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ALGEBRAIC APPROACH TO A RELATIVISTIC SCALAR POTENTIAL

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and

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Algebraic approach has been applied to a linear scalar potential in the Dirac equation. The results are compared with the published numerical results of Rein.

1. Introduction

Recently it has been shown by several authors[1-3] that scaling variational methods when combined with algebraic perturbation theory provides a simple and elegant method for solving the eigenvalue equations in Quantum Mechanics. In this method Lie algebra of some dynamical group (for example SO (2, 1)) is used to extract the eigenvalue.

This method has been extensively used to find the eigenvalues of Schrödinger equation with some well known potentials including various screened Coulomb[3-6] potentials. But so far this method has been applied to Schrödinger equation only save Dirac's equation with pure Coulomb potential which is, in any case, exactly solvable.

However, as we shall see, for a spherically symmetric potential the dynamical group approach can be applied to find the eigenvalues of Dirac equation both for vector and scalar potentials. In this note, however, the method will be applied to Dirac equation with a scalar potential. Though our analytical result can be applied to any spherically symmetric potential for the sake of numerical check

we shall derive the eigenvalues of scalar linear potential for which numerical published results are available.

2. *Results and discussion*

The Dirac equation for a scalar potential $V(r)$ can be written in the following form

$$(W + m + U) F = \frac{dG}{dr} + \frac{\chi}{r} G \tag{1}$$

$$(-W + m + U) G = \frac{dF}{dr} - \frac{\chi}{r} F \tag{2}$$

where

$$\chi = l \text{ when } j = l - \frac{1}{2}$$

$$\chi = -l - 1 \text{ when } j = l + \frac{1}{2}$$

and we have used the Dirac spinors.

$$\psi_{js}^{M} = \begin{cases} \left(\frac{1}{r}\right) F(r)\, Y_{ls}^{M} & l = j + \frac{s}{2} \quad s = \pm 1 \\ \left(\frac{1}{r}\right) G(r)\, Y_{l'j}^{M} & l' = j - \frac{s}{2}. \end{cases} \tag{3}$$

Eliminating F and writing $W = E + m$ we get from (1) and (2)

$$\frac{d^2G}{dr^2} - \frac{\chi(\chi+1)}{r^2} G = (U^2 + 2mU - E^2 - 2mE) G + \\ + \frac{1}{2m + E + W} \frac{dU}{dr} \left(\frac{dG}{dr} + \frac{\chi}{r}\right) G. \tag{4}$$

Now putting $G(r) = \exp\left\{\frac{1}{2} \log (2m + E + U(r))\right\} r\varphi(r)$, equation (4) can be written as (using the fact that $\chi(\chi + 1) = l(l + 1)$ and $U(r) = \lambda r$)

$$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} - \frac{l(l+1)}{r^2} \varphi = \Bigg[\lambda^2 r^2 + 2m\lambda r - E^2 - 2Em - \\ - \frac{3}{4} \frac{\lambda^2}{(2m + E + r)^2} + \frac{\lambda}{(2m + E + r)} \frac{\chi}{r} \Bigg] \varphi(r) = 0. \tag{5}$$

It should be mentioned here that the number l being the degree of the ordinary spherical harmonics in terms of which spherical harmonics with spin are expressed is identical with the azimuthal quantum number of Schrödinger theory. Now the non-compact SO (2, 1) algebra is realized in terms of the radial vector r and the operator Δ defined by

$$\Delta = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+1)}{r^2}. \tag{6}$$

The SO (2, 1) generators are given by

$$T_1 = -\frac{1}{2}(r\Delta + r) \tag{7a}$$

$$T_2 = -\mathrm{i}\left(1 + r\frac{\mathrm{d}}{\mathrm{d}r}\right) \tag{7b}$$

$$T_3 = -\frac{1}{2}(r\Delta - r). \tag{7c}$$

T_1, T_2, T_3 satisfy the following commutation relations

$$[T_1, T_0] = -\mathrm{i}T_3 \tag{8a}$$

$$[T_0, T_3] = -\mathrm{i}T_1 \tag{8b}$$

$$[T_3, T_1] = \mathrm{i}T_2 \tag{8c}$$

and the Casimir invariant is

$$Q = T_3^2 - T_1^2 - T_2^2. \tag{9}$$

Introducing the step operator

$$T_\pm = T_1 \pm \mathrm{i}\, T_0 \tag{10}$$

and denoting by $|lmn\rangle$ the orthonormal basis states of the representation space the irreducible representation is given by

$$T_3 |lmn\rangle = n |lmn\rangle \tag{11}$$

$$T_\pm |lmn\rangle = (l + 1 \pm n)(n - 1) |lmn \pm 1\rangle \tag{12}$$

$$Q |lmn\rangle = (T_3^2 - T_1^2 - T_2^2) |lmn\rangle = l(l+1) |lmn\rangle \tag{13}$$

(n takes the values 1, 2, 3, ... while l takes the values 0, 1, ..., $N - 1$ and $-1 \leq m \leq 1$).

