware that thermodynamics in general is a particular area of physics where variational principles play a crucial role [7,8]. The reason is that an investigation based on a mathematical theorem shows that the basic differential equations of irreversible thermodynamics, the transport equations cannot be derived from a usual Hamiltonian type variational principle. In this paper, we demonstrate that the NHCS concerning the self-adjoint extension of the near-horizon Hamiltonian can yield additional features within the framework of fractionally differentiated Lagrangian function where violation of energy conservation via weak dissipation is one of its basic ingredients. It is worth-mentioning that the gravitational field around the black hole may acts like a dissipative quantum channel and consequently, the energy is not conserved. In addition, there exist many theoretical arguments that the violation of Noether's charge and energy conservations is nothing than a signature for black hole formation [9]. To a good approximation, energy is conserved for a large black hole as it hardly radiates, but this is not the case for a small black hole due to Hawking radiation. The system of any size black hole and Hawking radiation conserves mass-energy. However, this is not the case for the black hole itself [10]. Moreover, information loss associated with breakdown of quantum mechanics laws and postulates leads apparently to violations of energy conservation. One may refer the energy loss to the presence of a friction force normally present in the geodesic equation.

As one of the main features of the fractional calculus of variations is the violation of energy, we are interested to explore the fractional dynamics of a black hole.

In reality, it is well believed today that fractional calculus is a quite irreplaceable means for description and investigation of classical and quantum complex dynamical system with holonomic as well as with nonholonomic constraints [11]. The study of fractional calculus (FC) opened new branches of thought and fills in the gaps of traditional standard calculus in ways that as of yet, no one completely assimilates or understands. FC describes more accurately the complex physical systems and at the same time, investigates more about simple dynamical systems. FC has recently been applied to many problems in physics, finance and hydrology, polymer physics, biophysics and thermodynamics, chaotic dynamics, chaotic advection, random Brownian walks, modeling dispersion and turbulence, viscoelastically damped structures, control theory, transfer equation in a medium with fractal geometry, stochasting modeling for ultraslow diffusion, kinetic theories, statistical mechanics, dynamics in complex media, wave propagation in complex and fractal media, astrophysics, cosmology, etc [11]. Today there exist many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types, including Grunwald-Letnikov fractional derivative, Caputo fractional derivative, etc., but the Riemann-Liouville derivative and integral are still the most frequently used. Although various fields of application of fractional derivatives and integrals are already well done, some others have just started in particular the study of fractional problems of the Calculus of Variations (COV) and respective Euler-Lagrange type equations is a subject of current strong research and investigations. Many different approaches were proposed in literature but most of them faced difficult mathematical manipulations although when dealing with "simple" dissipative problems (Riewe [12], Klimek [13], Agrawal [14], Baling and Avkar [15], Klimek [16]). The major problem with all these approaches is the presence of non-local fractional differential operators and the adjoint of a fractional differential operator used to describe the dynamics is not the negative of itself. Other complicated problems arise during the mathematical manipulations as the appearance of a very complicated Leibniz rule (the derivative of product of functions) and the non-presence of any fractional analogue of the chain rule.

Recently, we proposed a novel approach known as fractional action-like variational approach (FALVA) or fractionally differentiated Lagrangian function (FDLF) to model nonconservative dynamical systems where fractional time integral introduces only one parameter while in other models an arbitrary number of fractional parameters (orders of derivatives) appear [17-30]. The derived Euler-Lagrange equations are similar to the standard one but with the presence of fractional generalized external force acting on the system. No fractional derivatives appear in the derived equations. The conjugate momentum, the Hamiltonian and the Hamiltons equations are shown to depend on the fractional order of integration and vary as inverse of time. We review rapidly in what follows its basic concepts on classical and Riemann manifolds.

A)-Consider a smooth classical manifold (configuration space) and denote $L : R \times TM \to R$ be the smooth Lagrangian function (smooth map). For any piecewise smooth path $q : [t_0, t_1] \to M$ we define the fractional action integral by

$$S_L[q] = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} L(\dot{q}(\tau), q(\tau), \tau)\,(t-\tau)^{\alpha-1} \mathrm{d}\tau = \int_{t_0=0}^{t} L(\dot{q}(\tau), q(\tau), \tau)\,\mathrm{d}g_t(\tau)\,, \quad (1)$$

where $L(\dot{q}, q, \tau)$ is the Lagrangian weighted with $(t-\tau)^{\alpha-1}/\Gamma(\alpha)$ and

