ISSN 1330–0016
CODEN FIZBE7

MESON FORM FACTORS IN THE QUARK-BASED LINEAR σ MODEL

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Received 24 August 2003; Accepted 26 April 2004
Online 16 July 2004

The quark-level linear σ model (LσM) is employed to compute a variety of electromagnetic and weak observables of light mesons, including pion and kaon form factors and charge radii, charged-pion polarizabilities, semileptonic weak $K_{\ell 3}$ decay, semileptonic weak radiative pion and kaon form factors, radiative decays of vector mesons, and nonleptonic weak $K_{2\pi}$ decay. The agreement of all these predicted observables with experiment is striking. In passing, the tight link between the LσM and vector-meson dominance is shown. Some conclusions are drawn on the LσM in connection with lattice and renormalization-group approaches to QCD.

PACS numbers: 12.40.-y, 11.30.Rd, 13.40.Gp, 13.40.Hq, 13.20.Cz, 13.20.Eb, 13.25.Es, 12.40.Vv
UDC 539.12.1

Keywords: linear sigma model, quarks, mesons, pion, kaon, form factors, charge radii, charged-pion polarizabilities, weak decay

1. L$\boldsymbol{\sigma}$M and chiral Goldberger-Treiman relations

The NJL [1] and $Z = 0$ compositeness [2] relations $m_\sigma = 2\hat{m}$ and $g = 2\pi/\sqrt{N_c}$, with $\hat{m} \sim M_N/3$ and meson-quark coupling $g = 2\pi/\sqrt{3} = 3.628$, follow by nonperturbatively solving [3] the strong-interaction Nambu-type gap equations $\delta f_\pi = f_\pi$ and $\delta\hat{m} = m$ (where f_π is the pion decay constant and $\hat{m}$ is the nonstrange con-

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stituent quark mass) in quark-loop order, regularization schemes. For a more detailed description of the quark-level LσM, see Ref. [4], or the Appendices of Ref. [5]. Here, we collect instead powerful results of the LσM on meson form factors and related data for strong, e.m., and weak interactions.

This chiral LσM is based on the quark-level pion and kaon Goldberger–Treiman relations (GTRs) $f_\pi g = \hat{m} = (m_u + m_d)/2$ and $f_{\rm K} g = (m_s + \hat{m})/2$, for $f_\pi \approx 93$ MeV ($f_\pi \approx 90$ MeV in the chiral limit (CL) [6]), $f_{\rm K}/f_\pi \approx 1.22$, and $m_s \approx 1.44\,\hat{m}$. We begin in Sec. 2 by studying meson vector form factors and their measured charge radii. In Sec. 3 we survey charged-pion polarizabilities for $\gamma\gamma \to \pi\pi$, and compare the results with the LσM predictions. In Sec. 4 we study the semileptonic weak K_{l3} decays and the form factor $f_+(k^2)$ evaluated at $k^2 = 0$. Then in Sec. 5 we examine the radiative semileptonic weak form factors for $\pi^+ \to \mathrm{e}^+\nu\gamma$ and $\mathrm{K}^+ \to \mathrm{e}^+\nu\gamma$ decays, with the observed pion second axial-vector form factor implying a pion charge radius $r_\pi \sim 0.6$ fm, also found in Sec. 2 from data [7] and from the theoretical LσM. In Sec. 6 we return to the LσM and its link with vector-meson dominance (VMD). Finally, in Sec. 7 we begin by studying the $\Delta I = 1/2$ rule for two-pion decays of the kaon in connection with the σ as the pion's chiral partner, and end by showing that the mass of the now experimentally confirmed scalar κ meson is consistent with the observed K$\to 2\pi$ decay rate. We summarize our results and draw our conclusions in Sec. 8.

2. *Meson vector form factors and charge radii*

The charged-pion and kaon e.m. vector currents are defined as

$$\begin{aligned} \langle \pi^+(q')|V^\mu_{em}(0)|\pi^+(q)\rangle &= F_\pi(k^2)\,(q'+q)^\mu\,, \\ \langle K^+(q')|V^\mu_{\rm em}(0)|K^+(q)\rangle &= F_{\rm K}(k^2)\,(q'+q)^\mu\,, \end{aligned} \tag{1}$$

with $k_\mu = q'_\mu - q_\mu$. The former pion form factor $F_\pi(k^2)$ can be nonperturbatively characterized, through the LσM bootstrap, by the (constituent) quark udu and dud loop graphs of Fig. 1a, while the charged-kaon form factor $F_{\rm K}(k^2)$ is in a similar manner determined by the usu and sus loop graphs depicted in Fig. 1b. Even if each of the diagrams in Fig. 1 appears to be linearly divergent by naive power counting, gauge invariance enforces every single quark triangle (QT) to be merely logarithmically divergent. For details of the evaluation of these QT diagrams, see Ref. [5].