Eq. (5) can be written, after multiplying both sides by $(2m + E + U(r))^2$ in terms of the generators given in (7). However if we neglect terms of the order of $O\left(\frac{1}{2m}\right)^3$, then equation (5) simplifies a great deal and we get,

$$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} - \frac{\chi(\chi+1)}{r^2}\varphi(r) - \left[\lambda^2 r^2 + 2m\lambda r - e + \frac{\lambda\chi}{2mr}\right]\varphi(r) = 0 \quad (14)$$

where e, the reduced energy is given by

$$e = E^2 + 2Em - \frac{3\lambda^2}{16m^2}. \quad (15)$$

Now using the generators given in (7), (15) can be written as (after multiplying both sides by r)

$$\Omega\,|\alpha\rangle = 0 \quad (16)$$

where Ω is given by

$$\Omega = -2m(T_3 + T_1) + 2me(T_3 - T_1) - 4m^2\lambda(T_3 - T_1)^2 -$$

$$-(\chi + 1) - 2m\lambda(T_3 - T_1)^2. \quad (17)$$

Now let us perform the tilting[7] transformation given by

$$e^{-i\Theta T_2}\,\Omega(e)\,e^{i\Theta T_2}\,e^{-i\Theta T_2}\,|\psi\rangle = 0 \quad (18)$$

so that

$$\overline{\psi} = e^{-i\Theta T_2}\,|\psi\rangle \quad (19)$$

is the relation between the group state and the physical state. Taking

$$\overline{\Omega}(e, \Theta) = e^{-i\Theta T_2}\,\Omega(e)\,e^{i\Theta T_2} \quad (20)$$

equation (16) can be written as

$$\overline{\Omega}(e, \Theta)\,|\overline{\psi}\rangle = 0. \quad (21)$$

Under the transformation (20), T_1 and T_3 transform as follows

$$e^{-i\Theta T_2}\,T_3 e^{i\Theta T_2} = T_3 \cosh\Theta + T_1 \sinh\Theta \quad (22a)$$

$$e^{-i\Theta T_2}\,T_1 e^{i\Theta T_2} = T_1 \cosh\Theta + T_3 \sinh\Theta. \quad (22b)$$

Therefore $\bar{\Omega}(e, \Theta)$ is given by,

$$\bar{\Omega}(e, \Theta) = -2m(T_3 + T_1)\,e^{\Theta} + 2me(T_3 - T_1)\,e^{-\Theta} -$$

$$- 4m^2\lambda(T_3 - T_1)^2\,e^{-2\Theta} - 2m\lambda^2(T_3 - T_1)^3\,e^{-3\Theta} - (\chi + 1). \qquad (23)$$

We now use the group state

$$|\bar{\psi}\rangle = |lmn\rangle \qquad (24)$$

and write

$$\langle lmn|\,\bar{\Omega}(e, \Theta)\,|lmn\rangle = 0 \qquad (25)$$

to obtain the energy in terms of Θ. This gives,

$$-2mx - \lambda(\chi + 1)\,z + 2mexz^2 - 2m^2\lambda Az^3 - m^2\lambda Bz^4 = 0 \qquad (26)$$

where

$$x = n, \qquad y = l(l + 1), \qquad z = e^{-\Theta} \qquad (27)$$

and

$$A = 3x^2 - y, \qquad B = 5x^3 - 3xy + x. \qquad (28)$$

Equation (26) gives e as a function of Θ. As usual Θ is chosen as to make e a minimum i. e.,

$$\left.\frac{\partial e_{nl}(\Theta)}{\partial \Theta}\right|_{\Theta=\Theta_{nl}} = 0 \qquad (29)$$

and

$$\left.\frac{\partial^2 e_{nl}(\Theta)}{\partial \Theta^2}\right|_{\Theta=\Theta_{nl}} > 0. \qquad (30)$$

Eqs. (26) and (29) give

$$2m\lambda^2 Bz^4 + 2m^2\lambda Az^3 - \lambda(\chi + 1)\,z - 4mx = 0. \qquad (31)$$

