ISSN 1330–0016
CODEN FIZBE7

SINGULAR AND EXACTLY SOLVABLE POTENTIALS

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Received 25 November 2002; Accepted 8 December 2003
Online 19 April 2004

The potentials involving singularities considered in this work result from the construction starting with higher-order-generation superpotentials. The second generation with a single base as well as the third one with a double base will be discussed. According to the choice of these bases, a number of results, which may be interesting from the theoretical point of view, can be observed such as the partial breaking of the symmetry or the construction of a new type of exactly solvable potentials which show existence of eigenstates with positive eigenvalues. Extension to the third generation potential of the theorem on quasi-exact solvability will also be discussed.

PACS numbers: 03.64.Ge UDC 530.145

Keywords: higher-order superpotentials, singularities, second generation with a single base, third generation with a double base, breaking of symmetry, existence of eigenstates

1. Introduction

Exactly solvable potentials have the property that the entire set of eigenvalues and eigenfunctions can be reached analytically by algebra, while for quasi-exactly ones only part of them is available by the same methods.

Except perhaps for some very special cases, potentials involving singularities are not exactly solvable, and whenever the solution may exist, it must be obtained by other means, mostly numerical.

Singular potentials serving as models are nevertheless quite useful in many aspects of physics; some of them can also be tackled analytically. For instance, the conventional multi-term singular potentials of the form $\sum_m x^{-m}$ widely used in atomic physics can be analytically tractable from the point of view of collision theory. In fact, when the quantities m are integer numbers, an analytical expression of the phase shift can be constructed within the frame of the JWKB approximation [1] and more recently, it has also been shown that for non-integer (fractional) numbers, an analytical approach remains still possible from the point of view of the eigenspectra [2,3].

In the past decade and chiefly making use of the advances in supersymmetric quantum mechanics (see Refs. [4,5] and references therein) one may observe several attempts to analytically deal with singular potentials either directly or indirectly [6–13].

The type of singular potentials to be considered in this work differs from the conventional one in the sense that the singular term x^{-m} will be replaced by $\mathrm{d}\log[f_n(x)]/\mathrm{d}x$ in which the function $f_n(x)$ is a polynomial of order n. The zeros of this function ($f_n(x_i) = 0$) constitute the set of singularities of the problem.

The present paper describes a number of recently obtained results in this relatively new field of research. Three main points will be elaborated:

(a) As mentioned above, although the type of singular potentials considered here (they will be referred to as the second generation potentials) are generally not exactly solvable in the usual sense, and sometimes discarded a priori since they would lead to non-hermitian Hamiltonians, it was shown that under certain conditions, analytical and normalisable solutions can be attained in closed forms. As an illustration, the method was applied to the case of anharmonic oscillator and presented in Ref. [14]. For completeness, we present in this work a second example which can be handled in the same way but after some modifications. These two examples constitute two aspects of the multiple facets of the theory.

(b) If it is true that, taken separately, the second-generation potentials $V^{(n)+}$, $V^{(n)+}$ depending on the bases m, n, …, are singular and generally lead to non-hermitian Hamiltonians, it will be shown that by combining these bases according to specific rules, there is a possibility to generate a new type of potentials $V^{(n,m)}$ referred to as the third-generation potentials which, surprisingly, are free of singularities and furthermore exactly solvable in the conventional sense. From the theoretical point of view, this undoubtly can constitute a useful tool if one wishes to proceed a step further in the investigation. The examples discussed explain how to initiate the transition from the second to the third type of potentials.

(c) When the second generation potentials $V^{(n)+}$ are not exactly solvable, it was also shown that regardless of the choice of the base $|n\rangle$, and under certain conditions, a theorem on its quasi-exact solvability can be proved. It is then interesting to extend the same idea to the new situation concerning the third-generation potentials $V^{(n,m)}$ in order to see whether this theorem would remain valid and what can be its consequences.

In the following, unnecessary repetitions can be avoided by first keeping exactly the same notations and conventions as used in Ref. [14] (referred to as [I] from now on). These notations will be gradually modified according to the needs of the discussion as explained later.

2. *Theory*

Starting with a couple of $SU(2)$ partners $V^{\mp} = u^2 \mp u'$, u being the usual superpotential $u(x)$, we have already shown that if V^- is exactly solvable, the set of its eigenfunctions can be always be written as

$$\phi_n \quad = \quad X_n \exp\left(- \int u \mathrm{d}x \right), \tag{1}$$

where the X_n, referred to as the mixing functions ($n = 0, 1, \ldots$), must be solutions of the second order differential equation

$$X_n'' - 2uX_n' + E_n X_n \quad = \quad 0, \tag{2}$$

which therefore links the ordinary second-order differential equation with integrable systems [15,16].

