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LETTER TO THE EDITOR

ON THE GRAVITOMAGNETIC CLOCK EFFECT

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We show that the whole of the gravitomagnetic clock effect in the orbit of a spinning test particle which is revolving round a spinning central massive astrophysical body can be calculated using a gravitational spin-orbit coupling potential which involves the two spins and the orbital angular momentum of the test particle.

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General relativity predicts that two freely counter-revolving test particles in the exterior field of a central rotating mass take different periods of time to complete the same full orbit; the time difference is termed as the gravitomagnetic clock effect (GCE). Let us consider circular geodesics in the equatorial plane of a Kerr black hole of mass M and spin angular momentum J. Let t_+ denote the period of co-rotating (i.e. motion in the same sense of the rotation of the central body) motion and t_- denote that of counter-rotating motion along such an orbit according to asymptotically static inertial observers. Then

$$t_+ - t_- = \frac{4\pi J}{Mc^2} \,. \tag{1}$$

This result was first derived by Cohen and Mashhoon [1]. Subsequently, various theoretical aspects of this effect have been investigated [2 − 7]. On the observational

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side, the possibility of its detection has been considered by a number of authors [8 – 13].

A remarkable addition to the scenario of GCE came through the work of Faruque [14], namely, when the orbiting particle possesses spin, it contributes to the gravitomagnetic clock effect an additional amount given by $6\pi S/mc^2$, where S is the spin of the orbiting test body whose mass is m. Subsequently, Bini et al. [15] have shown that the clock effect can be classified in three simple situations: (1) co-rotating orbit with test particle spin up against counter-rotating orbit with test particle spin up, (2) co-rotating orbit with spin down against counter-rotating orbit with spin down, and (3) co-rotating orbit with spin up against counter-rotating orbit with spin down and vice versa. If we picture these situations, we see that actually in the first case the two spins $\vec{J}$ and $\vec{S}$ are parallel and in the second case they are antiparallel. In the third case an orbit of the first case is differentiated with an orbit of the second case. In this short article, we shall show, in a very concise way, how the clock effects in all these cases can be obtained using gravitational spin-orbit coupling. First of all we quote the results from Refs. [14, 15]:

Case I:

$$t_+ - t_- = \frac{4\pi J}{Mc^2} + \frac{6\pi S}{mc^2}\,, \tag{2}$$

Case II:

$$t_+ - t_- = \frac{4\pi J}{Mc^2} - \frac{6\pi S}{mc^2}\,, \tag{3}$$

Case III:

$$t_+ - t_- = \frac{4\pi J}{Mc^2}\,. \tag{4}$$

The derivation of these results was done by solving Mathisson-Papapetrou equations in Ref. [15], and the result in Eq. (3) was found in Ref. [14] by approximation procedures. However, it is shown in Refs. [16] and [17] that in the simple situation of circular motion of a spinning particle in the equatorial plane of a non-spinning central body, the second term in Eqs. (2) and (3) can be obtained using gravitational spin-orbit coupling. Subsequently, this line of derivation of clock effect is elaborated by Mashhoon and Singh [18]. As noted in Ref. [19], it is not possible to decide by the distant static observer whether he is measuring a time delay of spinning clocks in a non-rotating space-time or a time delay of non-spinning clocks moving on geodesics in a rotating space-time. That is, there is an equivalence. Hence, we can assume that the set of Eqs. (2 – 4) should follow from a single Hamiltonian containing spin-orbit coupling potentials where both spins are involved. As shown by Barker and O'Connel [20], the spin of the central source couples with the orbital angular momentum of the orbiting particle, and there is coupling between the spin and orbital angular momentum of the orbiter, too. Hence, in a scenario where the gravito-electric part of the field is approximated by the Newtonian potential, the total Hamiltonian can be written as

$$H = \frac{p^2}{2m} + m\Phi + V_{LJ} + V_{LS}\,, \tag{5}$$

where

$$V_{LJ} = \xi_1 \frac{G}{c^2 r^3} \vec{J}\cdot\vec{L}\,, \qquad V_{LS} = \xi_2 \frac{GM}{mc^2 r^3} \vec{S}\cdot\vec{L}\,, \tag{6}$$

where ξ_1 and ξ_2 are constants of the order of unity. Their exact value can not be ascertained as yet. However, the results of Eqs.(2−4) can be generated by taking $\xi_1 = 1$ and $\xi_2 = 3/2$. Hence, in what follows we will use these values. In a simple situation where $\vec{J}$, $\vec{S}$ and $\vec{L}$ all point either in the $+\hat{z}$ or $-\hat{z}$ direction, we can proceed to calculate the clock effects using the procedure followed in Ref. [16]. In the equatorial circular orbit, if $\vec{J}$ and $\vec{S}$ point along the $+\hat{z}$ direction and in the first consideration we take $\vec{L}$ to point also in the $+\hat{z}$ direction, then the equation of motion reads

$$-mr\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 \hat{r} = -\frac{GMm}{r^2}\hat{r} + \frac{2Gm}{c^2 r^2}\left(J + \frac{3M}{2m}S\right)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)\hat{r}\,. \tag{7}$$

When $\vec{J}$ points along the $+\hat{z}$ direction and $\vec{S}$ points along the $-\hat{z}$ direction, the equation of motion reads

$$-mr\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 \hat{r} = -\frac{GMm}{r^2}\hat{r} + \frac{2Gm}{c^2 r^2}\left(J - \frac{3M}{2m}S\right)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)\hat{r}\,. \tag{8}$$

Equation (7) corresponds to the first case we discussed above and Eq. (8) corresponds to the second case. The third case is generated with Eq. (7) and (8). We shall briefly sketch how Eq. (7) leads to the clock effect in Eq. (2).