The QT expressions [5] in the CL (i.e. $M \to 0$) should be compared to the CL non-perturbative LσM [3, 8] or NJL [9] results

$$F_\pi(k^2)^{\rm CL}_{\rm L\sigma M} = -4{\rm i}g^2 N_c \int_0^1 {\rm d}x \int {\rm đ}^4p \left[p^2 - \hat{m}^2 + x(1-x)k^2\right]^{-2}, \tag{2}$$

$$F_{\rm K}(k^2)^{\rm CL}_{\rm L\sigma M} = -4{\rm i}g^2 N_c \int_0^1 {\rm d}x \int {\rm đ}^4p \left[p^2 - m^2_{us} + x(1-x)k^2\right]^{-2}, \tag{3}$$

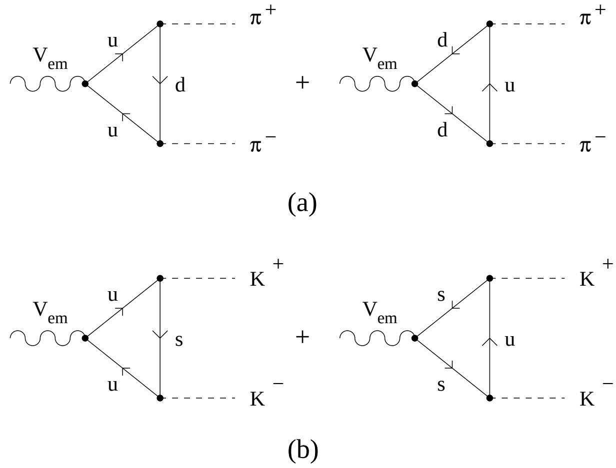

Fig. 1. VPP quark triangle graphs.

where $m_{us} = (m_s + \hat{m})/2$ and $đ^4p = \mathrm{d}^4p\ (2\pi)^{-4}$. The logarithmic divergence of these expressions has been guaranteed through a rerouting procedure [8, 10]. When $k^2 = 0$, these form factors become automatically normalized to unity, due to logarithmically divergent gap equations (LDGEs) [10, 11, 5], the GTRs in Sec. 1, and the definition of the pion and kaon decay constants $\langle 0|A_3^\mu|\pi^0\rangle = \mathrm{i}f_\pi q^\mu$, $\langle 0|A^\mu_{4-i5}|K^+\rangle = \mathrm{i}\sqrt{2}f_K q^\mu$, with $f_\pi \approx 93$ MeV and $f_K/f_\pi \approx 1.22$ [3, 10].

To proceed, given Eqs. (2) and (3), the π^+ and, as an obvious $SU(3)$ extension, K^+ charge radii are computed in the LσM as

$$\langle r^2_{\pi^+}\rangle^{\mathrm{CL}}_{\mathrm{L}\sigma\mathrm{M}} = 6\ \frac{\mathrm{d}F_\pi(k^2)}{\mathrm{d}k^2}\bigg|_{k^2=0} = \frac{8\mathrm{i}N_c}{(2\pi)^4}\,g^2\left(\frac{-\mathrm{i}\pi^2}{2\hat{m}^2}\right) = \frac{N_c}{4\pi^2 f_\pi^2} \approx (0.61\ \mathrm{fm})^2\ , \tag{4}$$

$$\langle r^2_{\mathrm{K}^+}\rangle^{\mathrm{CL}}_{\mathrm{L}\sigma\mathrm{M}} = 6\ \frac{dF_{\mathrm{K}}(k^2)}{dk^2}\bigg|_{k^2=0} = \frac{8\mathrm{i}N_c}{(2\pi)^4}\,g^2\left(\frac{-\mathrm{i}\pi^2}{2m^2_{us}}\right) = \frac{N_c}{4\pi^2 f_{\mathrm{K}}^2} \approx (0.49\ \mathrm{fm})^2\ . \tag{5}$$