Assuming exact solvability of V^-, higher-order generation superpotential can be constructed

$$v^{(n)} \quad = \quad u(x) - \frac{X_n'}{X_n}, \tag{3}$$

X_n being any solution of Eq. (1) and $|n\rangle$ any excited state. Note that the ground state $|0\rangle$ corresponds to $n = 0$; it does not bring anything new. In the following, $n{\neq}0$ and the state $|n\rangle$ refers to the “base” of the construction. The corresponding couple of partners $SU(2)$ potentials $V^{(n)\mp}$ is defined as

$$V^{(n)\mp} \quad = \quad {v^{(n)}}^2 \mp {v^{(n)}}' , \tag{4}$$

and will be referred to as the second-generation potential.

It has also been shown that $\phi_m^{(n)\mp}$, the set of eigenfunctions relative to $V^{(n)\mp}$, are of the form [15,16]

$$\begin{aligned} \phi_m^{(n)-} &= X_m^{(n)} \exp\left(- \int v^{(n)} \mathrm{d}x \right), \\ \phi_m^{(n)+} &= Y_m^{(n)} \exp\left(- \int v^{(n)} \mathrm{d}x \right), \qquad m = 1, 2, \ldots, \end{aligned} \tag{5}$$

with obvious notations, ϕ_m representing the m^{th} excited state. The second-generation mixing functions $X_m^{(n)}$, $Y_m^{(n)}$ are given by

$$\begin{aligned} X_m^{(n)} &= \frac{X_m}{X_n}, \\ Y_m^{(n)} &= \frac{1}{X_n}\left[X_m' - \frac{X_n'}{X_n} X_m \right], \end{aligned} \tag{6}$$

which correspond to the couple of potentials

$$\begin{aligned} V^{(n)-} &= V^- + E_n, \\ V^{(n)+} &= u^2 + u' + 2\frac{X_n'}{X_n}\left[\frac{X_n'}{X_n} - 2u \right] - E_n. \end{aligned} \tag{7}$$

From Eq. (5), it is seen that if $\phi_m^{(n)-}$ can always be normalised (note that $V^{(n)-}$ differs from V^- only by a constant), this will not be the case for the second eigenfunction $\phi_m^{(n)+}$, because of the singularities involving in the analytic expression of $Y_m^{(n)}$ (see Eq. (6)). The eigenspectrum of both $V^{(n)-}$ and $V^{(n)+}$ is of the form

$$E_m^{(n)} = -[E_m - E_n].$$

If both components $V^{(n)-}$ and $V^{(n)+}$ are equally exactly solvable, then we have here an example of double degeneracy, a consequence of $SU(2)$ symmetry according to Witten. However, as pointed out above, the presence of these singularities (theorem of Sturm-Liouville) make normalization impossible for the whole or partial set of $\phi_m^{(n)+}$, leading to complete or partial destruction of the pairings between the states of ($V^{(n)-}$, $V^{(n)+}$).

3. *Quasi-exact solvability*

It will be interesting to extend the present analysis by considering an alternative construction of the second-generation potentials $v^{(n)}$ with

$$v^{(n)} = u - t\frac{X_n'}{X_n}, \tag{8}$$

where u is the first-generation potential mentioned above, X_n is a set of mixing functions corresponding to the base $|n\rangle$ and t is an arbitrary parameter. After simplifications, the second-generation potential $V^{(n)-}$ is

$$V^{(n)-} = u^2 - u' + t(t-1)\left(\frac{X_n'}{X_n}\right)^2 + tE_n.$$

Obviously, these types of potentials are not exactly solvable, the set of singularities ($\{x = x_i\}$) which split the domain of the potential $V^{(n)-}$ into several separated parts would now play a crucial role in the search for solutions. The "strength" of the singularities (i.e. the coupling) will depend on the choice of both the magnitude of the parameter t and the base $|n\rangle$. Generally, these couplings are "strong" so that the boundary conditions to be imposed on the solutions should be $\phi(x_i) = 0$ [17,6]. Concerning the Schrödinger equation

$$\phi'' - V^{(n)-}\phi = E\phi\,,$$

one may discern two different situations depending on the magnitude of the parameter t:

1) If $0 < t < 1$, a theorem has already been presented in [I]. It states that independently of the choice of the base $|n\rangle$, this equation has two exact solutions

(ϕ_+, ϕ_-) which constitute then a "doublet". They are

$$\phi_+ = X_n^{1-t} \exp\Big(- \int u\mathrm{d}x \Big), \qquad E_+ = (1-2t)E_n,$$

$$\phi_- = X_n^{t} \exp\Big(- \int u\mathrm{d}x \Big), \qquad E_- = 0.$$

For completeness, the following remarks can be useful later:

(a) If x_i are the zeros of the polynomial X_n, then one can note that $\phi(x_i) \equiv 0$, confirming thus the prescription concerning the "strong coupling" case mentioned above.

