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LETTER TO THE EDITOR

ON GRAVITATIONAL AND NON-GRAVITATIONAL ACCELERATION

JANEZ STRAND

Department of Physics and J. Stefan Institute, University of Ljubljana, Slovenia

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The unidirectional motion of a particle in a static homogeneous gravitational field is studied together with the motion of the source in the proper frame of the particle. Then the nongravitational counterpart, e.g. a charged particle in a static homogeneous electric field, is considered. The comparison of both types of results establishes a formal symmetry which may be useful in the transition from special to general relativity.

A static homogeneous gravitational field for unidirectional motion is defined as a flat two-dimensional space-time which in the weak field approximation reduces to the corresponding Newtonian case. It is characterized by the parameter $\alpha = m_G g/m_I$ where m_G is the gravitational mass, m_I the inertial mass and g the Newtonian gravitational acceleration in the assumed homogeneous gravitational field. We measure time in units of c/α and distance in units of c^2/α. The space-time element in the non-inertial reference frame K_G, supported in this field, is

$$d\tau^2 = g_{00}(x)\, dt^2 + g_{11}(x)\, dx^2 \tag{1}$$

with g_{oo} in the first order being equal to the Newtonian limit

$$g_{00} = 1 - 2x + \dots \tag{2}$$

(With the usual units the corresponding equation reads $g_{00} = 1 - 2V/c^2 + \dots$ with the gravitational potential V.) The coordinate x, in general, is not equal to the distance from the origin.

Demanding that the Riemann curvature tensor vanishes the relation

$$-\frac{\mathrm{d}g_{00}^{1/2}}{\mathrm{d}x} = (-g_{11})^{1/2} \tag{3}$$

is obtained[1,2]. The condition (2) was taken into account and the sign of g_{11} chosen according to experience with inertial reference frames. The function $u(x)$,

$$g_{00} = u^2, \quad g_{11} = -\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2, \tag{4}$$

being arbitrary up to the condition (2) in the first order, determines the choice of coordinates. With it the space-time element (1) reads

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x^2 - u^2\,\mathrm{d}t^2 = \mathrm{d}r^2 - (1-r)^2\,\mathrm{d}t^2 \tag{5}$$

where we have introduced the distance r from the origin:

$$\mathrm{d}r = -\,\mathrm{d}u, \quad r = 1 - u. \tag{6}$$

We either chose the function $u = 1 - x$, i.e. $x = r$, or make any other choice of $u(x)$ but use the distance $r = \int_0^x (-g_{11})^{1/2}\,\mathrm{d}x$.

It should be noted that curved space-time is occasionally used to describe the static homogeneous gravitational field[3], this being consistent with the Einstein field equations supplemented by the cosmological term[4], whereas our description supposes a zero cosmological constant.

The worldline of a freely falling particle P_G is a geodesic and the corresponding equation of motion reads

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = \frac{1}{(1-r)^3}. \tag{7}$$

Its solution for the case that the particle P_G is released from the source S_G stationary at the origin at $\tau = t = 0$ is

$$\tau = \tanh t, \quad r = 1 - \frac{1}{\cosh t} \tag{8a}$$

and

$$(1 - r)^2 + \tanh^2 t = 1. \tag{8b}$$

S_G should be envisaged as a material releasing mechanism, at rest in K_G abstracted to a point.

The coordinate transformation[1,5,6)]

$$T = (1 - x) \sinh t, \quad 1 - X = (1 - r) \cosh t \tag{9}$$

leads from the reference frame $K_G(t, r)$ supported in the static homogeneous gravitational field to an inertial reference frame $K(T, X)$:

$$d\tau^2 = (1 - r)^2 \, dt^2 - dr^2 = dT^2 - dX^2. \tag{10}$$

Here X is the distance from the origin. Eq.(8a) inserted into the transformation (9) gives $T = \tau$ and $X = 0$, so K is the proper frame of particle P_G. In this inertial frame the motion of the source S_G is obtained by inserting $r = 0$ into (9). The equations

$$T = \sin t, \quad 1 - X = \cosh t \tag{11a}$$

and

$$(1 - X)^2 - T^2 = 1 \tag{11b}$$

show that the source S_G in K is moving hyperbolically.