Here we have evaluated the charge radii in the CL [3, 6, 12], with $f_\pi^{\mathrm{CL}} \approx 90$ MeV, $f_{\mathrm{K}}^{\mathrm{CL}} \approx 110$ MeV. If, on the other hand, we compute these charge radii using the detailed QT graphs in the CL, we get for the π^+ exactly the same result as in Eq. (4), while for the K^+ a complicated function of $\hat{m}$ and m_s is obtained [5, 13]. We may then expand the latter function in terms of the $SU(3)$-breaking parameter $\delta = (m_s/\hat{m}) - 1 \approx 0.44$, resulting in an expression previously derived in Ref. [14]. Taking into account the first three terms of this expansion, we may estimate the ratio r_{K}/r_π to be $\langle r^2_{\mathrm{K}^+}\rangle / \langle r^2_{\pi^+}\rangle \approx 1 - 5\delta/6 + 3\delta^2/5 \approx 0.750$, or $\langle r_{\mathrm{K}^+}\rangle / \langle r_{\pi^+}\rangle \approx 0.866$. Here we note that the observed pion charge radius is [7] $r_\pi = (0.642 \pm 0.002)$ fm, and the analogue charged-kaon charge radius is [15] $r_{\mathrm{K}} = (0.560 \pm 0.031)$ fm. If we take the experimental value $r_{\pi^+} \approx 0.64$ fm, the latter ratio implies $\langle r_{\mathrm{K}^+}\rangle \approx 0.556$ fm, which is in agreement with experiment. On the other hand, if we were to take the full, unexpanded expression for $\langle r^2_{\mathrm{K}^+}\rangle$ [5], we would get $\langle r_{\mathrm{K}^+}\rangle \approx 0.545$ fm, being still compatible with experiment.

Finally, note that the above field-theory versions of the charged-pion form factor can be recovered in an even simpler fashion by using a once-subtracted dispersion relation for the pion charge radius, yielding in the CL [3] $r_\pi^2 = N_c/4\pi^2(f_\pi^{\rm CL})^2 = 1/\hat{m}^2$, where we use the GTRs of Sec. 1, along with $g = 2\pi/\sqrt{N_c}$ from Sec. 1. This suggests that the tightly bound "fused" $\bar{q}q$ pion charge radius in the CL is $r_\pi^{\rm CL} = 1/\hat{m} = (197.3\ {\rm MeV\,fm})/(325\ {\rm MeV}) \approx 0.61$ fm, with $\hat{m}_{\rm CL} \approx 325$ MeV $\sim M_{\rm N}/3$, as expected from the GTR $m_{\rm CL} = f_\pi^{\rm CL} g \approx 90\ {\rm MeV} \times 3.628 \approx 325$ MeV.

3. *Charged-pion polarizabilities for $\gamma\gamma \to \pi\pi$*

Analyzing Crystal-Ball [16], MARK-II [17] and CELLO [18] $\gamma\gamma \to \pi\pi$ data, Kaloshin et al. [19−21] obtain for the charged electric pion polarizability $\alpha_{\pi^+} = (2.75 \pm 0.50) \times 10^{-4}$ fm^3. The LσM prediction follows from the model-independent value [22], given by $\alpha_{\pi^+} = (\alpha/8\pi^2 m_\pi f_\pi^2)\gamma$, where $\gamma \equiv F_{\rm A}(0)/F_{\rm V}(0)$. In Sec. 5 we find $\gamma = 2/3$. Hence, the LσM result $\alpha_{\pi^+}^{{\rm L}\sigma{\rm M}} = \alpha/12\pi^2 m_\pi f_\pi^2 \approx 3.9 \times 10^{-4}$ fm^3 is reasonably near the data above. Moreover, if we extend the LσM to $SU(3)$, the prediction for γ becomes $\gamma_{SU(3)}^{{\rm L}\sigma{\rm M}} \approx 0.58$ (see Sec. 5), yielding $\alpha_{\pi^+}^{{\rm L}\sigma{\rm M}} \approx 3.4 \times 10^{-4}$ fm^3, which is closer to the experimental value. Besides, the latter two LσM values for α_{π^+} are also compatible with, e.g., the prediction 3.6×10^{-4} fm^3 of a quark confinement model that gives good results for heavy-meson semileptonic form factors [23], too.

Another consistency check is the detailed quark-plus-meson-loop analysis $\alpha_{\pi^+}^{{\rm L}\sigma{\rm M}} = \alpha/8\pi^2 m_\pi f_\pi^2 - \alpha/24\pi^2 m_\pi f_\pi^2 = \alpha/12\pi^2 m_\pi f_\pi^2$ [24], requiring $\gamma^{{\rm L}\sigma{\rm M}} = 2/3$ from the model-independent value of α_{π^+} above.