Case I: $\vec{J}$ and $\vec{S}$ point along the $+\hat{z}$ direction

This case is described by Eq. (7) which is reduced to the following form

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 + \frac{2G}{c^2 r^3}\left(J + \frac{3M}{2m}S\right)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right) - \frac{GM}{r^3} = 0\,, \tag{9}$$

which is a quadratic equation having solutions, neglecting smaller term issuing under the square root, given by

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -\frac{G}{c^2 r^3}\left(J + \frac{3M}{2m}S\right) \pm \sqrt{\frac{GM}{r^3}}\,. \tag{10}$$

Here, the positive sign refers to the co-rotating orbit and the negative sign refers to the counter-rotating orbit with reference to the direction of $\vec{J}$. The second term in

Eq. (10) is larger than the first, so, in finding $\mathrm{d}t/\mathrm{d}\varphi$, we use the binomial theorem and obtain the following approximate formulae

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\varphi}\right)_+ = \frac{1}{\omega_K} + \frac{G}{c^2 r^3 \omega_K^2}\left(J + \frac{3M}{2m}S\right) = \frac{1}{\omega_K} + \frac{J}{Mc^2} + \frac{3}{2}\frac{S}{mc^2}\,, \tag{11}$$

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\varphi}\right)_- = -\frac{1}{\omega_K} + \frac{J}{Mc^2} + \frac{3}{2}\frac{S}{mc^2}\,, \tag{12}$$

where $\omega_K = \sqrt{GM/r^3}$ is the Keplerian angular frequency. Now, for the co-rotating orbit, we obtain the period of circular motion by integrating Eq. (11) from $\varphi = 0$ to $\varphi = 2\pi$. For the counter-rotating orbit, we obtain the period by integrating Eq. (12) from $\varphi = 0$ to $\varphi = -2\pi$. In this way, we obtain

$$t_+ = \frac{2\pi}{\omega_K} + \frac{2\pi J}{Mc^2} + \frac{3\pi S}{mc^2}\,, \tag{13}$$

and

$$t_- = \frac{2\pi}{\omega_K} - \frac{2\pi J}{Mc^2} - \frac{3\pi S}{mc^2}\,. \tag{14}$$

The period difference is

$$t_+ - t_- = \frac{4\pi J}{Mc^2} + \frac{6\pi S}{mc^2}\,. \tag{15}$$

Thus, we have found the result depicted in Eq. (2) for clock effect in co-rotating orbit with spin up against counter-rotating orbit with spin up.

Case II: $\vec{J}$ points along the $+\hat{z}$ direction and $\vec{S}$ points along the $-\hat{z}$ direction

This case is described by Eq. (8). Following the same procedure just described for the Case I, we obtain the time periods

$$t_+ = \frac{2\pi}{\omega_K} + \frac{2\pi J}{Mc^2} - \frac{3\pi S}{mc^2}\,, \tag{16}$$

$$t_- = \frac{2\pi}{\omega_K} - \frac{2\pi J}{Mc^2} + \frac{3\pi S}{mc^2}\,. \tag{17}$$

The period difference in this case is

$$t_+ - t_- = \frac{4\pi J}{Mc^2} - \frac{6\pi S}{mc^2}\,. \tag{18}$$

Thus we have found the result depicted in Eq. (3) for the clock effect in co-rotating orbit with spin down against counter-rotating orbit with spin down.

Case III: Co-rotating with spin up against counter-rotating with spin down or co-rotating with spin down against counter-rotating with spin up

In this case the time period in Eq.(17) is subtracted from the time period in Eq.(13) to get the clock effect. Or the time period in Eq.(14) is subtracted from the time period in Eq.(16). In both cases we obtain

$$t_+ - t_- = \frac{4\pi J}{Mc^2}\,. \tag{19}$$

Thus we have found Eq. (4) for the clock effect in the co-rotating with spin up against the counter-rotating with spin down or the co-rotating with spin down against the counter-rotating with spin up.

We now summarize: The gravitomagnetic clock effect in the orbit of a spinning particle which is revolving round a spinning central body contains two terms involving the two spins. In previous studies, the results were calculated by solving Mathisson-Papapetrou equations. Here, we have shown that whole set of the results can be derived using spin-orbit coupling potential. The signs of the terms came out here exactly as those in the previous studies. Hence, not only the contribution of the spin of the orbiting test particle, but also the contribution of the spin of the central body can be thought of as due to gravitational spin-orbit coupling. This gives us a complete view for the whole picture of the gravitomagnetic clock effect; this unified view is, as far as know, not shown elsewhere in literature.

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O GRAVITOMAGNETSKOM SATNOM UČINKU

Pokazujemo da se ukupni gravitomagnetski satni učinak za stazu ispitne čestice, koja se vrti i kruži oko središnjeg masivnog astrofizičkog tijela koje se vrti, može izračunati primjenom gravitacijskog potencijala vezanja spina i staze dviju vrtnji i staznog momenta impulsa ispitne čestice.