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LETTER TO THE EDITOR

TIME OPERATOR FOR A QUANTUM SINGULAR OSCILLATOR

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The problem of existence of a self-adjoint time operator conjugate to a Hamiltonian with SU(1,1) dynamical symmetry is investigated. In the space spanned by the eigenstates of the generator K_3 of the SU(1,1) group, the time operator for the quantum singular harmonic potential of the form $\omega^2x^2 + g/x^2$ is constructed explicitly, and shown that it is related to the time-of-arrival operator of Aharonov and Bohm. Our construction is fully algebraic, involving only the generators of the SU(1,1) group.

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The unequal role played by time as an observable in classical and quantum mechanics has been the source of controversy since the early days of quantum mechanics. The problem arises because we expect observables to be represented in quantum mechanics by self-adjoint operators. However, a well-known argument due to Pauli [1] stated that a self-adjoint time operator T conjugate to a self-adjoint Hamiltonian H could not be constructed if the spectrum of H is bounded from below.

Since then, the search for various time operators and the analysis of their self-adjointness and associated time-energy uncertainty relations have been the subject of a number of papers [2]. The general concensus is that no such operator exists. Recently, the validity of Pauli's objections has been critically evaluated [3], with the conclusion that there is no a priori reason to exclude the existence of self-adjoint time operators canonically conjugate to a semibounded Hamiltonian. For this and other similar reasons, it seems reasonable to investigate explicit constructions of time operators for various quantum mechanical systems.

In this letter we pose a general problem of finding an operator conjugate to a Hamiltonian with SU(1,1) dynamical symmetry. We assume that the Hamiltonian is linear in the generators K_1, K_2, K_3 of the su(1,1) algebra

$$\tilde{H} = \Omega_3 K_3 + \Omega_2 K_2 + \Omega_1 K_1, \tag{1}$$

where the (2+1)- dimensional constant vector $\mathbf{\Omega} \equiv (\Omega_1, \Omega_2, \Omega_3)$ has the norm $\Omega^2 = \Omega_3^2 - \Omega_2^2 - \Omega_1^2$. The group generators K_3 and $K_\pm = K_1 \pm \mathrm{i} K_2$ satisfy the commutation relations of the su(1,1) algebra

$$[K_3, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_3. \tag{2}$$

Our objective here is to construct an operator $\tilde{T}$ in terms of the generators $K_3, K_\pm$ that is conjugate to the Hamiltonian $\tilde{H}$ and satisfies $[\tilde{H}, \tilde{T}] = \mathrm{i}$.

In the following, we use the standard complete orthonormal basis states $|n, k>$ that diagonalize the compact generator K_3. These states are obtained from $|0, k>$ by the n-fold application of K_+

$$|n, k> = \sqrt{\frac{\Gamma(2k)}{\Gamma(2k+n)n!}} (K_+)^n |0, k>,$$
$$K_- |0, k> = 0, \qquad K_3 |n, k> = (n+k)|n, k>, \; n = 0, 1, 2, ... \tag{3}$$

The Bargman index k is related to the eigenvalue $k(k-1)$ of the quadratic Casimir operator $\hat{C} = K_3^2 - K_1^2 - K_2^2$.

We also need the Barut-Girardello coherent states [4], which are the eigenstates of K_-

$$K_- |z, k> = z|z, k>, \qquad |z, k> = \sum_{n=0}^{\infty} z^n \sqrt{\frac{\Gamma(2k)}{\Gamma(2k+n)n!}} |n, k>, \tag{4}$$

where z is an arbitrary complex number. These coherent states can also be written as an exponential operator acting on the vacuum state of K_-

$$|z, k> = e^{zK_+(K_3+k)^{-1}} |0, k> . \tag{5}$$

In deriving this expression, we have used the operator identity

$$[K_+(K_3+k)^{-1}]^n = K_+^n \frac{\Gamma(K_3+k)}{\Gamma(K_3+k+n)}. \tag{6}$$

Note also that the operator $K_+(K_3+k)^{-1}$ is canonical to K_-,

$$[K_-, K_+(K_3+k)^{-1}] = 1. \tag{7}$$

The eigenvalue problem [5] for our model Hamiltonian $\tilde{H}$,

$$\tilde{H}|\Psi(\lambda)> = \lambda|\Psi(\lambda)>, \qquad |\Psi(\lambda)> = \sum_{n=0}^{\infty} C_n(\lambda)|n, k>, \tag{8}$$

depends on the choice of the vector $\mathbf{\Omega}$.