Finally we comment on low-energy $\gamma\gamma \to 2\pi^0$ scattering, where there is no pole term, and the neutral polarizabilities α_{π^0}, β_{π^0} are much smaller than α_{π^+}, β_{π^+}. In Ref. [25] it was shown that a $\gamma\gamma \to 2\pi^0$ cross section of ~ 10 nb (generated by a $\sigma(700)$ meson pole) reasonably anticipated the later 1990 Crystal-Ball data [16] in the 0.3−0.7 GeV range.

4. *Semileptonic weak $\mathrm{K}_{\ell 3}$ decay and form-factor scale $f_+(0)$*

The semileptonic weak $\mathrm{K}^+ \to \pi^0 \mathrm{e}^+ \nu$ ($\mathrm{K}_{\ell 3}$) decay width is measured as [15]

$$\Gamma_{\mathrm{K}^+ \to \pi^0 \mathrm{e}^+ \nu} = \frac{\hbar}{\tau_{\mathrm{K}^+}} (4.87 \pm 0.06)\% = (25.88 \pm 0.32) \times 10^{-16}\ \mathrm{MeV}\,. \qquad (6)$$

Taking a q^2 form-factor dependence $f_+(q^2) = f_+(0)[1 + \lambda_+ q^2/m_\pi^2]$, the standard V−A (vector here) weak current predicts a $\mathrm{K}_{\ell 3}$ decay width ($y = m_{\pi^0}^2/m_{\mathrm{K}^+}^2$, $m_{\rm e} = m_\nu = 0$; see also Ref. [26])

$$\Gamma_{\mathrm{K}^+ \to \pi^0 \mathrm{e}^+ \nu} = \frac{G_{\rm F}^2\, |V_{us}|^2\, m_{\mathrm{K}^+}^5}{2\,\pi^3\, 768}\, f_+^2(0) \Big(0.5792 + 0.1600\, \frac{m_{\mathrm{K}^+}^2}{m_{\pi^0}^2} \lambda_+ + 0.01770\, \frac{m_{\mathrm{K}^+}^4}{m_{\pi^0}^4} \lambda_+^2 \Big)$$

$$= f_+^2(0)\ (25.90 \pm 0.07) \times 10^{-16}\ \text{MeV}\ , \tag{7}$$

where $G_F = 11.6639 \times 10^{-6}$ GeV^{-2}, $V_{us} = 0.2196 \pm 0.0026$, and $\lambda_+ = 0.0278 \pm 0.0019$ [15]. If we neglect here the term quadratic in λ_+, as e.g. done in Ref. [26], the leading factor in Eq. (7) becomes 25.80 instead of 25.90. Moreover, accounting for a nonvanishing electron mass yields a totally negligible correction of the order of 0.001%. In any case, comparison with the data in Eq. (6) clearly shows that the form-factor scale $f_+(0)$ must be near unity. However, electroweak radiative corrections to $\Gamma(K^+ \to \pi^0 e^+ \nu)$ are *not* negligible on the scale of the experimental errors in V_{us} and λ_+, giving rise to an enhancement of $|V_{us}|$ by more than 2% [27], suggesting that $f_+(0)$ should be a trifle less than unity.

As a matter of fact, the nonrenormalization theorem [28] *requires* the form factor $f_+(q^2)$ to be close to unity when $q^2 = 0$. Furthermore, in the infinite-momentum frame (IMF), tadpole graphs are suppressed and so [29] $1 - f_+^2(0) = \mathcal{O}(\delta^2) \approx 6\%$ is second order in $SU(3)$-symmetry breaking. Of similar order are, for example, $(m_\pi/m_K)^2 = 7.7\%$, and $(1 - f_K/f_\pi)^2 = 5\%$, for $f_K/f_\pi = 1.22$.

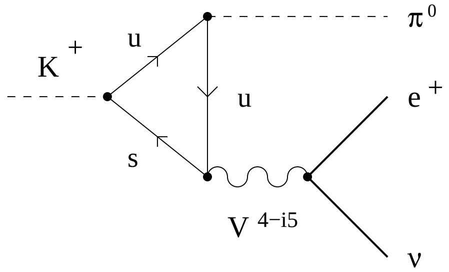

Fig. 2. Quark-loop contribution to $K^+ \to \pi^0 e^+ \nu$.

