A NEW APPROACH TO THE PROBLEM OF TRANSPORT IN AQUATIC ECOSYSTEMS

NEDŽAD LIMIĆ

Ruđer Bošković Institute, POB 1016, 41001 Zagreb, Croatia

Received 14 October 1991

UDC 530.19

Original scientific paper

So far the conventional diffusion equation has been practically the only model for the description of transport of substances in the sea or lake. The most serious drawbacks of this model are: (a) the infinite velocity of spreading of substance from a source and b) neglect of non-zero time correlations between velocity fluctuations. In this paper, a class of transport models is derived for which velocity of spreading is finite. In addition, a complete covariance function of velocity fluctuations is incorporated. New transport models are derived starting from the mass balance equation with stochastic velocity fields. Derivations of the new models are exact (without truncations). Finally, certain simplified versions are proposed for applications.

1. Introduction

In the present article the term diffusion equation or diffusion law is reserved for the linear diffusion equation with advection and extinction terms included. In ecological modelling the diffusion equation is usually considered as the transport model which follows from the mass conservation law by applying to it Fick's relationship between the flux and concentration gradient (see for instance Patten[1] and Jorgensen[2]). This approach is basically deterministic. Here we consider, as an alternative, the diffusion equation to be the result of statistical averaging of certain transport models subject to a stochastic velocity field. In the case that the random velocity field $\mathbf{v}$ is defined by the Brownian motion such stochastic transport models are

$$\dot{\mathbf{r}}(t) = \mathbf{v}(t, \mathbf{r}(t)),$$

describing the random motion $\mathbf{r}(t)$ of a particle[3], or

$$\frac{\partial C}{\partial t} + \mathbf{v}\nabla C = q,$$

describing the stochastic mass balance equation[4].

Let us mention here that the Brownian motion is used in a wider sense than in standard textbooks. The Brownian motion is usually associated with molecular motion. The corresponding average motion can be derived exactly. It is the well known diffusion equation for the concentration field. Since the diffusion equation is used for describing the transport on a much larger scale we can state that the Brownian motion is the underlying stochastic motion of water masses, i. e. it approximates the turbulent or eddy motion. As far as the diffusion equation is the basic model for describing the transport, we can state, that the Brownian motion is the basic model for turbulent motion of water. In this sense the Brownian motion is used here.

1.1. Non-stationary problems

The conventional, non-stationary diffusion equation is still extensively used in describing the non-stationary transport of substance and species in an aquatic environment. One of reasons for this is, perhaps, a lack of any better transport law which describes an average motion of substance in a turbulent medium. We are rarely aware of erronous usage of the diffusion equation in describing the transport in the sea. This is caused mainly because of a high cost of experiments that could provide us with adequate data to test non-stationary diffusion law. On the other hand, it is well known that the conventional diffusion equation cannot describe tracer experiments with ground water. The following observation cannot yet be correctly modelled:

(a) The spreading of a tracer plume around locations of sources i.e. immediately after the release of substance, must be described by the linear law while the duffusion equation predicts the quadratic law.

(b) The distribution of exit time or the first passage time differes strongly from the corresponding distributions which could be obtained from the conventional diffusion equation.

(c) From a source substance spreads with a finite velocity through the space. This apparent fact contradicts the infinite velocity of spreading of substance that follows from the diffusion equation.

The encountered shortcomings of the conventional diffusion law in description of tracer experiments with ground water has forced researchers in the field to try other methods. Monte Carlo methods are mostly exploited among new approaches to this problem.

1.2. Stationary problem

There are many cases of transport of substance in the coastal sea that can be considered as a stationary transport problem. Conditions for stationarity must be sought in those environmental systems for which yearly averages of inputs of various substances are practically constant over dozen of years. In such cases one expects that fluctuations of concentration field, as well as of other relevant physical and chemical fields, cancel in the average over a year and the resulting yearly averaged fields are practically the same for each year in a certain sequence of many years.

Data for testing stationarity of transport can be obtained more easily. They are collected and processed by standard experimental methodologies. Such types of experiments consist of the measurement of parameters during shorter periods in various seasons, during many years, and over the same set of sampling stations. Therefore, it is possible to test the stationary diffusion equation by using adequate data set (Legović et. al.[5], Limić et. al.[6]). The reliability of estimates of various parameters is satisfactory, so that the stationary diffusion equation can be accepted as a useful tool in describing stationary transport of substance in the coastal sea.

1.3. Objectives

A general transport model avoiding drawbacks a), b) and c) of 1.1 cannot be derived. This assertion will follow from our derivation of one, simple class of transport models that match requirements of 1.2. and escape shortcomings of 1.1.

We consider here a class of transport models which have the stationary form analogous to the stationary diffusion equation, and match (a)—(c) of 1.1. for non-stationary conditions. To match (a) a non-zero correlation for various time-moments must be imposed on velocity fluctuations. Let us remind that the covariance function of velocity components of the Brownian motion is proportional to the Dirac δ-function. To match (b) and (c) velocity fluctuations must have amplitudes that are bounded by a certain value. Let us point out that, contrary to this constraint, velocity fluctuations of the Brownian motion are normally distributed. Thus, they have a positive probability to exceed any finite value.

Our methodology is developed in Section 2 and it is applied there to generalize the conventional diffusion equation. In particular, if the motion is isotropic, the resulting equation for the concentration field contains the Taylor diffusion tensor. Various models of velocity fluctuations are considered in Section 3. A new class of transport models is derived in Section 4. Here, the methodology of Section 2 is applied. Certain simplifications are considered in Section 5 and their usefulness for applications is discussed.

New class of models that is derived here is based on the wave (telegraph) equation. It seams that Goldstein[7] was the first utilizing the telegraph equation for a description of transport of substance in a turbulent medium. A generalization of thy Goldstein model to the case of more dimensions is proposed by Bourett[8]. A similar equation is derived by Koch and Brady[9] by assuming a simple condition on the permeability covariance function. Results of the present work do not contradict results of the above mentioned authors. Our results are more general. More important, the results are derived from a unique supposition on velocity fluctuations by a method which is developed here. The method is quite general so that it reproduces the conventional diffusion equation as well.

2. *A generalization of diffusion law*

One of justifications of our method is a derivation of the diffusion equation by assuming the velocity fluctuations to be the Gaussian process. Apart from deriving a generalized diffusion equation the discussion of this section is important from the methodological point of view.

The velocity field has the following representation

$$\mathbf{v}(t, \mathbf{r}) = \mathbf{w}(\mathbf{r}) + \mathbf{f}(t, \mathbf{r}), \tag{2.1}$$

where $\mathbf{w} = \mathbf{E}(\mathbf{v})$ is the mean velocity field and $\mathbf{f}$ is a fluctuation, such that div $\mathbf{w} =$ = div $\mathbf{f} = 0$. The fluctuation is assumed to have the following representation

$$\mathbf{f}(t, \mathbf{r}) = \sum g_k(t)\, \varphi_k(\mathbf{r}), \tag{2.2}$$

where φ_k are deterministic velocity fields, div $\varphi_k = 0$, and g_k are stochastic processes. Here, the velocity fields φ_k are called modes. The form (2.2) is quite general. For instance, solutions of the Navier-Stokes equation are represented in this form. Of course, $g_k(t)$ are deterministic functions in the case of construction of numerical solution of the Navier-Stokes equation.

A generalization of the diffusion law follows easily from the model (2.1)—(2.2) if we assume that the amplitudes g_k in (2.2) are Gaussian processes. Let their statistics be defined in the following way. For even n the moments have form

$$\begin{aligned}\sigma_{k_1 k_2 \dots k_n}(t_1, t_2, \dots, t_n) &= \mathbf{E}(g_{k_1}(t_1)\, g_{k_2}(t_2) \dots g_{k_n}(t_n)) = \\ &= \int \mathrm{d}\lambda\, \varrho(\lambda)\, \mu_n(\lambda, t_1, t_2, \dots, t_n)\, b_{k_1}(\lambda)\, b_{k_2}(\lambda) \dots b_{k_n}(\lambda)\end{aligned} \tag{2.3}$$

while they are vanishing for odd n. Here, for each λ, the functions μ_n are well-known statistical moments of the order n of the Gaussian process (see for instance Gikhman and Skorokhod[10]). The integration extends over a subset Λ in a multi-dimensional space of variables $\lambda_1, \lambda_2, \dots$. The functions b_k could be incorporated into μ_n. The reason that they are separated from μ_n is as follows. The functions b_k, mean field $\mathbf{w}$, and velocity modes φ_k must fulfill the following assumption. The velocity fields $\mathbf{w}$ and

$$\mathbf{z}(\lambda, \mathbf{r}) = \sum b_k(\lambda)\, \varphi_k(\mathbf{r})$$

for all $\lambda \in \Lambda$, are either jointly collinear or they are constant. Apparently, this assumption is superfluous for $\mathbf{w} = 0$. Otherwise, this assumption enables us to derive the diffusion equation for $\mathbf{E}(C)$ even in the case of non-vanishing mean field $\mathbf{w}$.

Our derivation of the diffusion equation is based on the mass balance equation:

$$\frac{\partial C}{\partial t} + \mathbf{v} \nabla C = q \tag{2.4}$$

in a stochastic velocity field $\mathbf{v}$ which is defined by (2.1)—(2.3). The input distribution q is also assumed to be stochastic, but independent of $\mathbf{v}$.