(b) The two members of this doublet merge into a single one when $t = 1/2$. For other cases ($t > 1$ or $t < 0$), one is left with only one member (ϕ_+ or ϕ_-) since the other one cannot be normalised.

(c) Unicity of the existence of the "doublet" can be proved.

2) When $t = 1$, one may discern two cases:

(a)

$$V^{(n)-} \quad = \quad V^- + tE_n$$

which means that by construction, $V^{(n)-}$ is also exactly solvable.

(b) But its partner $V^{(n)+}$ (see (7)) is singular and generally leads to non-hermiticity except eventually for some special choice of the base $|n\rangle$. This is confirmed by the example concerning the case of anharmonic oscillator potentials already discussed in [I].

In the following, we present another example which displays a different facet of the method:

3.1. Example

Let

$$u(x) \quad = \quad a \coth x - \frac{b}{\operatorname{sh} x},$$

where a, b are arbitrary parameters.

The initial (first-generation) couple of potentials are

$$V^{\mp} \quad = \quad a^2 + \frac{a(a \mp 1) + b^2}{\operatorname{sh}^2 x} \mp b\big(1 \pm 2a\big)\frac{\operatorname{ch} x}{\operatorname{sh}^2 x}. \tag{9}$$

One can solve Eq. (2) with the change of variable

$$y = \frac{1}{2}\big(\operatorname{ch} x + 1\big),$$

so that Eq. (2) can be cast into the usual hypergeometric form

$$y\big(y-1\big)\frac{\mathrm{d}^2 X}{\mathrm{d}y^2} + \Big[\big(\alpha+\beta+1\big)y - \gamma\Big]\frac{\mathrm{d}X}{\mathrm{d}y} + \alpha\beta X = 0, \tag{10}$$

provided that

$$a = -\frac{1}{2}\big(\alpha+\beta\big), \qquad \gamma = -\big(a+b\big) + \frac{1}{2}.$$

We note that hypergeometric function $F(\alpha, \beta, \gamma; y)$, the solution of Eq. (10), reduces to a polynomial of order n if one takes $\alpha = -n$, $n = 0, 1, \ldots$. The corresponding eigenvalue of the state $|n\rangle$ is

$$E_n = -\big[\big(n-a\big)^2 - a^2\big]. \tag{11}$$

It can be seen that normalization of the eigenfunctions (see Eq. (1)) requires two conditions

$$b > a \qquad \text{and} \qquad n < a, \tag{12}$$

which implies that for a given value of the parameter a, the number of acceptable eigenstates must be limited by this second condition.

It is interesting to consider the first eigenstates $|0\rangle$, $|1\rangle$, $|2\rangle$. When $n = 0$, the hypergeometric function X_0 reduces to a constant with $E_0 = 0$. The analytic expression of this ground state is

$$\phi_0 = \frac{(\operatorname{sh} x)^{b-a}}{(1+\operatorname{ch} x)^b} F_0, \tag{13}$$

which exactly agrees with the result mentioned in Ref. [18].

On the other hand, for the next eigenstate $n = 1$, after some simple algebra, we have the following analytic expression of the mixing function

$$X_1 = A_1\left(1 - \frac{A_2}{A_1}\operatorname{ch} x\right), \tag{14}$$

in which

$$A_1 = \frac{1}{2}\frac{2(b-1)}{a+b-1/2}, \qquad A_2 = \frac{1}{2}\frac{2a+1}{a+b-1/2}. \tag{15}$$

Since $\mathrm{ch}\,x \geq 1$, and taking into account the first condition (12), which means that the polynomial X_1 involves one zero, the corresponding solution has one node and, therefore, must be attributed to the first excited eigenstate.

For the next case, $n = 2$, we find

$$\begin{aligned} X_2 &= \big(1 + \frac{1}{4}B_2 - \frac{1}{2}B_1\big) - \frac{1}{2}\big(B_1 - B_2\big)\,\mathrm{ch}\,x + \frac{1}{4}B_2\,\mathrm{ch}^2\,x, \\ B_1 &= 4\frac{a-1}{a+b-1/2}, \qquad B_2 = \frac{1}{2}B_1\frac{2a-1}{a+b-3/2}. \end{aligned}$$

More generally, the mixing functions X_n can always be represented by a polynomial of order n in terms of $\mathrm{ch}\,x$.

Contrarily to the first example discussed in [I], in the present example, and regardless of the choice of the base $|n\rangle$, the second-generation potential $V^{(n)+}$ does not lead to any acceptable solution (un-normalised) (see Eq. (6)).