Next we follow the (constituent) quark-model triangle graph of Fig. 2, with

$$\sqrt{2}\,\langle \pi^0 | V_\mu^{4-i5} | K^+ \rangle \;=\; f_+(t)(p_K + p_\pi)_\mu \;+\; f_-(t)(p_K - p_\pi)_\mu\ . \tag{8}$$

Note that, for this process, the f_- form factor can be disposed of, since it is weighted by $m_e \ll m_K$ [26], giving rise to a m_e^2/m_K^2 suppression of the corresponding contributions to $\Gamma(K^+ \to \pi^0 e^+ \nu)$. To test $SU(2)$-symmetry breaking in $K_{\ell 3}$ decays as in Eqs. (7) and (8) above, we note the present data consistency [15] of $\lambda_+(K^+_{e3}) = 0.0278 \pm 0.0019$, $\lambda_+(K^0_{e3}) = 0.0291 \pm 0.0018$, $\lambda_+(K^+_{\mu 3}) = 0.033 \pm 0.010$, and $\lambda_+(K^0_{\mu 3}) = 0.033 \pm 0.005$. Then, expanding in the $SU(3)$-breaking parameter $\delta = (m_s/\hat{m}) - 1$ (as already used in Sec. 2) and working in the soft-pion CL, the Feynman graph of Fig. 2 predicts [30] (recall the value of the meson-quark coupling $g \approx 3.628$ in Sec. 1) $f_+(0) = 1 - g^2\delta^2/8\pi^2 \approx 0.968$. This value slightly below unity is not only in agreement with the nonrenormalization theorem above, as $1 - f_+^2(0) = 1 - (0.968)^2 = 6.3\%$, but also quantitatively compatible with Eqs. (6) and (7), if we account for the mentioned radiative corrections contributing with about -1.5% to $f_+(0)$, and the experimental errors in V_{us} and λ_+. Here, we should add that using unitarity of the CKM matrix and the present experimental value for $|V_{ud}|$ [15] would result in an about 4% larger value for $|V_{us}|$ (see Ref. [27], second paper), thus calling in question the small error bars in $|V_{us}|$, and lending extra support to our conclusion that $f_+(0)$ is somewhat *less* than unity.

5. *Semileptonic weak radiative form factors for* $\pi^+ \to e^+\nu\gamma$ *and* $K^+ \to e^+\nu\gamma$

From Ref. [15], the $\pi^+ \to e^+\nu\gamma$ and $K^+ \to e^+\nu\gamma$ matrix elements are

$$M_V = \frac{-e\,G_F\,V_{qq'}}{\sqrt{2}\,m_P}\,\epsilon^\mu \ell^\nu F_V^P\,\epsilon_{\mu\nu\sigma\tau}\,k^\sigma q^\tau\ , \tag{9}$$

$$M_A = \frac{-ie\,G_F\,V_{qq'}}{\sqrt{2}\,m_P}\,\epsilon^\mu \ell^\nu \{F_A^P\,[(s-t)g_{\mu\nu} - q_\mu k_\nu] + R^P t\,g_{\mu\nu}\}\ , \tag{10}$$

where $V_{qq'}$ is the corresponding Cabibbo-Kobayashi-Maskawa (CKM) mixing-matrix element, ϵ^μ is the photon polarization vector, ℓ^ν is the lepton-neutrino current, q and k are the meson and photon four-momenta, respectively, with $s = q\cdot k$, $t = k^2$, and P stands for π or K. The weak vector (pion) form factor F_V^π in Eq. (9) and the second axial vector form factor R^π in Eq. (10) are model independent [31], with F_V^π determined only by conserved vector currents (CVC), and R^π related via the pion charge radius ($r_\pi = 0.642 \pm 0.002$ fm) to partially conserved (pion) axial currents (PCAC). Specifically, F_V^π was long ago determined by CVC [31], viz. $F_V^\pi(0) = \sqrt{2}m_{\pi^+}/8\pi^2 f_\pi \approx 0.027$, reasonably close to data [15] 0.017 ± 0.008. Furthermore, PCAC predicts (PCAC is manifest in the LσM [32, 33]) $R^\pi = m_{\pi^+} f_{\pi^+} r_{\pi^+}^2/3 = 0.064 \pm 0.001$, where $f_{\pi^+} = 130.7 \pm 0.1$ MeV [15] and we use $r_{\pi^+} = 0.642 \pm 0.002$ fm. The latter prediction is near data [34] $R^\pi = 0.059^{+0.009}_{-0.008}$.