$$\Gamma(\alpha) g_t(\tau) = t^{\alpha} - (t-\tau)^{\alpha}\,, \quad (2)$$

with the scaling properties

$$g_{\mu t}(\mu\tau) = \mu^{\alpha} g_t(\tau), \quad \mu > 0\,. \quad (3)$$

In reality, we consider a smooth action integral (a time smeared measure $\mathrm{d}g_t(\tau)$ on the time interval $[0, t] \in R^{+}$) which can be rewritten as the strictly singular Riemann-Liouville type fractional derivative Lagrangian

$$S_{\beta\in[0,1]}[q] = D_1^{-1+\beta} L(\dot{q}(t), q(t), t) \quad (4)$$

$$= \int_0^t L(\dot{q}(t), q(t), t) \frac{\mathrm{d}\tau}{(t-\tau)^{\beta}} \xrightarrow{\beta\to 0} \int_0^t L(\dot{q}(t), q(t), t)\mathrm{d}\tau\,, \quad (5)$$

and thereby retrieved the standard action integral or functional integral.

In this work, we have $\beta = 1-\alpha, \alpha \in (0,1)$. Such type of functionals is known in mathematical economy, describing, for instance, a so called “discounting” economical dynamics. The true fractional derivatives are also often, nowadays, used for describing so called “dissipative structures” appearing in nonlinear dynamical systems etc.

The fractional Euler-Lagrange equations associated to the fractional action integral (1) was proved to take the form

$$E_i(L) \equiv \frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial y^i}\right) = \frac{1-\alpha}{t-\tau}\frac{\partial L}{\partial y^i} = \frac{\partial R}{\partial y^i} = F_{y_i,\alpha} \tag{6}$$

where $R(y^i) \equiv (1-\alpha)L(y^i)/(t-\tau)$ is identified as the fractional Rayleigh dissipation function. The critical points are then solutions of the fractional Euler-Lagrange Eqs. (2) with $1 \leq 1 \leq N$.

B)-Given a smooth Riemann manifold M, we consider C^2 curves $q : [t_0, t] \to R$ which assign to every point in R^N an invertible symmetric matrix with entries g_{ij} (metric) with Lagrangian given by

$$L(q,\dot{q}) = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}(q)\dot{q}^i\dot{q}^j\,. \tag{7}$$

A path $q = q(\tau)$ makes the fractional action integral stationary if and only if its parametric equation $x^i = x^i(\tau)$ in any coordinate system (x^i) satisfies the fractional equation [21]

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{ik}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right) = \frac{1-\alpha}{t-\tau}\left(g_{ik}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right), \quad 1 \leq k \leq N\,. \tag{8}$$

If U is an open subset of R^n with path $\gamma : [t_1, t_2]$, where we may define the fractional length of γ by Eq. (2), then γ^k satisfy the fractional differential equation

$$\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}\tau^2} + \frac{\alpha-1}{\tau-t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}\tau} + \sum_{i,j}\frac{\mathrm{d}\gamma^i}{\mathrm{d}\tau}\Gamma^k_{ij}\frac{\mathrm{d}\gamma^j}{\mathrm{d}\tau} = 0\,, \tag{9}$$

where

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{\gamma}(\partial_i g_{jr} + \partial_j g_{ir} - \partial_r g_{ij})\, g^{rk} \tag{10}$$

is the Christoffel symbol. The one for geodesic motion is

$$\frac{\mathrm{d}^2x^k}{\mathrm{d}\tau^2} + \frac{\alpha-1}{\tau-t}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} + \Gamma^k_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} = 0\,. \tag{11}$$

Written as a system of first-order equations, the integral curves are

$$\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = v^k \tag{12}$$

$$\frac{\mathrm{d}v^k}{\mathrm{d}\tau} + \frac{\alpha-1}{\tau-t}v^k + \Gamma^k_{ij}v^iv^j = 0\,. \tag{13}$$

The term $F^k \equiv (1-\alpha)v^k/(\tau - t)$ is the decaying force term or the input weak decaying vector field. In fact, by defining $\dot{x}^\sigma \equiv \mathrm{d}x^\sigma/\mathrm{d}T = y^\sigma$ $(T \equiv \tau - t)$, Eq. (13) is identical to a Langevin equation with a time-dependent friction term in case a random source characterizing the properties of medium where motion occurs is applied. Equation (11) holds for any parametrised curve $\lambda \in R$

$$\frac{\mathrm{d}^2 x^k}{\mathrm{d}\lambda^2} + \frac{\alpha - 1}{\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} + \Gamma^k_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} = 0\,, \tag{14}$$

or in terms of v^k

$$\frac{\mathrm{d}v^k}{\mathrm{d}\lambda} + \frac{\alpha - 1}{\lambda}v^k + \Gamma^k_{ij}v^i v^j = 0\,. \tag{15}$$

Equations (11)-(15) represent the basic ingredient of our work, and we expect that our fractional formalism holds on the space of all self-adjoint extensions for analyzing the near-horizon conformal structure capable of describing black hole decay.