In order to solve (2.4) for C and calculate the mean field $\mathbf{E}(C)$, we transform (2.4) into another stochastic differential equation. After inserting the representation (2.1) into the transport model (2.4) we obtain an equation with two terms containing the space derivatives, div $(\mathbf{w}C) = \mathbf{w}\nabla C$ and div $(\mathbf{f}C) = \mathbf{f}\nabla C$. It is convenient for our further analysis to get rid of the term $\mathbf{w}\nabla C$ by using the function $u = \exp(tD)\, C$, $D = \mathbf{w}\nabla$, instead of the function C. The corresponding equation for u has the form:

$$\dot{u} + Su = p, \tag{2.5}$$

where

$$S(t) = \exp(tD)\, \mathbf{f}\nabla \exp(-tD), \qquad p = \exp(tD)\, q. \tag{2.6}$$

A solution of the stochastic differential equation (2.5) can be obtained by iterations:

$$u(t) = \sum_{n=0} (-1)^n u_n(t), \tag{2.7}$$

$$u_n(t) = \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots \mathrm{d}t_n \int_{-\infty}^{t_n} \mathrm{d}s\, S(t_1)\, S(t_2) \ldots S(t_n)\, p(s). \tag{2.8}$$

Let us suppose that $p(s) = \delta(s)\, p_0$, where p_0 is a state vector. We have to calculate

$$\mathbf{E}(u_n(t)) = \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n\, \mathbf{E}(S(t_1)\, S(t_2) \ldots S(t_n))\, p_0. \tag{2.9}$$

The assumptions (2.2) and (2.3) imply

$$\mathbf{E}(S(t_1)\, S(t_2) \ldots S(t_n)) = \sum_{\text{all comb. of } k_i} \mathbf{E}(g_{k_1} g_{k_2} \ldots g_{k_n}) \times \tag{2.10}$$

$$\prod_{j=1}^{n} \exp(t_j D)\, \nabla\varphi_{k_j} \exp(-t_j D) = \int \mathrm{d}\lambda\, \varrho(\lambda)\, \mu_n(\lambda, t_1, t_2, \ldots, t_n) \times$$

$$\sum_{\text{all comb. of } k_i} b_{k_1} b_{k_2} \ldots b_{k_n} \prod_{j=1}^{n} \exp(t_j D)\, \nabla\varphi_{k_j} \exp(-t_j D) =$$

$$= \int \mathrm{d}\lambda\, \varrho(\lambda)\, \mu_n(\lambda, t_1, t_2, \ldots, t_n)\, (\sum b_k \nabla\varphi_k)^n.$$

In this equality the co-linearity of $\mathbf{w}$ and $\mathbf{z}(\lambda)$ is used in order to commute exp $(\pm tD)$ and $\mathbf{z}(\lambda)\nabla$. By using the structure of the statistical moments of the Gaussian process (Gikhman and Skorokhod[10]) one obtains (Limić[11]):

$$v_k(\lambda, t) = \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n\, \mu_n(\lambda, t_1, t_2, \ldots, t_n) =$$

$$= \frac{1}{2^k k!} \left(\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2\, \mu_2(\lambda, t_1, t_2)\right)^k,$$

where $n = 2k$. Evidently, the function μ_2 is the covariance function of a process g_k. Now we have

$$\mathbf{E}(u(t)) = \sum_{n=0} (-1)^n \mathbf{E}(u_n(t)) = \sum_{k=0}^{\infty} \int \mathrm{d}\lambda\, \varrho(\lambda)\, v_k(\lambda, t)\, (\mathbf{z}(\lambda)\, \nabla)^{2k} p_0. \quad (2.11)$$

Let us define the function

$$U(\lambda, t, \mathbf{r}) = \sum_{k=0} v_k(\lambda, t)\, (\mathbf{z}(\lambda, \mathbf{r})\, \nabla)^{2k} p_0. \quad (2.12)$$

Then $\mathbf{E}(u(t)) = \int \mathrm{d}\lambda\, \varrho(\lambda)\, U(t, \lambda)$. Now we need the following equality

$$\frac{\partial}{\partial t} v_k(\lambda, t) = h(\lambda, t)\, v_{k-1}(\lambda, t), \quad (2.13)$$

where

$$h(\lambda, t) = \int_0^t \mu_2(t, s)\, \mathrm{d}s. \quad (2.14)$$

By using (2.14) we can derive a generalization of the diffusion equation for $U(\lambda, t, \mathbf{r})$

$$\frac{\partial}{\partial t} U(\lambda, t, \mathbf{r}) - h(\lambda, t)\, A(\lambda)\, U(\lambda, t, \mathbf{r}) = \mathbf{E}(p(t)) \quad (2.15)$$

where

$$A(\lambda) = (\mathbf{z}(\lambda, \mathbf{r})\, \nabla)^2 = \sum_{i,j=1}^{2} \partial_i z_i(\lambda, \mathbf{r})\, z_j(\lambda, \mathbf{r})\, \partial_j \quad (2.16)$$

is a differential operator of the second order. Let us turn back to $C(t, \mathbf{r}) = \exp(-Dt)\, u(t, \mathbf{r})$. By inserting C into (2.11)—(2.16) we obtain

$$\mathbf{E}(C(t, \mathbf{r})) = \int \mathrm{d}\lambda\, \varrho(\lambda)\, C(\lambda, t, \mathbf{r}), \quad (2.17)$$

$$\frac{\partial}{\partial t} C(\lambda, t) + \mathbf{w}\nabla\, C(\lambda, t) - h(\lambda, t)\, A(\lambda)\, C(\lambda, t) = \mathbf{E}(q(t)). \quad (2.18)$$

In this way we have derived a generalized diffusion equation. The mean concentration field $\mathbf{E}(C(t, \mathbf{r}))$ is an integral of components $C(\lambda, t, \mathbf{r})$, $\lambda \in \Lambda$, where each component $C(\lambda, t)$ satisfies a diffusion-like equation with time-variable diffusion tensor

$$a_{ij}(\lambda, t, \mathbf{r}) = h(\lambda, t)\, z_i(\lambda, \mathbf{r})\, z_j(\lambda, \mathbf{r}), \quad (2.19)$$

where z_i are the components of velocity field $\mathbf{z}$. This tensor can be straightforwardly related to the Taylor diffusion tensor for isotropic diffusion (Taylor[12], Monin and Yaglom[13], Batchelor[14]).

Properties of the obtained model can be easily examined in the one-dimensional case. For the sake of simplicity we assume that $\varrho(\lambda) = \delta(\lambda - \lambda_0)$ for some $\lambda_0 \in \Lambda$, in order to get rid of the integration over Λ. In this case the velocities $\mathbf{w}$ and $\mathbf{z}$ must be constant. We assume the Gaussian processes g_k to be stationary so that their covariance function has the form $\mu_2(t, s) = R(t - s)$. Let us define the function:

$$e(t) = 2z^2 \int_0^t h(t_1)\, dt_1 = 2z^2 \int_0^s dt_1 \int_0^{t_1} dt_2 R(t_1 - t_2).$$

Then we have

$$\mathbf{E}(C(t, \mathbf{x})) = \int ds \int dy\, Y(t - s, \mathbf{x} - \mathbf{y})\, \mathbf{E}(q(s, \mathbf{y})) \tag{2.20}$$

where

$$Y(t, \mathbf{x}) = \frac{1}{\sqrt{2\pi e(t)}} \exp\left\{-\frac{(\mathbf{x} - \mathbf{w}t)^2}{2e(t)}\right\}. \tag{2.21}$$

The conventional diffusion law follows by choosing $z^2 R(t - s) = a\delta(t - s)$, where a is the conventional diffusion constant.

Three particular cases of the covariance function R are interesting to discuss:

(i) $\int_0^t R(t_1)\, dt_1 \sim t^{-\varrho}$, $\varrho > 1$, as t increases;

(ii) $z^2 \int_0^\infty R(t) = a > 0$;

(iii) $\int_0^t R(t_1)\, dt_1$ diverges as t increases.

In all three cases $e(t)$ behaves as t^2 for small t. As time tends to infinity, the function $e(t)$ tends to a positive number, behaves as at, and increases faster than t, for the respective three cases (i)—(iii). The case (ii) corresponds to the conventional diffusion law. The number a can be interpreted as an asymptotic diffusion constant and the solution (2.20) exhibits the linear law for small times and the standard quadratic law for the larger times. This property stresses the non-triviality of the considered model (2.20) in the class of exactly derivable models for the transport of substance in a stationary velocity field which is defined by (2.1)—(2.3). Thus, (2.20) is an extension of the conventional diffusion law for which the covariance function of velocity fluctuations is a smooth function.

An illustration of properties (i)—(iii) is given in Fig. 1. Let $R(t) = \int \exp(i\lambda t)\, S(\lambda)\, d\lambda$ be the covariance function and consider the following cases

$$\begin{aligned}
&\text{(a)}\ S(\lambda) = \frac{3}{4}\lambda^2 \Theta(1 - \lambda^2), && e_1(t) = (3/2)(1 - t^{-1}\sin t),\\
&\text{(b)}\ S(\lambda) = 2^{-1}\Theta(1 - \lambda^2), && e_2(t) = t\int_0^{t/2} s^{-2}(\sin s)^2\, ds,\\
&\text{(c)}\ S(\lambda) = \delta(\lambda) && e_3(t) = 2^{-1}t^2.
\end{aligned} \tag{2.22}$$

In all three cases we have $e(t) \approx t^2/2$ for small t. The curves $Y(t, \mathbf{x}) = \text{const}$ are drawn in the left part of Fig. 1., while the graphs of $Y(t, 1)$ are given in the right part of this figure.

Let us stress that the smoothness of the covariance function of velocity fluctuations has removed the drawback (a) stated in Introduction but not the drawbacks (b) and (c).