We consider two cases [6]:

a) $\Omega^2 > 0,\ \Omega_3 > 0$, when $\tilde{H}$ can be transformed by means of the unitary operator to a standard form $H = U^\dagger \tilde{H} U = \Omega K_3$. The energy spectrum is discrete and bounded from below;

b) $\Omega^2 = 0,\ \Omega_3 > 0$ when $\tilde{H}$ can be transformed to $H = U^\dagger \tilde{H} U = \Omega_3(K_3 - K_1)$. In this case, the energy spectrum is continuous and bounded from below.

Let us first consider the time-operator problem for a particle moving in a repulsive singular potential of the Calogero-Moser type [7]. The motion is described by the Hamiltonian

$$H = \frac{1}{2}(p^2 + \frac{g}{x^2}),\ g > 0. \tag{9}$$

This Hamiltonian is interesting for several reasons:

i) It is scale invariant and has the full conformal group as a dynamical symmetry group [8] with the generators H, $D = -(xp + px)/4$, and $K = \frac{1}{2}x^2$, which obey the algebra

$$[H, D] = \mathrm{i}H,\ [K, D] = -\mathrm{i}K,\ [H, K] = 2\mathrm{i}D \tag{10}$$

with a constant Casimir operator $\hat{C} = \frac{1}{2}(HK + KH) - D^2 = g/4 - 3/16$.

ii)The spectrum of H is positive, continuous, and bounded from below, with a non-normalizable ground state [8].

iii) It can be easily extended to the well-known one-dimensional N-body problem of Calogero-Moser [7].

iv) Recently, it has been observed that the dynamics of particles near the horizon of a black hole is also associated with this Hamiltonian [9].

If we now identify

$$\begin{aligned} K_1 &\equiv S = \frac{1}{2}(\omega K - \frac{1}{\omega}H), \\ K_2 &= D, \\ K_3 &\equiv R = \frac{1}{2}(\omega K + \frac{1}{\omega}H), \end{aligned} \tag{11}$$

it can be seen that the conformal algebra (10) is isomorphic to the algebra of $SU(1,1) \sim O(2,1) \sim SL(2,\mathcal{R})$ with the Bargman index $k = \frac{1}{2}(1 + \sqrt{g + \frac{1}{4}})$. We note that $H = \omega(K_3 - K_1)$ and $\omega K = K_3 + K_1$ are related to K_- as follows

$$e^{-\omega K} H e^{\omega K} = -2\omega K_- \,. \tag{12}$$

The energy eigenstates of $H|E> = E|E>$ are thus seen to be proportional to the Barut-Girardello coherent states [4,10] with $z = -E/2\omega$

$$|E> = e^{\omega K}| - \frac{E}{2\omega}, k> \,. \tag{13}$$

Since $\lim_{E\to 0} < x|E > = < x|e^{\omega K}|0, k > \propto \omega^k x^{2k-1/2}$, the eigenstate $< x|E >$ in the limit $E \to 0$ is not normalizable. The dificulty arises from the oscillating behaviour of $< x|E >$ at large distances [8].

Combining the relations (7) and (12), we find that the operator

$$T(\omega) = -\frac{\mathrm{i}}{2\omega} e^{\omega K} K_+ (K_3 + k)^{-1} e^{-\omega K}, \quad T^\dagger(\omega) \neq T(\omega) \tag{14}$$

has the property $[H, T(\omega)] = \mathrm{i}$ and can be interpreted as a possible time operator conjugate to H. Since ω is a free parameter, $T(\omega)$ generates for $\omega \neq 0$ an uncountable number of different time operators canonically conjugate to H. It is important to point out that the solution of $[H, T] = \mathrm{i}$ is not unique. Any $\bar{T} = T + \phi(H)$, with arbitrary ϕ, satisfies the same canonical commutation relation.

In the limit $\omega \to 0$, we find that

$$T(\omega) \to \frac{\mathrm{i}}{2\omega} + D\frac{1}{H} + \mathrm{i}\frac{k}{H} + \mathcal{O}(\omega) \tag{15}$$

becomes singular. To avoid this singularity, we use the concept of a minimal Hermitian solution and choose

$$T = lim_{\omega\to 0} \frac{1}{2}[T(\omega) + T^\dagger(\omega)] = \frac{1}{2}(\frac{1}{H}D + D\frac{1}{H}) \tag{16}$$

as a possible time operator conjugate to H.