2. *Black hole decay as fractional geodesic equation*

Given a continuous Abelian group [5] for a $4D$ black hole

$$G \equiv \left\{ g(M^2) = \mathrm{e}^{2\pi M^2} \;:\; M^2 \in R \right\}, \tag{16}$$

the group invariant metric can be written as

$$\mathrm{d}s^2 = \mathrm{Trace}(g^{-1}\mathrm{d}g)^2 = 16\pi^2 M^2 (\mathrm{d}M)^2\,. \tag{17}$$

In fact, G acts as a group of transformation in the space of all self-adjoint extensions denoted by Ω. It was pointed in Ref. [3] that M is a good candidate to describe Ω. Having this in mind, for any parametrised curve $M(\lambda) \in \Omega \;/\; \lambda \in R$, the fractional geodesic equation in Ω is written as

$$\frac{\mathrm{d}^2 M}{\mathrm{d}\lambda^2} + \frac{\alpha - 1}{\lambda}\frac{\mathrm{d}M}{\mathrm{d}\lambda} + \Gamma^M_{MM}\frac{\mathrm{d}M}{\mathrm{d}\lambda}\frac{\mathrm{d}M}{\mathrm{d}\lambda} = 0\,, \tag{18}$$

where $\Gamma^M_{MM} = M^{-1}$. This equation describes how the mass of the black hole changes with respect to λ when weak dissipations are present. Equation (18) may be written as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\lambda^{\alpha-1}\frac{\mathrm{d}M}{\mathrm{d}\lambda}\right) + \frac{\lambda^{\alpha-1}}{M}\left(\frac{\mathrm{d}M}{\mathrm{d}\lambda}\right)^2 = 0\,. \tag{19}$$

With $Y = M\mathrm{d}M/\mathrm{d}\lambda$, one can easily prove that

$$\frac{\mathrm{d}M}{\mathrm{d}\lambda} = \frac{C\lambda^{1-\alpha}}{M}\,, \quad 0 < \alpha < 1\,, \tag{20}$$

where C is an integration constant. Equation (20) describes in fact how the mass of the black hole changes with respect to λ up to an overall undetermined constant C. In order to obtain a decaying process to compensate the energy loss, one possible realistic choice is $\lambda = M^{-1}T + a$, where a is a constant. Equation (20) reduces to

$$\frac{\mathrm{d}M}{\mathrm{d}T} = \frac{D}{M^{3-\alpha}}(T + aM)^{1-\alpha}, \quad 0 < \alpha < 1\,, \tag{21}$$

where D is a constant of integration. We thus have a fractional description of the decay of black holes which depends on the constant a. For $\alpha = 1$ or $T \ll 1$, we find the standard result for black hole decay. Equation (21) represents a deviation from the standard case. For the particular case $a = 0$, $M \propto -T^{(2-\alpha)/(4-\alpha)}$, the mass of the black hole (decreases/increases) with time as a power-law ($D < 0/D > 0$, respectively)[1]. This solution holds also at late times or very large black hole mass. An interesting feature may appear at late times and very low black hole mass if $a < 0$. This yields $M \propto T^{(2-\alpha)/(4-\alpha)}$ and surprisingly, the mass of the black hole may increase with time. As pointed previously, the mass of a black hole may grow by accretion or decrease by emission of Hawking radiation. Surprisingly, light black holes (low mass) may accrete energy and grow larger. This growth may stop if we define from Eq. (21), and for the particular case $a < 0$, the characteristic mass $M^* = -aT$ for which $\mathrm{d}M/\mathrm{d}T = 0$. M^* represents the mass of a critical black hole, growing linearly with time [31]. With the choice $\lambda = MT + a$, one obtains

$$\frac{\mathrm{d}M}{\mathrm{d}T} = \frac{D}{(MT + a)^{1-\alpha}}, \quad 0 < \alpha < 1\,, \tag{22}$$

and thus for $\alpha = 1$, we find a consistent relation with mass conservation as it is expected. If for instance, we choose $a = 0$, one easily finds $M \propto T^{\alpha/(2-\alpha)}$ and thus for $0 < \alpha < 1$, M increases slowly with time. In fact, one may allow the black hole mass to grow slowly with time but one may alter the scale-length of the black hole potential in galaxy dynamics [32]. In reality, virtual pairs of particles near a black hole can steal energy from the gravitational field and energy of new particles that came from gravitational energy of black hole, thus a black hole mass must decrease. Thus, we favour Eq. (21) with respect to Eq. (22).