If the velocity fluctuations are assumed to be Gaussian processes the Taylor diffusion tensor[1 2)] is naturally associated to stochastic mass balance equation (2.4). Actually, we obtained a general diffusion law which is valid for non-isotropic velocity fluctuations as well. In the case of isotropic fluctuations the Taylor diffusion tensor can be directly related to fluctuations in the following way. Let $\underline{\mathfrak{r}}(\tau, \mathbf{r}) = \int_0^{\tau} \mathbf{f}(s, \mathbf{r})\, \mathrm{d}s$ be the relative displacement of a particle which was at $\mathbf{r}$ in the moment $t = 0$. Apparently, $\mathbf{E}(\underline{\mathfrak{r}}) = 0$. The Taylor diffusion tensor for the considered case is defined by

$$\mathbf{E}(\underline{\mathfrak{r}}_i(\tau, \mathbf{r})\, \underline{\mathfrak{r}}_j(\tau, \mathbf{r})) = \int_0^{\tau}\int_0^{\tau} \mathbf{E}(f(s_1, \mathbf{r}) f(s_2, \mathbf{r}))\, \mathrm{d}s_1\, \mathrm{d}s_2 =$$

$$= \int_0^{\tau} \mathrm{d}s_1 \int_0^{\tau} \mathrm{d}s_2 \int_{\Lambda} \mathrm{d}\lambda\, \varrho(\lambda)\, \mu_2(\lambda, s_1 - s_2)\, z_i(\lambda, \mathbf{r})\, z_j(\lambda, \mathbf{r}) =$$

$$= \int \mathrm{d}\lambda\, \varrho(\lambda)\, e(\lambda, \tau)\, z_i(\lambda, \mathbf{r})\, z_j(\lambda, \mathbf{r}).$$

3. *Random velocity fields with bounded variations*

Again we represent the velocity field in the form (2.1), (2.2). However, the statistical moments of fluctuations cannot be defined by the Gaussian process since the amplitudes of fluctuations in such case can be arbitrarily large.

A general supposition on the stochastic amplitudes in (2.2) cannot help us to derive a simple transport model (partial differential equations). Therefore, it is fundamental in our derivation to define a sub-class of processes g_k and velocity modes φ_k enabling us to reach our goal. This class is described as follows.

The function

$$\chi(\gamma_1, \gamma_2, \ldots, \gamma_n, t_1, t_2, \ldots, t_n) = \int_{\Lambda} \mathrm{d}\lambda \varrho(\lambda) \cos\Big(\sum_i b_i(\lambda, t_i)\, \gamma_i\Big), \qquad (3.1)$$

where ϱ is a probability density and b_i are real-valued functions, is the characteristic function of n stochastic processes. Let us denote them jointly by $G(t) = (g_1(t), g_2(t), \ldots, g_n(t))$. As before, variable λ formally stands for multi-dimensional variable space $\lambda_1, \lambda_2, \ldots$, and Λ is its subset. Statistical moments of the odd orders vanish, while the even ones have the form

$$\sigma_{i_1 i_2 \cdots i_n}(t_1, t_2, \ldots, t_n) = \int \mathrm{d}\lambda \varrho(\lambda)\, b_{i_1}(\lambda, t_1)\, b_{i_2}(\lambda, t_2) \ldots b_{i_n}(\lambda, t_n). \qquad (3.2)$$

The components of the random process $G(t)$ are denoted by $g_k(t)$. Apparently, for a fixed t, each g_k is a random variable. In the case of continuous functions b_l, the probability $P(x_1, x_2)$ that $g_k(t)$ is in the interval (x_1, x_2) is equal

$$P(x_1, x_2) = \int\limits_{\mathbb{B}(x_1, x_2)} d\lambda \varrho(\lambda),$$

where $\mathbb{B}(x_1, x_2)$ is the set of $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $b_k(\lambda, t)$ has a value in the interval (x_1, x_2).

The same relationship as in the previous Section is valid among the functions b_k, the mean current field $\mathbf{w}$ and the teh modes φ_k, i. e. for each λ the velocity field

$$\mathbf{z}(\lambda, t, \mathbf{r}) = \sum b_k(\lambda, t) \varphi_k(\mathbf{r}), \tag{3.3}$$

and $\mathbf{w}$ are either constant or colinear. The abstract definition of velocity fluctuations (3.1)—(3.3) may or may not look realistic. Therefore, we wish to discuss features of this velocity fluctuations and give some examples.

First we consider the covariance matrix of fluctuations

$$\mathfrak{C}_{ij}(t_1, t_2, \mathbf{r}_1, \mathbf{r}_2) = \int d\lambda \varrho(\lambda) z_i(\lambda, t_1, \mathbf{r}) z_j(\lambda, t_2, \mathbf{r}_2). \tag{3.4}$$

In the case of a spatially variable mean field $\mathbf{w}$, the matrix $\mathfrak{C}$ is singular, because of the commutativity of $\mathbf{w}$ and $\mathbf{z}(\lambda)$. However, if either $\mathbf{w} = 0$ or $\mathbf{w}$ and $\mathbf{z}$ are spatially constant fields, this matrix does not need to be singular, i. e. it can fulfill the following inequality:

$$\sum \bar{\xi}_i \mathfrak{C}(t_i, t_j, \mathbf{r}_i, \mathbf{r}_j) \xi_j = \int d\lambda \varrho(\lambda) \left| \sum \xi_i z_i(\lambda, t_i, \mathbf{r}_i) \right|^2 \geq \varkappa \sum |\xi_i|^2$$

for any two complex numbers ξ_i and some positive $\varkappa$.

Example 3.1. Let us consider the two-dimensional space $\mathbb{R}^2$, with the following velocity fields $z(\lambda)$:

$$\mathbf{z}(\lambda, t, \mathbf{r}) = K_1 \cos(\mu\lambda t) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \text{for } \lambda \in [-\lambda_A, 0],$$

$$\mathbf{z}(\lambda, t, \mathbf{r}) = K_2 \cos(\mu\lambda t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \text{for } \lambda \in [0, \lambda_A],$$

where μ is a parameter. Let $\varrho(\lambda) = (2\lambda_A)^{-1}$ for $|\lambda| \leq \lambda_A$, and zero otherwise. Then

$$\mathfrak{C}_{ij}(t_1, t_2, \mathbf{r}_1, \mathbf{r}_2) = 2^{-1} K_1^2 \lambda_A^{-1} \delta_{i1} \delta_{j2} \int\limits_{-\lambda_A}^{0} \varrho(\lambda) \cos(\mu\lambda t_1) \cos(\mu\lambda t_2) d\lambda +$$

$$+ 2^{-1} K_2^2 \lambda_A^{-1} \delta_{i2} \delta_{j2} \int\limits_{\lambda_A}^{0} \varrho(\lambda) \cos(\mu\lambda t_1) \cos(\mu\lambda t_2) d\lambda. \tag{3.5}$$

with $\mathfrak{C}_{12} = \mathfrak{C}_{21} = 0$. For $t_{1,2}$ tending to infinity, but so that

$$|t_1 - t_2|/(t_1 + t_2) \to 0,$$

we obtain asymptotically a familiar covariance matrix:

$$\mathfrak{C}_{11}(t_1, t_2, \mathbf{r}_1, \mathbf{r}_2) = K_1^2 \frac{\sin(\mu\lambda_A(t_1 - t_2))}{\mu\lambda_A(t_1 - t_2)},$$

$$\mathfrak{C}_{22}(t_1, t_2, \mathbf{r}_1, \mathbf{r}_2) = K_2^2 \frac{\sin(\mu\lambda_A(t_1 - t_2))}{\mu\lambda_A(t_1 - t_2)}. \tag{3.6}$$

Example 3.2. We shall give now a structure of the fluctuations $\mathbf{f}$, construct the correponding field $\mathbf{z}(\lambda, t)$ and joint probability distributions of the stochastic vector process $G(t)$. This construction is not simple but it is very instructive. Let $\{\mathbf{r}_{kl}\}_{1\,1}^{\infty\infty}$ be a sequence of positions (grid-knots), Ψ be a function with a compact support (a finite element) and let the basis $(\Psi_{kl}\}_{1\,1}^{\infty\infty}$, $\Psi_{kl}(\mathbf{r}) = \Psi(\mathbf{r} - \mathbf{r}_{kl})$, have the following simple property

$$\sum \Psi_{kl}(\mathbf{r}) = 1 \qquad \text{for each } \mathbf{r} \in \mathbb{R}^2.$$

For instance $\psi(\mathbf{r}) = \mathfrak{f}(x)\,\mathfrak{f}(y)$, where $\mathfrak{f}(x) = 2^{-2}(2 - x^2)$ for $|x| < 1$, $\mathfrak{f}(x) = 2^{-2}(2 - |x|)^2$ for $1 < |x| < 2$, and zero otherwise. Now we define two systems of stream functions

$$s_{kl,p}(\mathbf{r}) = \begin{Bmatrix} y\Psi_{kl}(x, y) & \text{for } p = 1 \\ -x\Psi_{kl}(x, y) & \text{for } p = -1 \end{Bmatrix}.$$

Apparently,

$$\sum s_{kl,1}(\mathbf{r}) = y, \qquad \sum s_{kl,-1}(\mathbf{r}) = x.$$

The velocity modes are defined by $\boldsymbol{\varphi}_{kl,p} = \operatorname{curl} s_{kl,p}$. We define the measure ϱ on $\mathbb{R}^\infty$ in the following way

$$\varrho(\lambda_0, \{\lambda_{kl,p}\}) = \begin{cases} 1 & \text{for } \lambda_0, \lambda_{kl,p} \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Thus $\Lambda = \prod \times [0, 1] \subset \mathbb{R}^\infty$. Let $\mathfrak{m}$ be one-to-one mapping of indices k,l onto the set $\mathbb{N} = \{1,2, \ldots\}$ so that we can define the ordering in the set of all indices (k, l, p) by the mapping $(k, l, p) \to p\,\mathfrak{m}(k,l)$. For instance we can choose $\mathfrak{m}(k, l) = k + (k + l - 1)(k + l - 2)/2$. Let $I^- = [0, 1/2]$ and $I^+ = (1/2, 1]$. To define $b_{kl,p}$ we need a sequence of uniformly bounded non-negative functions $\beta_{kl,p}$ on $[0, 1]$, the sequence of functions

$$\chi_{kl,p}(\lambda) = \begin{cases} \pm 1 & \text{if } \lambda_0, \lambda_{m.n,q} \in [0, 1] \text{ for } \mathfrak{m}(m\ n) < \mathfrak{m}(k\ l) \text{ and } \lambda_{kl,p} \in I^+ \\ 0 & \text{otherwise} \end{cases}$$

and the sequence of uniformly bounded non-negative functions $a_{kl,p}$ on $\mathbb{R}$. Then we consider

$$a(\lambda, t) = \sum a_{kl,p}(t)\, \chi_{kl,p}(\lambda)$$

$$b_{kl,p}(\lambda, t) = a(\lambda, t)\, \beta_{kl,p}(\lambda_0), \tag{3.7}$$

$$z(\lambda, t, \mathbf{r}) = a(\lambda, t) \sum \beta_{kl,p}(\lambda_0)\, \varphi_{kl,p}(\mathbf{r}).$$

Each function $\chi_{kl,p}$ depends only on a finite number of variables. Thus, $\chi_{11,p}$ depends on λ_0 and $\lambda_{11,p}$, i. e. on two variables, $\chi_{22,p}$ depends on λ_0, $\lambda_{11,\pm 1}$, $\chi_{12,\pm 1}$, $\lambda_{21,\pm 1}$ and $\lambda_{22,p}$, i. e. on 8 variables e. t. c. The following property of these functions is used for farther analysis: $\int \varrho(\lambda)\, \mathrm{d}\lambda\, \chi_{kl,p}(\lambda)\, \chi_{mn,q}(\lambda) = 0$ for $p\,\mathfrak{m}\,(k, l) \neq q\,\mathfrak{m}\,(m, n)$.