4. The third generation

The corresponding superpotential is defined by

$$v^{(m,n)} = v^{(n)} - \frac{Y_m^{(n)\prime}}{Y_m^{(n)}}, \tag{16}$$

where $v^{(n)}$ refers to the second-generation potential (see Eq. (3)) and $Y_m^{(n)}$ is the mixing function Y defined in (6) for the second generation.

As above, consider the system of coupled first-order differential equation

$$\phi^{(m,n)\prime} + F^{(m,n)}\phi^{(m,n)} = 0\,, \tag{17}$$

in which

$$\phi^{(m,n)} = \big(\phi_1^{(m,n)}, \phi_2^{(m,n)}\big)^\dagger, \qquad F^{(m,n)} = \begin{pmatrix} v^{(m,n)} & d^{(m,n)} \\ 0 & v^{(m,n)} \end{pmatrix}.$$

With obvious notation, the mixing function is defined as

$$\phi_{1,r}^{(m,n)-} = X_r^{(m,n)}\phi_2^{(m,n)}. \tag{18}$$

The third-generation couple of potentials $V^{(m,n)\mp}$ are

$$V^{(m,n)\mp} = v^{(m,n)^2} \mp v^{(m,n)\prime}\,, \tag{19}$$

and consider the Schrödinger equation relative to the first component $\phi_{1,r}^{(m,n)-}$ ($r = 0, 1, 2, \ldots$)

$$\phi_{1,r}^{(m,n)''} - V^{(m,n)-}\phi_{1,r}^{(m,n)} = E_r^{(m,n)}\phi_{1,r}^{(m,n)}. \tag{20}$$

It is exactly solvable if the mixing function $X_r^{(m,n)}$ is a solution of the differential equation

$$X_r^{(m,n)''} - 2\left(v^{(n)} - \frac{Y_m^{(n)'}}{Y_m^{(n)}}\right)X_r^{(m,n)'} + E_r^{(m,n)}X_r^{(m,n)} = 0. \tag{21}$$

In Appendix, it will be shown that this solution must be of the form

$$X_r^{(m,n)} = \frac{Y_r^{(n)}}{Y_m^{(n)}}. \tag{22}$$

For the second component $V^{(m,n)+}$, one may proceed in the same way and introduce a second mixing function $Y_r^{(m,n)}$ defined by

$$\phi_{1,r}^{(m,n)+} = Y_r^{(m,n)}\phi_2^{(m,n)}. \tag{23}$$

Here again, exact solvability of the Schrödinger equation means that the quantity $Y_r^{(m,n)}$ must satisfy the following equation

$$\begin{aligned} Y_r^{(m,n)''} &- 2\left(v^{(n)} - \frac{Y_m^{(n)'}}{Y_m^{(n)}}\right)Y_r^{(m,n)'} \\ &- \left[2\left(v^{(n)'} - \left(\frac{Y_m^{(n)'}}{Y_m^{(n)}}\right)'\right) - E_r^{(m,n)}\right]Y_r^{(m,n)} = 0. \end{aligned} \tag{24}$$

Noting that the last equation merely results from the differentiation of Eq. (21). Therefore,

$$Y_r^{(m,n)} = X_r^{(m,n)'} = \left(\frac{Y_r^{(n)}}{Y_m^{(n)}}\right)', \tag{25}$$

and the eigenfunction corresponding to the potential $V^{(m,n)+}$ is of the form

$$\phi_r^{(m,n)+} = \left(\frac{Y_r^{(n)}}{Y_m^{(n)}}\right)' \exp\left(-\int v^{(m,n)}\mathrm{d}x\right). \tag{26}$$

4.1. Discussion

Since the problem now involves simultaneously the two sets of singularities originating from the choice of the two bases $|n\rangle$, $|m\rangle$, normalisation of these solutions must be considered more carefully:

1) For the first solution $\phi_{1,r}^{(m,n)-}$, making use of the relations (22) and (18), one has

$$\phi_{1,r}^{(m,n)-} = Y_r^{(n)} \exp\left(- \int v^{(n)} \mathrm{d}x \right).$$

where $v^{(m,n)}$ is given in (16). One can thus conclude that normalisation of the solution $\phi_r^{(m,n)-}$ is always possible if the choice of the base $|n\rangle$ is such that the second-generation component $\phi^{(n)+}$ is independent of the choice of the second base $|m\rangle$. The legitimacy of this result can be checked by computing the third-generation component $V^{(m,n)-}$ which, after simplifications, is simply

$$V^{(m,n)-} = V^{(n)+} + const, \tag{27}$$

(see Appendix).

2) In order to deal with the second solution and see whether normalisation is also possible, one can use another approach based on the Wronskian formalism.