If we reinstate Newton's constant, the black hole entropy is simply given by $S = 4\pi GM^2$ and consequently increases with time if M grows with time. One then expects that the temperature of an accretion disk orbiting a black hole should decrease with increasing mass or entropy. For $M^* = -aT$, $S = 4\pi GM^{*2}$ increases linearly with time while the entropy is constant (adiabatic process)[2]. More generally, one can extend the usual linear black hole entropy to a more generalized

[1] In Ref. [5], the constant D was chosen negative, based on the expectation that the back-reaction effects will cause the mass of the black hole to decrease in order to compensate for this energy loss.

[2] For a decreasing black hole mass, the entropy may remain constant if we assume that the gravitational constant increases with time by the same order. Increasing gravitational constant is widely discussed in the literature. (see for example Refs. [33[, [34] and [35] and references therein).

fractional formula given by $S \propto T^{2(2-\alpha)(4-\alpha)}$. Note that $m = (4-2\alpha)/(4-\alpha) < 1$ for $0 < \alpha < 1$ and consequently, the entropy increases slowly with time.

For a very massive black hole, the metric takes the form

$$\mathrm{d}s^2 \approx 16\pi^2 M^2 \left(1 - \frac{3}{16\pi^2 M^2}\right), \tag{23}$$

with the corresponding fractional geodesic equation

$$\frac{\mathrm{d}^2 M}{\mathrm{d}\lambda^2} + \frac{\alpha-1}{\lambda}\frac{\mathrm{d}M}{\mathrm{d}\lambda} + \left(\frac{1}{M} + \frac{3}{16\pi M^3}\right) = 0\,. \tag{24}$$

with the following corresponding solution

$$\frac{\mathrm{d}M}{\mathrm{d}\lambda} = \frac{E\lambda^{1-\alpha}}{M}\ \exp\left(\frac{3}{32\pi M^2}\right), \tag{25}$$

which gives for $\lambda = M^{-1}T + a$

$$\frac{\mathrm{d}M}{\mathrm{d}T} \approx \frac{E(M^{-1}T+a)^{1-\alpha}}{M^2}\ \left(1 + \frac{3}{32\pi M^2}\right). \tag{26}$$

E is a constant of integration[3]. For $\alpha = 1$, we find the standard Bekenstein-Hawking decay rate formula for large black hole mass. At very late times and for a massive black hole, one can easily deduce that the mass of the black hole decreases with time as a power-law ($E < 0$), but may increase for $\alpha < 0$. In this last case, the entropy grows with time. Notice that from a classical point of view, the event-horizon Schwarzschild radius in the expanding Universe should grow with time due to the isotropy and homogeneity of the cosmos, which, consequently causes increasing entropy for black holes [36]. Again, for the particular case $\alpha < 0$, the characteristic mass $M^* = -aT$ leads to $\mathrm{d}M/\mathrm{d}T = 0$ when the entropy is constant.

3. Conclusions

The fractional action-like variational approach with its corresponding fractional geodesic equation yields important additional features concerning the growth of black holes with time, their corresponding fractional entropy and the fractional generalization of the second law of thermodynamics. The fractional mechanism described in this work may give some important features concerning primordial black holes in the early Universe predicted in some inflationary models, non-local quantum black hole thermodynamics [37], their geometrical aspects and evolution [38]. Further details are under progress.

[3] In Ref. [5], the constant is chosen also negative for the same assumption as stated in the previous footnote.

Acknowledgements

The author would like to thank the anonymous referees for their useful comments and suggestions.

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PRIMJENA DIJELNOG UČINU-SLIČNOG VARIJACIJSKOG NAČELA ZA RAST CRNIH JAMA I ENERGIJU SKUPLJANJA

U okviru dijelnog učinu-sličnog varijacijskog načela, nedavno uvedenog ovim autorom, raspravljaju se neki zanimljivi izgledi i odlike rasta crnih jama i energije skupljanja, umjesto isparavanja crnih jama.

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