It is easy to derive the joint probability densities and distributions for random variables $g_{m_1}(t_1), g_{m_2}(t_2), \ldots, g_{m_n}(t_n)$, $m_i = p\,\mathfrak{m}\,(k, l)$. We have the following respective expressions:

$$p_{t_1, t_2, \ldots, t_n}(x_1, x_2, \ldots, x_n) = \frac{1}{2} \int \mathrm{d}\lambda\, \varrho(\lambda) \times$$

$$\times \left[\prod_{k=1}^{n} \delta(x_k - b_{m_k}(\lambda, t)) + \prod_{k=1}^{n} \delta(x_k + b_{m_k}(\lambda, t))\right].$$

$$F_{t_1, t_2, \ldots, t_n}(x_1, x_2, \ldots, x_n) = \frac{1}{2} \int \mathrm{d}\lambda\, \varrho(\lambda) \times$$

$$\times \left[\prod_{k=1}^{n} \Theta(x_k - b_{m_k}(\lambda, t)) + \prod_{k=1}^{n} \Theta(x_k + b_{m_k}(\lambda, t))\right].$$

Since the functions $|b_{kl,p}|$ are uniformly bounded by a constant M, we have $F(x_1, x_2, \ldots, x_n) = 0$ if at least one x_i is less than $-M$, and $F(x_1, x_2, \ldots, x_n) = 1$ if all $x_i > M$.

The covariance function can also be constructed. From (3.4) we have

$$\mathfrak{C}_{ij}(t_1, t_2, \mathbf{r}_1, \mathbf{r}_2) = \int_{\Lambda} \mathrm{d}\lambda\, \varrho(\lambda)\, a(\lambda, t_1)\, a(\lambda, t_2) \sum \beta_{kl,p}(\lambda_0)\, \beta_{mn,q}(\lambda_0) \times$$

$$\times (\varphi_{kl,p}(\mathbf{r}))_i\, (\varphi_{mn,q}(\mathbf{r}))_j.$$

The integration over Λ can be split into an integral of λ_0 over $[0, 1]$ affecting only functions β, and the multiple integral over λ_m. The former integral results in the covariance matrix function $\mathfrak{C}$ depending only on space variables, while the latter one defines the covariance function $\mathfrak{R}$ depending only on the variable t. It is instructive to give an example of $\mathfrak{R}$ so that one sees that we can get familiar covariance functions. Let $a_{p\mathfrak{m}(k,l)}(t) = \exp(-|t - p\,\mathfrak{m}\,(k, l)|)$ and let us use simply the index m instead of $p\,\mathfrak{m}\,(k, l)$.

Then

$$\mathfrak{R}(t_1, t_2) = \sum_{m,n\neq 0} \exp(-|t_1 - m| - |t_2 - n|)\Big(\prod_{l=-\infty}^{\infty} \int_{I_l} d\lambda\Big) \chi_m(\lambda)\, \chi_n(\lambda) =$$
$$= \sum_{m,n\neq 0} \exp(-|t_1 - m| - |t_2 - n|)\, \delta_{mn} = \tag{3.8}$$
$$= (\mathfrak{x}(t_1, t_2) + \varepsilon(t_1, t_2)) \exp(-|t_1 - t_2|) - \exp(-|t_1| + |t_2|),$$

where $\mathfrak{x}$ is a periodic function with positive lower and upper bounds and the function ε tends to zero as t_1 or t_2 tend to infinity.

Example 3.3. The function a of the previous example is uniformly bounded in t because this property have functions $a_m(t)$. The sign of a can be positive or negative because χ_m can have any sign. We could choose χ_m to have only non-negative values. In such a case the function a is a non-negative. Here we give an example of χ_m having this property

$$\chi_{kl,p}(\lambda) = \begin{cases} 1 \text{ if } \lambda_0, \lambda_{mn,q} \in I^+ \text{ for } \mathfrak{m}(m,n) < \mathfrak{m}(k,l) \text{ and } \lambda_{kl,p} \in I^- \\ 0 \text{ otherwise} \end{cases}$$

The structure of function a can be easily described as follows. Let $\lambda = \lambda_0$, $\{\lambda_{kl,p}\} \subset \Lambda$, i. e. $\lambda_0, \lambda_{kl,p} \in [0, 1]$. Let k_1, l_1 be the smallest number such that $\lambda_{kl,p} \in I^+$ for $\mathfrak{m}(k,l) < \mathfrak{m}(k_1,l_1)$ and $\lambda_{k_1 l_1,p} \in I^-$. Then

$$a(\lambda, t) = a_{p\mathfrak{m}(k_1,l_1)}(t)\, \chi_{p\mathfrak{m}(k_1,l_1)}(\lambda) = a_{p\mathfrak{m}(k_1,l_1)}(t).$$

In particular, $a(\lambda, t)^{-1}\, \partial a(\lambda, t)/\partial t = a_m(t)^{-1}\, \partial a_m(t)/\partial t$, where $m = p\mathfrak{m}(k_1, l_1)$.

4. *Some general transport models*

Our derivation of new transport models is based on the procedure which is developed in Section 2 and which is applied there to the derivation of generalized diffusion equation. There is no need to repeat all the steps of this procedure in the present case. Therefore, we wish to focus upon the essential parts of this procedure. The main step consists in relating the (diffusion) equation (2.15) for each λ to the series (2.11) which formally represent the mean field $\mathbf{E}(u(t))$ for each λ. This step is possible to carry out because the coefficients v_k of the series (2.11) fulfill a simple equality (2.13). Evidently, this equality resembles the diffusion equation. Thus, we have to focus our attention upon the coefficients v_k in the present case, too.

The stochastic velocity field is defined by (2.1), (2.2) with the fluctuations $\mathbf{f}$ defined as in Section 3. The stochastic mass balance equation (2.4) is treated analogously as along lines (2.5)—(2.9). Instead of (2.10) we now have

$$\mathbf{E}(S(t_1)\, S(t_2) \dots S(t_n)) =$$
$$= \int d\lambda\, \varrho(\lambda)\, \mu_n(\lambda, t_1, t_2, \dots, t_n) \prod_{k=1}^{n} \Big(\sum_{m=1}^{\infty} b_m(\lambda, t_k)\, \nabla\varphi_m\Big) \tag{4.1}$$

where μ_n are statistical moments of the order n of some stochastic process.

It is instructive to consider two main cases, the one for time-independent functions b_k and the other for which b_k dependent on time. For time-independent b_k we have ϱ time independent covariance function of components of velocity fluctuations. Obviously, this case is not very promising for a realistic description of stochastic motion in the sea or lake and it has only a methodological value.

4.1. Case of time-independent fluctuations

The differential operators (4.1) of the order n are now time-independent so that we have

$$\mathbf{E}\,(S\,(t_1)\,S\,(t_2)\,\dots\,S\,(t_n)) = \int \mathrm{d}\lambda\,\varrho(\lambda)\,(\mathbf{z}\,(\lambda)\,\nabla)^{2k}$$

where the field $\mathbf{z}$ is defined by (3.3). The multiple time-integrals in (2.9) can be easily calculated so that we have

$$\mathbf{E}\,(u_n\,(t)) = \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2\,\dots\,\mathrm{d}t_n \int_{-\infty}^{t_n} \mathrm{d}s\,\mathbf{E}\,(S\,(t_1)\,S\,(\mathrm{d}t_2\,\dots\,S\,(t_n))\,p\,(s) =$$

$$= \int_{\Lambda} \mathrm{d}\lambda \int_{0}^{t} \mathrm{d}s\varrho\,(\lambda)\,\nu_k\,(\lambda,\,t-s)\,(\boldsymbol{\nu}\,(\lambda)\nabla)^{2k}\,\nabla p\,(s),$$

where $k = n/2$ and

$$\nu_k\,(\lambda,\,t) = \frac{1}{(2k)!}\int_{0}^{t} (t-s)^{2k}\,\mathrm{d}s.$$

Hence, $\mathbf{E}\,(u\,(t)) = \Sigma\,\mathbf{E}\,(u_n\,(t))$ has the expression

$$\mathbf{E}\,(u\,(t)) = \int \mathrm{d}\lambda \int \mathrm{d}s\,\varrho(\lambda) \sum_{k=0} \nu_k\,(\lambda,\,t-s)\,A\,(\lambda)^k\,p\,(s)\,\mathrm{d}s = \int \mathrm{d}\lambda\,\varrho(\lambda)\,U\,(\lambda,\,t)$$

where

$$A\,(\lambda) = (\mathbf{z}\,(\lambda)\,\nabla)^2, \quad U\,(\lambda,\,t) = \int \mathrm{d}s\,\Sigma\,\nu_k\,(\lambda,\,t-s)\,A\,(\lambda)^k\,p\,(s)$$

as in (2.16) and (2.12), respectively. Due to the basic relationship

$$\ddot{\nu}_{k+1} - \nu_k = 0,$$

we have

$$\frac{\partial^2}{\partial t^2}\,U\,(\lambda,\,t) - A\,(\lambda)\,U\,(\lambda,\,t) = \frac{\partial}{\partial t}\,\mathbf{E}\,(p\,(t)).$$