5. The Wronskian formalism

In relation (6), one can note that

$$Y_m^{(n)} = \frac{1}{X_n^2}\left(X_m' X_n - X_n' X_m\right), \tag{28}$$

and the Wronskian is by definition

$$W(X_m, X_n) = X_m' X_n - X_n' X_m.$$

We write $W(m,n)$ so that

$$Y_m^{(n)} = \frac{1}{X_n^2} W(m,n). \tag{29}$$

Using Eqs. (26) and (29), the analytic expression of the eigenfunction $\phi_r^{(m,n)+}$ is

$$\phi_r^{(m,n)+} = \frac{1}{X_n}\left[\frac{\mathrm{d}W(r,n)}{\mathrm{d}x} - \frac{\mathrm{d}W(m,n)}{\mathrm{d}x}\frac{W(r,n)}{W(m,n)}\right] \exp\left(- \int u \mathrm{d}x \right). \tag{30}$$

Noting that

$$\frac{\mathrm{d}W(r,n)}{\mathrm{d}x} = X_r'' X_n - X_n'' X_r, \qquad \frac{\mathrm{d}W(m,n)}{\mathrm{d}x} = X_m'' X_n - X_n'' X_m,$$

the above result can be written in a more transparent form

$$\phi_r^{(m,n)+} = \big[AX_r'' + BX_r' + CX_r\big] \exp\left(-\int u\mathrm{d}x\right), \tag{31}$$

in which, as can be verified after some simple algebra,

$$A = 1, \quad B = -\frac{1}{W(m,n)} \frac{\mathrm{d}W(m,n)}{\mathrm{d}x},$$
$$C = \frac{1}{W(m,n)}\Big[X_m'' X_n' - X_n'' X_m'\Big]. \tag{32}$$

The solution (32) is not always normalisable because of the set of singularities provided by the function $W(m,n)$. However, since the choice of the two bases $|n\rangle$, $|m\rangle$ is still arbitrary, it will become meaningful if this choice is such that the Wronskian W cannot be equal to zero (i.e. it must be either positive or negative everywhere).

The third-generation second-component potential can be, after simplifications, written in the form

$$V^{(m,n)+} = u^2 - u' + 2\big(E_m - E_n\big)\frac{X_m}{W^2}\Big[X_n W' - 2X_n' W\Big] + 2E_n - E_m. \tag{33}$$

5.1. Discussion

1) The potential is free of singularities if the Wronskian complies to the above condition which dictates the choice of the bases $|m\rangle$, $|n\rangle$. Since the quantities X_m, X_n are polynomials of order m and n, assuming the Fuchsian case and using the Sturm-Liouville theorem, it can be shown (Ref. [17], p. 311) that, for two successive states (i.e. $n+1$, n), their "zeros" must be simple and alternating. More explicitly, this means that if x_i are the zeros of X_n and x_j that of X_{n+1}, then one must have

$$x_i < x_j < x_{i+1} \qquad ,$$

which shows the W must be different from zero.

2) If one considers inversion of the bases, i.e. $m \rightleftharpoons n$, one may note that

$$W(m,n) = -W(n,m)\,. \tag{34}$$

In principle, we may have two potentials $V^{(m,n)+}$, $V^{(n,m)+}$. This does not bring anything new since it can be verified that

$$V^{(m,n)+} \quad = \quad V^{(n,m)+} + E_m - E_n \, ,$$

so that with the exchange of their origins, the eigenspectra are the same, which is confirmed by examining the symmetry relative to the inversion of the relations (31) and (32).

3) It can also be verified that for the two special cases $r = m$, $r = n$,

$$\phi_m^{(m,n)+} \quad = \quad \phi_n^{(m,n)+} = 0 \, ,$$

which means that two eigenvalues $E_m^{(m,n)}$, $E_n^{(m,n)}$ do not exist in the eigenspectrum.

4) After rearrangement, one can write

$$E_r^{(m,n)} \quad = \quad -\big(E_r - 2\big(E_m - E_n\big)\big) \tag{35}$$

to discern three different parts of the eigenspectrum.

(a) If $E_r > 2\big(E_m - E_n\big)$ (assuming that E_r increase with r), one has a spectrum corresponding to bound states and isotropic with the original one defined by $\{E_r\}$.

(b) The two deleted states corresponding to $E_m^{(m,n)}$, $E_n^{(m,n)}$.

(c) When $E_r < 2\big(E_m - E_n\big)$ one has positive eigenvalue states for which the normalisation of the solutions can be achieved.

5.2. Example I

For illustration of the theory from the practical point of view, we discuss the case of harmonic oscillator which had served as theoretical model in a wide range of topics ranging from atomic and statistical to particle physics (see for instance Ref. [20]).