Now we turn to the non-gravitational counterpart, e.g. a particle P_N with mass m_I and charge q in a homogeneous electric field $\mathscr{E}$. The motion is characterized by the parameter $\alpha' = q\mathscr{E}/m_I$ which should have the same value as α in the gravitational case. We use the same dimensionless units as in the gravitational case. α' in this case represents the proper acceleration, i.e. the acceleration of the particle relative to its instantaneous rest frame. The motion with a constant proper acceleration is not necessarily linked with a charged particle in a homogeneous electric field but can be due, e.g. to the action of rockets on a space-ship. Thus, in the inertial reference frame K of special relativity with $d\tau^2 = dT^2 - dX^2$ the equation of motion reads

$$\frac{d^2X}{d\tau^2} = X + 1. \tag{12}$$

Its solution for the case that the particle P_N is released from the source S_N at the origin $\tau = t = 0$ is the hyperbolic motion

$$T = \sinh \tau, \quad X = \cosh \tau - 1 \tag{13a}$$

and

$$(X + 1)^2 - T^2 = 1. \tag{13b}$$

Again, S_N is envisaged as a material point, e.g. the tip of a cathode, at rest in K.

If the solution (13a) is inserted into the transformation (9) with X replaced by $-X$ we get $r = 0$. Therewith the non-gravitationally accelerated reference frame $K_N(t, r)$ is introduced as the proper frame of the particle P_N. Inserting $X \doteq 0$ into (9) we get for motion of the source S_N

$$T = \tanh t, \quad r = 1 - \frac{1}{\cosh t} \tag{14a}$$

and

$$(1 - r)^2 + \tanh^2 t = 1. \tag{14b}$$

The motion of the particle P_G in the reference frame K_G supported in the static homogeneous gravitational field (8) corresponds to the motion of the source S_N in the proper frame K_N of the non-gravitationally accelerated particle P_N (14). Likewise the motion of the source S_G in the freely falling inertial reference frame K (11) corresponds to the motion of the charged particle P_N in the inertial frame K at rest with respect to the sources of the static homogeneous electric field (13). Therewith we have established a characteristic skew symmetry (Table 1). In both non-inertial frames K_G and K_N the symmetry

Reference frame	gravitational field	electric field
non-inertial	falling particle P_G	source S_N
inertial	source S_G	accelerated particle P_N

Table 1: The skew symmetry in describing the motion of a particle in a static homogeneous gravitational field and in the static homogeneous electric field and of the sources in both rest frames of particles.

concerns the distance r. The description of motion in these frames by way of coordinates depends on the choice of the function $u(x)$. The symmetry is upheld with respect to the coordinates if in K_G and in K_N the same function is chosen. For the choice $u = 1/\cosh x$ in all four frames the motion is hyperbolic*.

In fact, a two-dimensional space-time can incorporate neither mass nor a gravitational field[8]. Thus, the static homogeneous gravitational field is a model that can be realized approximately only, e.g. in a Schwarzschild space-time in radial direction far away of the central body[5] or in a region immediately above the center of a large disk-shaped galaxy[1] or in a cavity completely surrounded by a continuous distribution of mass[9].

* The symmetry of motion in considered not the symmetry of the fields; e. g. a charged source S_G does not emit electromagnetic radiation in K_G but does so in K[1,7] whereas a charged source S_N does not emit in K but does so in K_N.

We have "derived" the principle of equivalence in the original form (a reference frame at rest in a gravitational field is physically equivalent to an accelerated reference frame in a space free of gravitation)[10] as well as in a more contemporary form (in a freely falling local Lorentz frame all non-gravitational laws of physics take their special-relativistic form)[11] for a static homogeneous gravitational field, i.e. for flat space-time. (If α should equal g the gravitational mass equal the inertial mass[9]: $m_G = m_I$.) Generalizing it to curved space-time both frames can be distinguished by measuring tidal forces[9,12]. Whether this measurement is considered local or not depends on convention.

Finally, we have to admit that the envisaged symmetry is a formal one, restricted to the equations quoted, and not a symmetry of interactions, since e.g. electromagnetism (a spin-1 field) violates the principle of equivalence. Furthermore this symmetry is not of much interest in the theory of gravitation since flat space-time, according to a contemporary definition[13], belongs to the realm of special relativity. Nevertheless, the symmetry may be instructive at the intermediate stage, on the way from special to general relativity.

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O GRAVITACIJSKOM I NEGRAVITACIJSKOM UBRZAVANJU

JANEZ STRNAD

Oddelek za fiziko in Institut J. Stefan, Univerza v Ljubljani, Ljubljana, Slovenija

UDK 531.5

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Istraživano je kretanje materijalne točke u statičkom homogenom gravitacijskom polju i kretanje izvora u vlastitom promatračkom sistemu te točke kao i kretanje naelektrizirane materijalne točke u statičkom homogenom električkom polju i kretanje odgovarajućeg izvora kao primjer negravitacijskog ubrzavanja. Uspoređujući rezultate primijetimo simetriju koja može biti poučna pri prijelazu od specijalne ka općoj teoriji relativnosti.