For $\mathbf{w} = \mathbf{0}$ we have $C\,(t,\,\mathbf{r}) = \int \mathrm{d}\lambda\,\varrho(\lambda)\,U\,(t,\,\lambda)$. For $\mathbf{w} \neq \mathbf{0}$ we have to substitute the function C instead of U. By repeating (2.16)—(2.18) we obtain the expression for solution C:

$$\mathbf{E}\,(C\,(t,\,\mathbf{r})) = \int \mathrm{d}\lambda\,\varrho(\lambda)\,C\,(\lambda,\,t,\,\mathbf{r}), \tag{4.2}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{w}\nabla\right)^2 C\,(\lambda,\,t) - A\,(\lambda)\,C\,(\lambda,\,t) = \left(\frac{\partial}{\partial t} + \mathbf{w}\nabla\right)\mathbf{E}\,(q\,(t)). \tag{4.3}$$

Thus, if the mean field $\mathbf{w}$ were zero, the governing equation for $C(\lambda, t)$ is the wave equation rather than the diffusion equation (2.18).

4.2. *A case of time-dependent fluctuations*

We assume now

$$b_k(\lambda, t) = a(\lambda, t)\, b_k(\lambda) \tag{4.4}$$

having in mind that this structure allows familiar covariance functions as has been discussed in Section 3. The function a must be positive on Λ to ensure correct definitions of various quantities in the following. The velocity modes φ_k and the amplitudes b_k define the field

$$\mathbf{z}(\lambda, t, \mathbf{r}) = a(\lambda, t) \sum b_k(\lambda)\, \varphi_k(\mathbf{r}) = a(\lambda, t)\, \zeta(\lambda, \mathbf{r}).$$

Hence

$$\mathbf{E}(S(t_1)\, S(t_2) \dots S(t_n)) = \int \mathrm{d}\lambda\, \varrho(\lambda) \left[\prod_{j=1}^{n} a(\lambda, t_j)\right] (\zeta(\lambda)\nabla)^n$$

and therefore

$$\mathbf{E}(u(t)) = \sum_{n=0} \mathbf{E}(u_n(t)) = \sum_{n=0} \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \dots \mathrm{d}t_n \int_{-\infty}^{t_n} \mathrm{d}s$$

$$\mathbf{E}(S(t_1)\, S(\mathrm{d}t_2) \dots S(t_n))\, p(s) = \sum_{k=0} \int \mathrm{d}\lambda \int \mathrm{d}s\, \varrho(\lambda)\, \nu_k(\lambda, t, s)\, (\zeta(\lambda)\nabla)^{2k} p(s),$$

where $\nu_k(\lambda, t, s)$ are functions that are defined by the following recursion

$$\nu_0(\lambda, t, s) = \Theta(t - s),$$

$$\nu_k(\lambda, t, s) = \int_{-\infty}^{t} a(\lambda, t_1)\, \mathrm{d}t_1 \int_{-\infty}^{t_1} a(\lambda, t_2)\, \mathrm{d}t_2\, \nu_{k-1}(\lambda, t_2, s).$$

Apparently, we have now the following basic equality among ν_k:

$$\frac{\partial}{\partial t}\left(a(\lambda, t)^{-1} \frac{\partial}{\partial t} \nu_k(\lambda, t, s)\right) = a(\lambda, t)\, \nu_{k-1}(\lambda, t, s).$$

Then, by the same argumentations as previously, we have

$$\mathbf{E}(C(t, \mathbf{r})) = \int \mathrm{d}\lambda\, \varrho(\lambda)\, C(\lambda, t, \mathbf{r}), \tag{4.5}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{w}\nabla\right)^2 C(\lambda, t) - a(\lambda, t)^{-1} \left(\frac{\partial}{\partial t} a(\lambda, t)\right) \left(\frac{\partial}{\partial t} + \mathbf{w}\nabla\right) C(\lambda, t) -$$

$$- A(\lambda) C(t, \lambda) = \left(\frac{\partial}{\partial t} + \mathbf{w}\nabla\right) \mathbf{E}(q(t)), \tag{4.6}$$

and

$$A(t, \lambda) = a(t, \lambda)^2 (\zeta(\lambda) \nabla)^2 = (\mathbf{z}(\lambda, t)) \nabla)^2, \tag{4.7}$$

4.3. *A general time-dependent fluctuation*

We wish to include the functions μ_n in our analysis. Therefore, we assume that the statistical moments of $g_k(t_k)$ for even n have the following form

$$\mathbf{E}(g_{k_1}(t_1) g_{k_2}(t_2) \dots g_{k_n}(t_n)) = \int d\lambda\, \varrho(\lambda)\, \mu_n(\lambda, t_1, t_2, \dots, t_n) \times$$

$$\times \prod_{k=1}^{n} \left(\sum_{m=1}^{\infty} b_m(\lambda, t_k) \nabla \varphi_m\right).$$

Let us define functions $a_k(\lambda, t)$ by

$$v_0(\lambda, t, s) = \Theta(t - s),$$

$$v_k(\lambda, t, s) = \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{2k-1}} dt_{2k}\, \mu_n(\lambda, t_1, t_2, \dots, t_{2k})\, \Theta(t_{2k} - s).$$

To relate v_k and v_{k-1} by a relationship which enables us to derive a differential equation for $\mathbf{E}(u)$, we have to impose certain conditions on μ_n. Let there exist a function $p(\lambda, t)$ such that

$$p(\lambda, t) \frac{\partial}{\partial t} (p(\lambda, t)^{-1} \mu_{2k}(\lambda, t, t_2, \dots, t_{2k}))\,|_{t_2 = t} = a(\lambda, t)^2 \mu_{2(k-1)}(\lambda, t_3, t_4, \dots, t_{2k}).$$

In this case we have (4.5), (4.7) while (4.6) is replaced by the following differential equation

$$\left(\frac{\partial}{\partial t} + \mathbf{w}\nabla\right)^2 C(\lambda, t) - \frac{\partial}{\partial t} \ln p(\lambda, t) \left(\frac{\partial}{\partial t} + \mathbf{w}\nabla\right) C(\lambda, t) -$$

$$- A(\lambda)\, C(\lambda, t) = \left(\frac{\partial}{\partial t} + \mathbf{w}\nabla\right) \mathbf{E}(q(t)).$$

4.4. *Finite versus infinite velocity of spreading*

A finite velocity of spreading of an information from a source is the most interesting feature of the derived transport models. Let us remind that the diffusion equation has not this property.

We assume that the input is located at $\mathbf{r}_0$. The concentration field caused by the input is a weighted average of the partial concentration fields $C(\lambda, t, \mathbf{r})$. For each λ, the substance is transported in two opposite directions from $\mathbf{r}_0$ by the velocities $\mathbf{w} + \mathbf{z}(\lambda, t)$ and $\mathbf{w} - \mathbf{z}(\lambda, t)$. Hence $\mathbf{z}(\lambda, t)$ is the diffusion velocity. A finite velocity of spreading of an information from sources follows from the uniform boundedness of $\mathbf{z}$ on Λ. The uniform boundedness of $\mathbf{z}$ can follow from various criteria on b_k and φ_k. It suffices to suppose the uniform boundedness of the sequence $\{b_k(\lambda)\}$ on Λ and the absolute convergence of $\sum \varphi_k$ on $\mathbb{R}^n$ (There should also exist div $(\sum \varphi_k)$). The finite velocity is ensured if the diffusion velocity $\mathbf{z}(\lambda, t)$ is uniformly bounded on Λ. On the other hand we can obtain the infinite velocity as well by allowing $\mathbf{z}(\lambda, t)$ to be unbounded on Λ. For instance, solutions of the conventional diffusion equation follow also from models of fluctuations of Section 3. Let us consider one-dimensional case with $\mathbf{w} = \mathbf{0}$, input $q(t, x) = \delta(t)\,\delta(x)$ and diffusion velocity independent of x. If the diffusion velocity z were time-independent the corresponding solution of (4.6) would be

$$C(\lambda, t, x) = 2^{-1}\,\Theta(t)\,[\delta(tz(\lambda) - x) + \delta(tz(\lambda) + x)]. \tag{4.8}$$

However, we need time-dependent diffusion velocities. Let $a(\lambda, t) = t^{-1/2}$, $z(\lambda, t, x) = a(\lambda, t)(\lambda/2)^2$. It turns out that the corresponding solution of (4.6) is

$$C(\lambda, t, x) = 2^{-1}\,\Theta(t)\,\{\delta(\lambda\sqrt{t} - x) + \delta(\lambda\sqrt{t} + x)].$$

If the probability distribution is $\varrho(\lambda) = (2\pi a)^{-1/2}\exp(-\lambda^2/(2a))$ we obtain finally

$$C(t, \mathbf{x}) = \int \mathrm{d}\lambda\,\varrho(\lambda)\,C(\lambda, t, x) = (8\pi a)^{-1/2}\,\Theta(t)\int \mathrm{d}\lambda\,\exp(-\lambda^2/(2a))$$

$$[\delta(\lambda\sqrt{t} - x) + \delta(\lambda\sqrt{t} + x)] = (2\pi a t)^{-1/2}\exp(-x^2/(2at)).$$

i. e. the solution of the diffusion equation with input

$$q(t, x) = \delta(t)\,\delta(x).$$

In this simple example the diffusion velocity $z(\lambda, t, x)$ is not a bounded function on $\Lambda = (-\infty, \infty)$. Consequently, our conclusion is that the velocity of spreading is finite if the fluctuations (2.2) have uniformly bounded realizations and infinite in the opposite case. Furthermore, as it has been discussed in Section 2, the finiteness of velocity doesn't depend on the covariance function of fluctuations.