Let $u = \frac{1}{2}x$, so that X_n can be identified as the Hermite polynomials $H_n(x)$.

1) Except for the special case $n = 1$, the second-generation potentials $V^{(n)+}$, although mathematically tractable, do not provide normaliseable solutions, except for the exact "doublet" mentioned earlier. However, it always tends to the initial harmonic oscillator potential at infinity.

2) Using the tabulated functions (see for instance Ref. [19], p. 411), we give first some analytic expressions of the Wronskian $W(m, n)$ which are always

different from zero everywhere

$$\begin{aligned} W(2,1) &= x^2+1, \\ W(3,2) &= x^4+3, \\ W(4,3) &= x^6-3x^4+9(x^2+1), \\ W(5,6) &= x^8-8x^6+30x^4+45. \end{aligned}$$

They give rise to the following exactly solvable potentials of the third-generation (some of them may have multiple-well structure)

$$\begin{aligned} V^{(2,1)+} &= \frac{1}{4}x^2-\frac{1}{2}+4\frac{x^2-1}{(x^2+1)^2}, \\ V^{(3,2)+} &= \frac{1}{4}x^2+\frac{1}{2}-8\frac{x^4-9}{(x^4+3)^2}, \\ V^{(4,3)+} &= \frac{1}{4}x^2+\frac{3}{2}+\frac{F_{10}}{W(4,3)^2}, \\ V^{(5,4)+} &= \frac{1}{4}x^2+\frac{5}{2}+\frac{F_{14}}{W(5,4)^2}. \end{aligned}$$

F_{10} and F_{14} are polynomials of order 10 and 14, respectively, and will not be reproduced here. These expressions also tend to the harmonic oscillator potential at infinity. Note that since the eigenvalue $E_n = n$, the number of eigenstates with positive eigenvalue must not exceed n in the present construction ($m > n$).

5.3. *Example II*

We construct the third-generation potential $V^{(m,n)+}$ from the initial potential defined in (9). Obviously, $V^{(m,n)+}$ also depends on the parameters a and b, which are arbitrary but must be subjected to the condition (12) (i.e., $b > a$).

Using the result (29), we consider the Wronskian $W(2,1)$ for the simplest case ($m = 2$, $n = 1$) where the mixing functions X_2, X_1 are already given by (14) and (16).

For the needs of the demonstration, it will be sufficient to consider only a special case in which

$$a = \frac{1}{4}b, \qquad b > 2\,.$$

The reason for this choice will be clarified below.

Therefore, the analytic expressions of the Wronskian W can be written as

$$W(2,1) \;=\; -\frac{1}{2}\,\mathrm{sh}\,x\left\{F_3(b)-\frac{B_2}{5b-2}\left[\big(3b+2\big)\,\mathrm{ch}\,x+\big(\frac{b}{2}-1\big)\,\mathrm{ch}^2\,x\right]\right\},$$

where $F_3(b)$ is a function of third order in terms of the parameter b. The requiring $F_3(b) = 0$, after simplifications, leads to the following third-order algebraic equation

$$b^3 - 9b^2 + 4.6b - 1.8 \quad = \quad 0.$$

The reason for the above special choice is that this equation always has a real and unique solution

$$b = 12.55 \qquad \text{and} \quad \text{therefore} \qquad a = 3.13.$$

Further, we write the analytic expression of the Wronskian W as

$$W(2,1) \quad = \quad -\frac{1}{2} \operatorname{sh} x \, G_2(\operatorname{ch} x),$$

where

$$G_2(\operatorname{ch} x) \quad = \quad \operatorname{ch} x \big(0.331 + 0.044 \operatorname{ch} x\big) > 0.$$

Returning now to the results (31) and (32), which display the complete analytic expression of the eigenfunctions, one can observe that the second and third terms always involve the differentiation $\mathrm{d}X_i/\mathrm{d}x$ $(i = m, n)$, so that the factor $\operatorname{sh} x$ can be removed making the result analytic everywhere.

Finally, using the result (33), the third-generation potential $V^{(2,1)+}$, after simplifications, can be written as

$$V^{(2,1)+} \quad = \quad V^- + \alpha \frac{f_6(\operatorname{ch} x)}{\operatorname{sh}^2 x \, G_2^2(\operatorname{ch} x)} + \beta \frac{X_2(\operatorname{ch} x)}{G_2(\operatorname{ch} x)} \,,$$

where V^- is given in (9), α, β are mere numerical factors, $f_6(\operatorname{ch} x)$ is a polynomial of order 6 which will not be displayed here.

The interesting observation is that as $x \to \pm\infty$, $V^{(2,1)+} \to V^- + const$ as expected from the theory.