Eq. (31) gives z and using this value of z in (26) we obtain e from which E can be obtained by using the relation (15). This method of treating Θ as a variational parameter is just the scaling variational method. This has the added advantage that the lowest approximation satisfies both the virial theorem and the Hellmann-Feynman theorem[8]. The correction to the zeroth order levels are obtained by treating the non-diagonal terms in (23) as perturbation terms and using a method[9] briefly described in the following. It is evident from equation (23) that all the matrix elements of $\bar{\Omega}(e, \Theta)$ are linear with respect to e. Therefore we can write

$$\langle n \pm i, l|\,\bar{\Omega}(e, \Theta)\,|n, l\rangle = a_{n\pm i,n} + b_{n\pm i,n}\,e$$

$$(n = 0, 1, 2, \ldots \quad i = 0, 1, 2, \ldots) \qquad (32)$$

where the coefficients a and b are easily calculated by using equations (11) to (14). Since $\bar{\Omega}(e, \Theta)$ leaves the l-value invariant, the n-th eigenstate $|a_{n,l}\rangle$ belonging to a particular l-value can be expanded in terms of the zeroth order eigenfunctions $|i, l\rangle$ which are the SO (2, 1) group states i. e.,

$$|\psi_{n,l}\rangle = \sum_{i=0}^{\infty} C_i \, |i, l\rangle. \tag{33}$$

Substituting (33) into (25) we thus obtain with the help of (32)

$$\sum_{j=0}^{\infty} \sum_{i=j-2}^{j+2} (a_{ji} + b_{ji} e_{n,l}) \, C_i \, |j, l\rangle = 0. \tag{34}$$

Since the states $|j, l\rangle$ ($j = 0, 1, 2, \ldots$) constitute a basis each coefficient of $|j, l\rangle$ in (34) should vanish separately. For $j = n$ the solution of the resulting secular equation with respect to $e_{n,l}$ yields

$$e_{n,l} = -\frac{a_{nn} C_n + \sum_{i \neq n} a_{ni} C_i}{b_{nn} C_n + \sum_{i \neq n} b_{ni} C_i} \tag{35}$$

and for $j \neq n$, the secular equation can be solved with respect to C_j yielding

$$C_j = -\frac{\sum_{i \neq j} a_{ji} C_i + \sum_{i \neq j} b_{ji} C_i e_{n,l}}{a_{jj} + b_{jj} e_{n,l}}. \tag{36}$$

Indeed, denoting by $e_{n,l}^{(k)}$ and $C_j^{(k)}$, the k-th order approximation of $e_{n,l}$ and C_j, respectively, the following Gauss-Seidel iterative scheme can be prescribed as follows

$$e_{n,l}^{(k)} = -\frac{a_{nn} + \sum_{i \neq n} a_{ni} C_i^{(k)}}{b_{nn} + \sum_{i \neq n} b_{ni} C_i^{(k)}} \quad (k \geq 0) \tag{37}$$

$$C_j^{(k)} = -\Big(\sum_{i<j} a_{ji} \, C_i^{(k)} + \sum_{i<j} b_{ji} \, C_i^{(k)} \, e_{n,l}^{(k-1)} + \sum_{i>j} a_{ji} \, C_i^{(k-1)} +$$

$$+ \sum_{i>j} b_{ji} \, C_i^{(k-1)} \, e_{n,l}^{(k-1)}\Big) \times (a_{jj} + b_{jj} \, e_{n,l}^{(k-1)})^{-1} \quad (j \neq n) \qquad (k \leq 1) \tag{38}$$

$$C_n^{(k)} = 1 \qquad (k \geq 1) \tag{39}$$

and the initial C-values are given by $C_i^{(0)} = \delta_{in}$.

For any Θ value for which the scheme (36)—(38) is convergent, we must find that $\lim_{k \to \infty} e_{n,l}^{(k)} = e_{n,l}$. But the rate of convergence is effectively Θ dependent. In Table 1, we compare our results with those of Rein et al.[10].

TABLE 1.

$m/\sqrt{\lambda}$	$W_0/\sqrt{\lambda}$	$W^p/\sqrt{\lambda}$	$W_1/\sqrt{\lambda}$	$W_2/\sqrt{\lambda}$
1	2.42	2.94	3.21	3.78
	(2.40)	(3.00)	(3.32)	(3.99)
4	5.14	5.56	5.78	6.28
	(5.08)	(5.58)	(5.85)	(6.44)
6	7.04	7.42	7.64	8.08
	(6.98)	(7.40)	(7.67)	(8.22)

* without second order correction.

Energy eigenvalues (normalized by $\sqrt{\lambda}$) of the ground state and two excited s states as well as the first p state.

The values given in parentheses are those given by Rein[10], W is the total energy $W = E + m$.

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Primijenjen je algebarski pristup na linearni skalarni potencijal u Diracovoj jednadžbi. Rezultati su uspoređeni s objavljenim numeričkim rezultatima Reina.