4.5. *Properties of solutions*

In the case of one space dimension, the generalized diffusion equation and models of this section have note useful properties than the conventional diffusion equation. Solutions are weighted averages of the corresponding, generalized diffusion equations and the wave (telegraph) equation, respectively. In two or three dimensions the derived models have a feature which seems unfavorable at the first sight. For each λ, the corresponding diffusion or telegraph equation contains the second order differential operator $A(\lambda) = \sum \partial_i z_i(\lambda)\,z_j(\lambda)\,\partial_j$ which is degenerate, i. e. $A(\lambda)\,u = 0$ for non-trivial functions u. Such functions are those, which have same values along directions of the velocity field $\mathbf{z}(\lambda)$ while generally they vary across streamlines of $\mathbf{z}(\lambda)$. This is not the case with the conventional diffusion or wave equations since these equations contain the Laplacian

Δ. However, we should not forget that this seeming disadvantage occurs for a fixed λ, while diffusion velocities $\mathbf{z}(\lambda)$ have various directions as λ varies over Λ. The following example in two dimensions is very instructive for the present discussion.

Let $\mathbf{w} = \mathbf{0}$, $\Lambda = \Lambda_1 \times \Lambda_2 \subset \mathbb{R}^2$, Λ_1 arbitrary in $\mathbb{R}$ and $\Lambda_2 = (0, \pi)$. Let $\lambda = (\beta, \gamma)$, $\varrho(\lambda) = \varrho_1(\beta)\,\varrho_2(\gamma)$. We assume now that the diffusion velocity $\mathbf{z}(\lambda)$ has the magnitude $\sigma(\beta)$ and direction γ so that $z_1(\lambda) = \sigma(\beta)\cos\gamma$, $z_2(\lambda) = \sigma(\beta)\sin\gamma$. For a fixed λ and the input $\delta(t)\,\delta(x)\,\delta(y)$ we have the solution

$$C(\lambda, t, \mathbf{r}) = 2^{-1}\,\Theta(t)\,\delta(x\sin\gamma - y\cos\gamma)\,[\delta(t\sigma(\beta) - (x\cos\gamma + y\sin\gamma)) + \delta(t\sigma(\beta) + (x\cos\gamma + y\sin\gamma))]$$

which follows from (4.8) after performing the rotation for the angle γ. Let us integrate this solution with respect to γ. The resulting concentration field depends only on the magnitude $\sigma(\beta)$ of the diffusion velocity. For instance, if $\varrho_2(\gamma) = 1/\pi$, we have

$$C(\beta, t, \mathbf{r}) = (2r\pi)^{-1}\,\Theta(t)\,[\delta(t\,\sigma(\beta) - r) + \delta(t\sigma(\beta) + r)] = 2^{-1}\,\Theta(t)\,\delta(t^2\sigma(\beta)^2 - r^2), \tag{4.9}$$

where $r = |\mathbf{r}|$. Before we proceed to discuss this result, we make an analogous transformation with solutions of the generalized diffusion equation. The same assertion on $\mathbf{w}$, Λ, ϱ and $\mathbf{z}$ results in

$$C_d(\beta, t, \mathbf{r}) = (\pi r\sqrt{2\pi e(t)})^{-1}\exp(t^2\sigma(\beta)^2 - r^2). \tag{4.10}$$

The expressions (4.9) and (4.10) describe spreading of substance from a source in all directions with the same velocity.

Let us compare features of solutions (4.2)—(4.6) with features of solutions of the conventional and generalized diffusion equation regarding the properties (a)—(c) of Introduction. The feature (a) with regard to diffusion equations is already discussed in Section 2. Let us see that the linear law of spreading is also exhibited by the here derived transport models. In the case of the generalized diffusion equation, the feature (a) was derived from expression of fundamental solutions. The same can be done for the transport equations (4.2)—(4.7). Since for each λ the governing equations are one-dimensional wave (telegraph) equations, it suffices to consider the simplest case of (4.2), (4.3) with $\mathbf{w} = 0$ and a constant diffusion velocity. The fundamental solution for this case[15] is $\mathfrak{E}(t, x) = (2\lambda)^{-1}\,\Theta(t)\,\Theta(t^2 z(\lambda)^2 - x^2)$. The linear law of spreading follows from $r = t\,|z(\lambda)|$ and implies the validity of both features, (a) and (b).

5. *Applications*

Before we proceed to develop transport models for applications we have to include the extinction of substance from the water column. We assume that the input is not independent of concentration. To remain within the class of linear models we assume the following form of input, $q = q_1 + q_2$, where $q_1 \geq 0$ and it does not depend on the concentration, while $q_2 = - kC$[5]. Here k is called the extinction constant. It is related to the half-life T of the extinction by $T = \ln(2)/k$. There exist two ways to generalize developed models in order to include this new form of input. This could be done from the very beginning in the derivation of transport models by replacing $\mathbf{w}\nabla$ with $k + \mathbf{w}\nabla$ every- where in expressions of Sections 2 and 4. It can be done also, straightforwardly, by replacing C and q in the derived models of Sections 3 and 4 with $\exp(kt)\, q(t, \mathbf{r})$ and $\exp(kt)\, C(t, \mathbf{r})$, respectively. In either case we obtain

$$\mathbf{E}(C(t, \mathbf{r})) = \int \mathrm{d}\lambda\, \varrho(\lambda)\, C(\lambda, t, \mathbf{r}), \tag{5.1}$$

$$\left(\frac{\partial}{\partial t} + k + \mathbf{w}\nabla\right)^2 C(\lambda, t) - a(\lambda, t)^{-1} \left(\frac{\partial}{\partial t} a(\lambda, t)\right) \left(\frac{\partial}{\partial t} + k + \mathbf{w}\nabla\right) C(\lambda, t) -$$

$$- A(\lambda, t)\, C(\lambda, t) = \left(\frac{\partial}{\partial t} + k + \mathbf{w}\nabla\right) \mathbf{E}(q(t)) \tag{5.2}$$

Let us focus upon two-dimensional models since they are most interesting. In the example of the previous section we considered two-dimensional models with constant parameters. After integration over all directions of diffusion velocity in space we obtained the solution (4.9). Formally, this is a fundamental solution of wave equation in three dimensions for which the input is equal to $\delta(t)\, \delta(r)\, \delta(z)$, $r = (x^2 + y^2)^{1/2}$, and for which the result is considered on the hyperplane $z = 0$. Thus two-dimensional models after the integration over all directions of diffusion velocity, turn to be three-dimensional models. The same is true for the generalized diffusion equation. The solution (4.10) is formally a solution of three-dimensional generalized diffusion equation with a particular input. It is difficult to say whether we are on loss or gain with regard to computations by increasing the space dimension for one instead of solving many equations in the original space. The only way out is to postulate a simple model for application while keeping realistic features of the considered models.

An analysis of properties of solutions of (5.1) (5.2) for general forms of the functions a and q is out of the scope of this work. We know that realistic inputs are those which are different from zero for a short time-interval or they are periodic functions. Analogously, we assume that diffusion velocity is either constant or periodic with the same period as input. We have in mind that these periods describe daily or seasonal variability of diffusion and input. To make further discussion as brief as possible we assume that periods are strictly λ-independent. With these suppositions we have the following conclusion. For large t the solution $C(\lambda, t, \mathbf{r})$ tends either to a time-independent function $C_\infty(\lambda, \mathbf{r})$ or a periodic function $C_\infty(\lambda, t, \mathbf{r})$. Let us calculate the corresponding integrals $C(t, \mathbf{r})$ of (5.2). In the former case this integral tends to a time-independent function of $\mathbf{r}$. In the latter case it tends to a periodic function.

In this section we are interested to find conditions which allow stationary transport if the input is also stationary. Therefore, we assume that the function a

is t-independent In this case the operator $A(\lambda)$ is also t-independent. This assumption simplifies the model, so that we have now the system consisting of (5.1) and

$$\left(\frac{\partial}{\partial t} + k + \mathbf{w}\nabla\right)^2 C(\lambda, t) - A(\lambda)\, C(\lambda, t) = \left(\frac{\partial}{\partial t} + k + \mathbf{w}\nabla\right) \Theta(t)\, q(\mathbf{r}). \quad (5.3)$$

The time dependency of input is simple. The (mean) distribution of input is factored into the product of a step function and a time independent distribution of $\mathbf{r}$.

The system (5.1), (5.3) could be proposed, or even postulated, as a model for applications if it were suitable for numerical evaluations. Its unfitness for numerical evaluations follows from necessity to solve (5.3) for various values of multivariable λ. We feel that we cannot get rid of the integration with respect to λ completely. Diffusivity results from the weighted integral of solutions for various diffusion velocity $\mathbf{z}(\lambda)$. Therefore, λ-dependence of the diffusion velocity $\mathbf{z}$ in the proposed models is an essential characteristic. Our intention is to reduce this dependency as much as possible.