5.3.1. **Discussion**

Since we must have $m < a$, $(a = 3.13)$, there are in principle four states $|r\rangle$, $r = 0, 1, 2, 3$, but as the two states $r = 2$, $r = 1$ should not be taken into account as explained above, we are left with two states $|0\rangle$, $|3\rangle$ which correspond to two eigenvalues -8.25 and $+1.26$.

6. The quasi-exact solvability

When our choice of the bases does not comply with the rule above (i.e., when $m \neq n + 1$), the potential $V^{(m,n)-}$ may involve singularities so that the preceding treatment become inadequate.

It will be interesting to test the theorem on quasi-exact solvability which has already been proved for the second-generation potentials of the type $V^{(n)+}$.

In the present case, the superpotential $v^{(m,n)}$ will be defined as

$$v^{(m,n)} = v^{(n)} - t\frac{Y_m^{(n)\prime}}{Y_m^{(n)}} .$$

$v^{(n)}$, $Y_m^{(n)}$ are defined above (Eqs. (3) and (6)), t is an arbitrary parameter.

Following exactly the same line of reasoning used for the second-generation potentials (see [I]), let

$$X^{(m,n)} = \frac{F}{Y_m^{(n)S}} .$$

For the moment, F is an unspecified function, S is an arbitrary parameter. The problem is now reduced to the evaluation of the following quantity

$$\left[X^{(m,n)\prime\prime} - 2\left(v^{(n)} - t\frac{Y_m^{(n)\prime}}{Y_m^{(n)}}\right)X^{(m,n)\prime}\right]X^{(m,n)^{-1}} = A + B + C ,$$

where

$$A = \frac{1}{{Y_m^{(n)}}^{S-1}}\left[F'' - 2v^{(n)}F'\right],$$

$$B = S\frac{F}{Y_m^{(n)S}}\left[Y_m^{(n)\prime\prime} - 2v^{(n)}Y_m^{(n)\prime}\right],$$

$$C = \frac{Y_m^{(n)\prime}}{Y_m^{(n)}}\left[2\big(t-S\big)F' + \big[S\big(S+1\big) - 2tS\big]\frac{Y_m^{(n)\prime}}{Y_m^{(n)}}F\right].$$

Taking $C = 0$, one must have

$$F = Y_m^{(n)}, \qquad S = \begin{cases} S_+ = 2t \\ S_- = 1 \end{cases}$$

and therefore

$$A + B = \frac{1}{{Y_m^{(n)}}^{S-1}}\big(1 - S_\pm\big)\big(2v^{(n)\prime} + E_m - E_n\big) ,$$

so that the potential $V^{(m,n)-}$ now depends on $S_\pm$, i.e.,

$$_\pm V^{(m,n)-} = {v^{(m,n)}}^2 - v^{(m,n)\prime} + \big(1 - S_\pm\big)\big(2v^{(n)\prime} + E_m - E_n\big).$$

More explicitly and after simplifications,

$$
\begin{aligned}
{}_+V^{(m,n)-} &= {v^{(n)}}^2 + \big(1-2t\big){v^{(n)}}' + t\big(t-1\big)\left[\frac{{Y_m^{(n)}}'}{Y_m^{(n)}}\right]^2 + t\big(E_m - E_n\big), \\
{}_-V^{(m,n)-} &= {v^{(n)}}^2 + \big(2t-1\big){v^{(n)}}' + t\big(t-1\big)\left[\frac{{Y_m^{(n)}}'}{Y_m^{(n)}}\right]^2 + t\big(E_m - E_n\big),
\end{aligned}
$$

and the corresponding exact eigenfunctions are

$$
\begin{aligned}
{}_+\phi &= W^{1-t}(m,n) X_n^{2t-1} \exp\left(- \int u \mathrm{d}x \right), \\
{}_-\phi &= W^{t}(m,n) X_n^{1-2t} \exp\left(- \int u \mathrm{d}x \right).
\end{aligned}
$$

The first case requires that

$$
\frac{1}{2} < t < 1\,,
$$

and the second one

$$
0 < t < \frac{1}{2}.
$$

Note also that ${}_+V ={}_- V$ when $t = 1/2$. In other words, if the theorem on quasi-exact solvability remains operational in the transition from the second to the third generation, one can note the removal of the existence of the exact "doublet" observed for the second generation and appearance of a couple of potentials ${}_+V$, ${}_-V$.

This is the reason why, for the third generation, the theorem must be reformulated as follows.

Theorem: For the third generation, regardless of the choice of the two bases and under certain conditions which however mutually exclude themselves, it is always possible to construct two exact solutions corresponding to a pair of singular potentials.

7. *Conclusion*

Although singular potentials a priori do not seem adaptable to current research in physics since, except for special cases, they usually lead to non-hermitian problems, the present approach may offer a somewhat different point of view and new perspectives.