We argue that $[H, D] = \mathrm{i}H$ is the key relation [11,12] for defining the time operator. In fact, by requiring that $HT + TH = 2D$, we can immediately deduce that the commutators of H and T must have the form

$$[H, T] = \mathrm{i}. \tag{17}$$

In the limit $g \to 0$, we have $H \to H_0$ and $T \to T_0$, so that

$$[H_0, T_0] = \mathrm{i}, \tag{18}$$

where $H_0 = p^2/2$ and

$$T_0 = \frac{1}{2}\left(\frac{1}{H_0}D + D\frac{1}{H_0}\right) = -\frac{1}{2}\left(x\frac{1}{p} + \frac{1}{p}x\right) \tag{19}$$

is the time-of-arrival operator of Aharonov and Bohm [13]. The operators T and T_0 can also be related to each other by means of a unitary operator [14] that transforms $H \to H_0$, so that we obtain

$$H = UH_0U^\dagger, \qquad T = UT_0U^\dagger, \qquad U = e^{-i\pi K_3} e^{\mathrm{i}\pi K_3^0}, \tag{20}$$

where $K_3^0 = K_3(g = 0)$.

Finally, we consider the quantum singular harmonic oscillator of the Calogero-Sutherland type [15], which is proportional to the K_3 generator of the $SU(1,1)$ group

$$H_{CS} = 2\omega K_3. \tag{21}$$

To construct the time operator for H_{CS}, we first observe the relationship [16] between H_{CS} and the Hamiltonian for the ordinary harmonic oscillator, $H_h = H_0 + \omega^2 K = H_{CS}(g=0)$

$$H_{CS} = U_1 H_h U_1^{-1}, \qquad U_1 = e^{-K_-}\ e^{K_-^0}, \tag{22}$$

where $K_-^0 = K_-(g=0)$. The time operator for H_h was constructed and discussed earlier in Refs. [12,17]. Its construction is simple if we observe that the Casimir operator with $k = 3/4$ can be used to express the operator K in the form

$$K = T_0 H_0 T_0 + \frac{1}{16H_0} = QH_0Q - \frac{\mathrm{i}}{2}Q, \qquad Q = -T_0 + \frac{\mathrm{i}}{4\mathrm{H}_0}. \tag{23}$$

Then the Hermitian operator

$$T_h = \frac{1}{2}(T_h(Q) + T_h^\dagger(Q)) \tag{24}$$

satisfies $[H_h, T_h] = \mathrm{i}$, where

$$T_h(Q) = \frac{1}{\omega} arctg(\omega Q). \tag{25}$$

It is now easy to see that the time operator for the Hamiltonian H_{CS} is

$$T_{CS} = U_1 T_h U_1^{-1}. \tag{26}$$

In conclusion, we have presented an algebraic method of constructing Hermitian operators conjugate to a Hamiltonian with $SU(1,1)$ dynamical symmetry. The time operator for the quantum singular harmonic potential is constructed explicitly and shown that it is related to the time-of-arrival operator, T_0 of Aharonov and Bohm. The question whether time operators thus constructed are self-adjoint operators in Hilbert space requires a careful examination of their spectra and eigenfunctions. The eigenvalue problem of the operator T_0 can be solved in the momentum space [2,18]. It is not self-adjoint and its eigenfunctions are not orthogonal. The same conclusion can be reached for the time operator T owing to the relation (20). For T_h and T_{CS}, this problem is still open [19,20].

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VREMENSKI OPERATOR KVANTNOG SINGULARNOG OSCILATORA

Istražujemo problem postojanja samo-adjungiranog vremenskog operatora koji je konjugiran hamiltonijanu sa SU(1,1) dinamičkom simetrijom. Konstruirali smo eksplicitno vremenski operator za kvantni singularni harmonički potencijal dan sa $\omega^2 x^2 + g/x^2$ u prostoru na svojstvenim stanjima generatora K_3 SU(1,1) grupe, i pokazali da je povezan s Aharonov–Bohmovim operatorom vremena pristizanja. Naša je konstrukcija vremenskog operatora potpuno algebarska i uključuje samo generatore SU(1,1) grupe.