The integration of the diffusion equation (5.3) over Λ does not produce an equation for $C(t, \mathbf{r}) = \int \mathrm{d}\lambda\, \varrho(\lambda)\, C(\lambda, t, \mathbf{r})$ because there is a term containing the product of $A(\lambda)$ and $C(\lambda, t, \mathbf{r})$. Our intention is to use the mean value theorem in the form

$$\int_{\Lambda_1} \mathrm{d}\lambda\, \varrho(\lambda)\, A(\lambda)\, C(\lambda, t, \mathbf{r}) = A(\lambda(t, \mathbf{r})) \int_{\Lambda_1} \mathrm{d}\lambda\, \varrho(\lambda)\, C(\lambda, t, \mathbf{r})$$

and approximate the operator $A(\lambda(t, \mathbf{r}))$ with $A(\lambda')$ with certain λ'. Here the set Λ_1 is a subset of Λ. The second and third examples of Section 3. are very useful for a justification of this approximation. The function $a(\lambda, t)$ of these examples has the form

$$a(\lambda, t) = \sum_{m \neq 0} \exp(-|t - m|)\, \chi_m(\lambda).$$

Let $\lambda \in \Lambda_1 = \Lambda / I_0$ be fixed. Then for each t, there exists an m such that the function $a_m(t) = \exp(-|t - m|)$ has a maximal value. Hence $a(\lambda, t) \approx \pm a_m(t)$. This implies that $a(\lambda, t)^2$ is a function having values in the interval [0, 1]. The average value of $a(\lambda, t)^2$ with respect to the measure ϱ tends to the periodic function

$$\int_{\Lambda_1} \mathrm{d}\lambda\, \varrho(\lambda)\, a(\lambda, t)^2 = \mathfrak{r}(t, t) = (\sinh(1))^{-1} \cosh(1 + 2([t] - t)) - 1.$$

In this way the operators $A(\lambda) = a(\lambda, t)^2 \sum \partial_i \zeta_i(\lambda)\, \zeta_j(\lambda)\, \partial_j$ of Examples 3.2 and 3.3 can be approximated by const $\times \sum \partial_i \zeta_i(\lambda_0)\, \zeta_j(\lambda_0)\, \partial_j$.

The variable λ_0 of Example 3.2 or 3.3. is from the interval (0, 1). This simple choice of λ was caused by our intention to make the examples as simple as possible. Now we can turn to a general choice of λ_0 and say that λ_0 is a multiple variable $\lambda_0 = (\lambda_1, \lambda_2, \ldots)$. For instance λ_1 can define the magnitude of diffusion velocity, while λ_2 can define the direction (angle).

We are ready to perform the last step in formulation a model for applications in two dimensions. We start with the working supposition that the current field

$\mathbf{v} = \mathbf{w} + \mathbf{f}$ describes a two-dimensional motion in the three-dimensional space, i. e. $\mathbf{w}$ and $\mathbf{f}$ are x_3-independent. The same supposition must be valid for q, i. e. q is also x_3-independent. This working supposition does not change any of previous result or conclusions. In particular, after averaging over directions of diffusion velocity we obtain a solution which formally satisfies the three-dimensional transport model. Now we express the solution by means of the fundamental solution of three-dimensional equation and integrate over x_3 since the input is x_3-independent. The result is a solution which formally satisfies the two-dimensional model. The two-dimensional model is the basis for our further discussion. We must have in mind that the model is derived from the working supposition that $\mathbf{v}$ is a two-dimensional flow. With this in mind, the integration over directions of diffusion velocity could be carried out and the resulting model is a two-dimensional telegraph equation:

$$C(t, \mathbf{r}) = \int \mathrm{d}\lambda\, \varrho(\lambda)\, C(\lambda, t, \mathbf{r}),$$

$$\left(\frac{\partial}{\partial t} + k + \mathbf{w}\nabla\right)^2 C(\lambda, t) - \sigma(\lambda)^2 \Delta C(\lambda, t) = \left(\frac{\partial}{\partial t} + k + \mathbf{w}\nabla\right) \Theta(t)\, q(\mathbf{p}). \tag{5.4}$$

Here, ∇ and Δ are two-dimensional gradient and Laplacian. The λ-dependency is contained only in the magnitude $\sigma(\lambda)$ of diffusion velocity. The measure ϱ is now a probability measure of a single real variable λ.

In order to demonstrate certain properties of the proposed model (5.4) we must restrict our discussion to the case of constant parameters. In this case solutions can be obtained in an explicit form.

Let us suppose that the input has the following simple form $q(t, \mathbf{r}) = \Theta(t)\, q(\mathbf{r})$, where q is a smooth function with a compact support. The constant mean field $\mathbf{w}$ (drift velocity) is assumed to have non-zero component w_1. By using the fundamental solution of the wave equation in two dimensions we can obtain solutions of (5.4). Thus, for a fixed λ we have

$$C_{w,k}(t, \mathbf{r}) = B_{w,k}(t, x - wt, y) \tag{5.5}$$

where

$$B_{w,k}(t, \mathbf{r}) = (2\pi\sigma)^{-1} \int_0^t \mathrm{d}s \int_{|\mathbf{r} + \mathbf{w}s - \mathbf{p}| < \sigma(t-s)} \mathrm{d}\mathbf{p} \exp(-k(t-s)) \times$$
$$\times [\sigma^2 (t-s)^2 - (\mathbf{r} - \mathbf{p})^2]^{-1/2} (\delta(s)\, q(p_1 + w_1 s, p_2) + kq(p_1 + w_1 s, p_2) +$$
$$+ w_1 \partial_1 q(p_1 + w_1 s, p_2)). \tag{5.6}$$

In particular, we have

$$C_{0,0}(t\ \mathbf{r}) = (2\pi\sigma)^{-1} \int_{|\mathbf{r} - \mathbf{r}'| < \sigma t} (\sigma^2 t^2 - |\mathbf{r} - \mathbf{r}'|^2)^{-1/2} q(\mathbf{r}')\, \mathrm{d}\mathbf{r}', \tag{5.7}$$

$$C_{k,0}(t, \mathbf{r}) = \exp(-kt)\, C_{0,0}(t, \mathbf{r}) +$$
$$+ (2\pi\sigma)^{-1} k \int \mathrm{d}s \exp(-k(t-s)) \int_{|\mathbf{r} - \mathbf{r}'| < \sigma(t-s)} (\sigma^2 (t-s)^2 - |\mathbf{r} - \mathbf{r}'|^2)^{1/2} q(\mathbf{r}')\, \mathrm{d}\mathbf{r}'. \tag{5.8}$$

There exists a conservation law that is implied by (5.4). After integrating the differential equation of (5.4) over the whole space $\mathbf{R}^2$ we obtain

$$Q(t) = \int_{\mathbb{R}^2} C_{w,k}(\lambda, t, \mathbf{r})\,\mathrm{d}\mathbf{r},$$

$$\left(\frac{\partial}{\partial t} + k\right) Q(t) = \Theta(t) \int_{\mathbb{R}^2} q(\mathbf{r})\,\mathrm{d}\mathbf{r}$$

so that

$$Q(t) = k^{-1}\,\Theta(t)\,[1 - \exp(-kt)] \int_{\mathbb{R}^2} q(\mathbf{r})\,\mathrm{d}\mathbf{r}. \tag{5.9}$$

We can find asymptotic forms of solutions as $t \to \infty$. The solution (5.7) vanishes asymptotically for each fixed $\mathbf{r}$, but $Q(t) = \Theta(t)\,|t|\,\int q(\mathbf{r})\,\mathrm{d}\mathbf{r}$ in accordance with (5.9). Similarly, we obtain the following asymptotic form for (5.8)

$$C_{k,0,\infty}(\mathbf{r}) = \frac{k}{2\pi\sigma^2} \int_{\mathbb{R}} K_0\left(\frac{k}{\sigma}\,|\mathbf{r} - \mathbf{r}'|\right) \mathrm{d}\mathbf{r}' \tag{5.10}$$

where K_0 is the Bessel function. Analogously

$$C_{k,w,\infty}(\mathbf{r}) = \frac{1}{2\pi\sigma^2} \int_0^{t} \mathrm{d}s \exp(-k(t-s)) \int_{|\mathbf{r} + \mathbf{w}(t-s) - \mathbf{p}| < \sigma(t-s)}$$

$$(\sigma^2 (t-s)^2 - |\mathbf{r} - \mathbf{w}(t-s) - \mathbf{p}|^2)^{-1/2} (kq(\mathbf{p}) + w_1 \partial_1 q(\mathbf{p}))\,\mathrm{d}\mathbf{p}. \tag{5.11}$$

The obtained asymptotic forms (5.10) and (5.11) are convenient for making a comparison between the models of this work and conventional stationary transport model. If drift is absent the conventional stationary model is defined by the stationary diffusion equation

$$-a\Delta C + kC = q.$$

Its solution C coincides with solution (5.10) if we interpret the turbulent diffusion coefficient as $a = \sigma^2/k$. In this way we have obtained the first remarkable result. The non-stationary model (5.4) has advantages as described by (a)—(c) of Introduction. Its restriction to stationary motion with the vanishing drift velocity, $\mathbf{w} = \mathbf{0}$, has the form

$$C_\infty(\mathbf{r}) = \int_{\Lambda} \mathrm{d}\lambda\, \varrho(\lambda)\, C_\infty(\lambda, \mathbf{r}), \tag{5.12}$$

$$\left(-\frac{\sigma(\lambda)^2}{k}\,\Delta + k\right) C_\infty(\lambda, \mathbf{r}) = q(\mathbf{r}), \tag{5.13}$$

which is in a perfect agreement with conventional stationary transport model.