It has been shown that concerning the second-generation singular potential $V^{(n)}$, an appropriate choice of the base $|n\rangle$ may in some cases leads to exactly solvable problems. When this is not the case, introduction of the parameter t can, in certain conditions, lead to quasi-exactly solvable problems with the presence of an exact "doublet".

For the third-generation potentials, which involve two bases, an appropriate choice of these bases, following certain specific rules, may lead to potentials of a new type which are free of singularities and exactly solvable.

On the other hand, it can be seen that for the transition from the second to the third generation potentials, extension of the theorem on quasi-exact solvability remains valid, however, with different consequences.

Appendix

Differentiating (17) and making use of the mixing function $X^{(m,n)}$, one obtains the following form

$$\phi_{1,r}^{(m,n)''} - V^{(m,n)-}\phi_{1,r}^{(m,n)} = \frac{X_r^{(m,n)''} - 2v^{(m,n)}X_r^{(m,n)'}}{X_r^{(m,n)}}\phi_{1,r}^{(m,n)}, \qquad (36)$$

which shows that solvability means that $X_r^{(m,n)}$ must be solution of the equation

$$X_r^{(m,n)''} - 2v^{(m,n)}X_r^{(m,n)'} + E_r^{(m,n)}X_r^{(m,n)} = 0. \qquad (37)$$

Substituting (22) into (37) and after simplifications

$$X_r^{(m,n)''} - 2v^{(m,n)}X_r^{(m,n)'} = \frac{1}{Y_r^{(n)}}\big(Y_r^{(n)''} - 2v^{(n)}Y_r^{(n)'}\big) - \frac{Y_r^{(n)}}{Y_m^{(n)2}}\big(Y_m^{(n)''} - 2v^{(n)}Y_m^{(n)'}\big).$$

On the other hand, it has been shown in previous work that the quantities $Y_r^{(n)}$, $Y_m^{(n)}$ must satisfy [16]

$$Y_r^{(n)''} - 2v^{(n)}Y_r^{(n)'} = \big(2v^{(n)'} - E_r^{(n)}\big)Y_r^{(n)}, \qquad E_r^{(n)} = E_r - E_n,$$
$$Y_m^{(n)''} - 2v^{(n)}Y_m^{(n)'} = \big(2v^{(n)'} - E_m^{(n)}\big)Y_m^{(n)}, \qquad E_m^{(n)} = E_m - E_n.$$

Therefore,

$$X_r^{(m,n)''} - 2v^{(m,n)}X_r^{(m,n)'} = \big(E_r - E_m\big)X_r^{(m,n)} = E_r^{(m,n)}X_r^{(m,n)}, \qquad (38)$$

and the quantity $E_r^{(m,n)}$ thus constitutes the eigenspectrum of the Schrödinger equation.

Counter proof: From (22) and (18), the set of eigenfunctions can be written as (for a state $|r\rangle$)

$$\phi_{1,r}^{(m,n)-} = \frac{Y_r^{(n)}}{Y_m^{(n)}} \exp\left(- \int \left(v^{(n)} - \frac{{Y_m^{(n)}}'}{Y_m^{(n)}}\right) \mathrm{d}x \right),$$

which, from (6), is

$$\phi_{1,r}^{(m,n)-} = \left(X_r' - \frac{X_n'}{X_n} X_r\right) \exp\left(- \int u \mathrm{d}x \right),$$

which means that $\phi_{1,r}^{(m,n)-}$ is similar to $\phi_{1,r}^{(m,n)+}$.

The analytical expression for $V^{(m,n)-}$ is

$$V^{(m,n)-} = u^2 + u' + 2\frac{X_n'}{X_n}\left[\frac{X_n'}{X_n} - 2u\right] - \left(E_m - 2E_n\right),$$

that is, up to a constant, the two potentials $V^{(m,n)-}$ and $V^{(n)+}$ are similar which confirms the above statement.

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SINGULARNI I EGZAKTNO RJEŠIVI POTENCIJALI

Potencijali sa singularnostima koji se razmatraju u ovom radu slijede iz postavki za generacije superpotencijala višeg reda. Raspravlja se druga generacija s jednom osnovicom i treća generacija s dvije osnovice. Ovisno o odabiru osnovica, nalazi se niz rezultata koji mogu biti zanimljivi sa stanovišta teorije, kao djelomično kršenje simetrije ili postavka nove vrste egzaktno rješivih potencijala koji pokazuju postojanje svojstvenih stanja s pozitivnim svojstvenim vrijednostima. Raspravlja se također proširenje teorema o kvazi-egzaktnoj rješivosti za potencijale treće generacije.