The conventional diffusion constant a can be interpreted as that value $\sigma(\lambda_0)^2/k$ for which we have

$$a = \frac{\sigma(\lambda_0)^2}{k} = [k \int_{\Lambda} \mathrm{d}\lambda\, \varrho(\lambda)\, \sigma(\lambda)^{-2}]^{-1}.$$

In this case the error function

$$\varepsilon(\mathbf{r}) = (-a\Delta + k)\, C_\infty(\mathbf{r}) - q(\mathbf{r}) = k \int_{\Lambda} \mathrm{d}\lambda\, \varrho(\lambda) \left[1 - \frac{\sigma(\lambda_0)^2}{\sigma(\lambda)^2}\right] C_\infty(\lambda, \mathbf{r})$$

vanishes in average

$$\int_{\mathbb{R}^2} \varepsilon(\mathbf{r})\, \mathrm{d}\mathbf{r} = 0.$$

For non-zero drift velocity $\mathbf{w} \neq \mathbf{0}$, the present models differ significantly from conventional diffusion model. Again we have (5.12), while the differential equation has the following form

$$\left(-\frac{\sigma(\lambda)^2}{k}\Delta + \frac{1}{k}(\mathbf{w}\nabla)^2 + 2\mathbf{w}\nabla + k\right) C_\infty(\lambda, \mathbf{r}) = \left(1 + \frac{1}{k}\,\mathbf{w}\nabla q(\mathbf{r})\right). \quad (5.14)$$

Let us consider the case with constant $\mathbf{w} = (w, 0)$ so that we have the following differential operator

$$L = -\frac{\sigma(\lambda)^2 - w^2}{k}\frac{\partial^2}{\partial x^2} - \frac{\sigma(\lambda)^2}{k}\frac{\partial^2}{\partial y^2} + 2w\frac{\partial}{\partial x} + k \quad (5.15)$$

on the left hand side of (5.14). If $\sigma(\lambda) > w$, this operator resembles (5.14). This operator is elliptic in the case of $\sigma(\lambda) > w$ and hyperbolic in the opposite case. Such drastic change of the basic type of differential operator (5.15) is a novelty of derived models.

We can illustrate some properties of solutions of (5.14) if we assume that q is y-independent so that $C_\infty(\lambda, \mathbf{r})$ is also y-independent. In this case (5.14) reduces to a second order differential equation in x, which can be easily solved. Let us take the following values of parameters:

$$w = 2\ \mathrm{cm/s}, \quad k = 10^{-6}\ \mathrm{s}^{-1}, \quad \sigma = \lambda\ \mathrm{cm/s},$$

$$q(x) = 1 - |x| \text{ for } |x| < 1 \text{ km and zero otherwise,} \quad (5.16)$$

$$\varrho(\lambda) = 16\lambda\,(3/2 - \lambda)/9 \text{ for } \lambda \in (0, 3/2) \text{ and zero otherwise.}$$

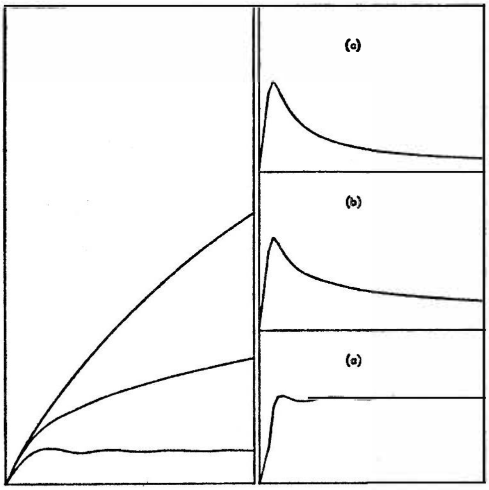

Fig. 1. Solutions of the one-dimensional generalized diffusion equation (2.20). The input is at $x = 0$, with an instantaneous intensity in $t = 0$. The left part illustrates isolines of equal concentrations for various forms of the function $e(t)$ that are defined by (2.22). The right part illustrates concentrations of $C(t, 1)$ for the same forms of functions $e(t)$.

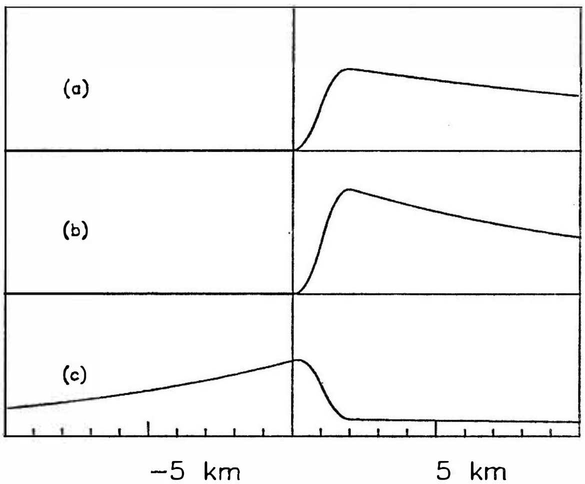

Fig. 2. Solutions of the stationary model (5.14) for which parameters are defined by (5.16). Illustration represent the function: a) $C_\infty(0, x)$, b) $\int d\lambda . \varrho(\lambda) C_\infty(\lambda, x)$, and c) $C_\infty(3, x)$.

The solution $C_\infty(0, x)$ is represented in Fig. 2a, solution $\int d\lambda \varrho(\lambda) C_\infty(\lambda, x)$ in Figure 2b, and $C_\infty(3, x)$ in Figure 2c. It is interesting that concentration values of the (b) case exceed concentration values of (a) case, i. e. that the fluctuations have increased values of concentration field. Solutions of the conventional model do not have this feature.

6. *Conclusion*

The mass balance equation for transport of substance in a velocity field becomes a stochastic differential equation if the velocity field is stochastic. This equation can be taken as a starting object in deriving various transport laws for the mean (average) concentration field. Resulting equations are conventional diffusion equation, generalized diffusion equation and a certain class of new models describing spreading of substance by a finite velocity. The class is rich enough to allow familiar covariance functions of velocity fluctuations. Numerical methods, that could be efficiently used for application to coastal sea, do not exist yet.

Acknowledgement

This research is partially supported by UNEP/IOC, Mediterranean Action Plan, within the project: Modelling non-stationary transport of pollutants in coastal seas. and the Ministry for Science of Republic of Croatia under the project № 1—07—145.

References

1) B. C. Patten (editor), *System analysis and simulation in ecology*, Part III, Academic Press, New York, 1975 2. S. E. Jorgensen (editor), *State-of-the-art in ecological modelling*, Proceedings of the Conference on ecological modelling, Copenhagen, 3. A. Friedman, *Stochastic differential equations and applications*, Vol. 1, Academic Press, New York, 1975;
4) N. Limić, Appl. Math. Model. **14** (1990) 549;
5) T. Legović, N. Limić and V. Valković, Estuarine, Coastal and Shelf Science **30** (1990) 619;
6) N. Limić and M. Ahel, *Estimation of input and extinction rate of petroleum hydrocarbons in coastal waters using a mathematical transport model*, Preprint IRB, 1991;
7) S. Goldstein, Quart. J. Mech. and Appl. Math **4** (1951) 129;
8) R. Bourett, Canad. J. Phys. **38** (1960) 665;
9) D. L. Koch and J. F. Brady, Phys. Fluids **31** (1988) 965;
10) I. I. Gihkman and A. V. Skorokhod, *The theory of stochastic processes*, Vol. I (in Russian), Nauka, Moscow, 1971;
11) N. Limić, J. Math. Biol. **27** (1989) 105;
12) G. I. Taylor, J. Aeronaut. Sci. **4** (1937) 311;
13) A. S. Monin and A. M. Yaglom, *Statistical hydrodinamics*, Part II, (in Russian) Nauka, Moscow, 1967;
14) G. K. Batchelor, *The theory of homogeneous turbulence*, Cambridge, Univ. Press, 1955;
15) V. S. Vladimirov, *Generalized functions of mathematical physics*, Nauka, Moscow, 1979.

List of symbols

$a(\lambda, t)$	a common factor in $b_k(\lambda, t) = a(\lambda, t)\, b_k(\lambda)$
$A(\lambda)$	the diffusion operator $\Sigma \partial_i z_i(t, \mathbf{r})\, z_j(t, \mathbf{r})\, \partial_j$
$b_k(\lambda)$, $b_k(\lambda, t)$	components of $\varphi_k(\mathbf{r})$ in the representation of $\mathbf{z}(\lambda, t, \mathbf{r})$
$C(t)$, $C(t, r)$	the mean concentration of substance, $C = \mathbf{E}(C)$
$C(\lambda, t)$, $C(\lambda, t, r)$	concentration with the probability $\varrho(\lambda)\, d\lambda$
$\delta(t)$, $\delta(r)$	the Dirac δ-function of t and r, respectively
$\mathbf{E}$	the mathematical expectation

$\mathbf{f}$, $\mathbf{f}(t, \mathbf{r})$	velocity fluctuation
$\varphi_k(\mathbf{r})$	a mode of deterministic velocity field
$g_k(t)$	stochastic processes
$\Theta(t)$	the function, $\Theta(t) = 1$ for $t > 0$ and zero otherwise
k	an index; the extinction constant of substance
q, $q(t, \mathbf{r})$	a distribution of input of substance
$\mathbf{r}$	a point in space
$\varrho(\lambda)$	probability density distribution
$\mathbf{v}$, $\mathbf{v}(t, \mathbf{r})$	a stochastic velocity field
$\mathbf{z}(\lambda)$, $\mathbf{z}(\lambda, t, \mathbf{r})$	a diffusion component of velocity field with the probability $\varrho(\lambda)\,d\lambda$
$\mathbf{w}$, $\mathbf{w}(\mathbf{r})$	the mean velocity field, $\mathbf{w} = E(\mathbf{v})$

NOVI PRISTUP PROBLEMU TRANSPORTA U FLUIDU

NEDŽAD LIMIĆ

Institut »Ruđer Bošković«, p. p. 1016, 41001 Zagreb

UDK 530.19

Originalni znanstveni rad

Praktički se do sada jedino difuziona jednadžba koristila kao model za opis transporta tvari u moru ili jezeru. Ozbiljni nedostaci toga modela su a) beskonačna brzina prostiranja substance od izvora kroz prostor i b) izostavljanje korelacija brzinske fluktuacije za razna vremena. U ovom je radu izvedena klasa modela transporta za koje je brzina prostiranja substance konačna. Osim toga je ugrađena potpuna kovariaciona funkcija u model. Nova klasa modela je izvedena iz zakona očuvanja mase u konzervativnom slučajnom polju brzina. U izvodima nema aproksimacije ili izostavljanja članova. Za potrebe primjene predložena su neka pojednostavljenja.