To apply the LσM theory, we consider the quark-plus-meson-loop graphs of Fig. 3. Then the ratio $\gamma = F_A(0)/F_V(0)$ is predicted as [35] $\gamma^{L\sigma M} = 1 - 1/3 = 2/3$, with $F_A^\pi(0) = \sqrt{2}m_\pi[(8\pi^2 f_\pi)^{-1} - (24\pi^2 f_\pi)^{-1}] = \sqrt{2}m_\pi/12\pi^2 f_\pi \approx 0.0179$. So this form-factor ratio divided by the CVC value of $F_V^\pi(0)$ above gives $\gamma^{L\sigma M} = 0.0179/0.027 \approx 0.66$, compatible with 2/3 and with data [15]: $\gamma_{\rm data} = (0.0116 \pm 0.0016)/(0.017 \pm 0.008) = 0.68 \pm 0.33$. With hindsight, this ratio $\gamma^{L\sigma M} = 2/3$ is near the original current-algebra (CA) estimate 0.6 found in Ref. [36], and exactly the same γ found in Sec. 3 from the LσM.

Extending the above LσM picture to $SU(3)$ symmetry, we first assume a scalar nonet pattern below 1 GeV (e.g., $f_0(600)$ (σ), $\kappa(800)$, $f_0(980)$, $a_0(980)$) as found from a kinematic IMF scheme [37], or from a dynamical coupled-channel unitarized model [38]. Then, inclusion of $a_0 a_0 \eta_{NS}$ ("NS" means nonstrange), KKκ, and $\kappa\kappa$K meson-loop graphs, besides the $\pi\pi\sigma$ graph of Fig. 3, lowers the prediction for γ to $\gamma^{L\sigma M}_{SU(3)} \approx 0.58$ [39], which is still in accordance with data and the CA estimate.

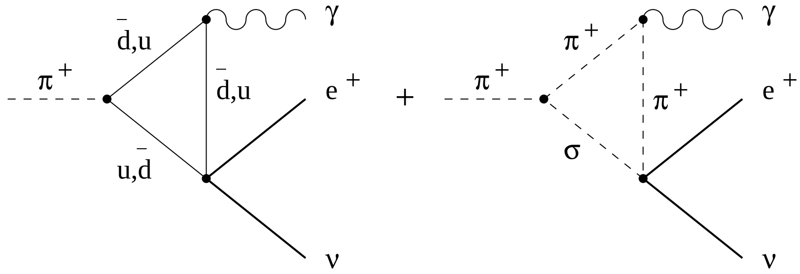

Fig. 3. Quark- and meson-loop contribution to $\pi^+ \to \gamma e^+\nu$.

Coming now to the $K^+ \to e^+\nu\gamma$ form factor, an $SU(3)$ LσM quark-plus-meson-loop analysis, involving uus, $\pi\pi\kappa$, a_0a_0K, KK$f_0(980)$, $\kappa\kappa\eta_S$ ("S" stands for strange), KKa_0, $\kappa\kappa\pi$, KKσ, and $\kappa\kappa\eta_{NS}$ loops, predicts [39], at $k^2 = 0$, $|F_V^K(0) + F_A^K(0)|_{L\sigma M} \approx 0.109 + 0.044 = 0.153$, close to the $K^+ \to e^+\nu\gamma$ data [15] $|F_V^K(0) + F_A^K(0)|_{data} = 0.148 \pm 0.010$. An $SU(3)$ LσM theory is reasonably detailed [11] due to resonances below 1 GeV, but the LσM kaon form-factor sum is easily tested via the data. The same is true for the pion form-factor values above, partly based on the measured pion charge radius [7] $r_\pi = 0.642 \pm 0.002$ fm.

6. VMD: LσM via VPP and VPV or PVV loops

We first confirm the (crucial) value of the pion charge radius [7] $r_\pi = 0.642 \pm 0.002$ fm via Sakurai's vector-meson-dominance (VMD) prediction [40] $r_\pi = \sqrt{6}/m_\rho \approx 0.63$ fm. Recall that the tightly bound $\bar{q}q$ chiral pion in Sec. 2, with constituent quark mass $\hat{m} \approx 325$ MeV (near $\hat{m} \approx M_N/3$), has CL charge radius $r_\pi^{CL} = 1/\hat{m} \approx 0.61$ fm. So the close agreement with Sakurai's value means we must take the VMD scheme along with the LσM as the basis of our chiral theory.

The ρ^0 form factor predicts, from $udu+$ dud quarks loops in the CL (see Fig. 4),

$$g_{\rho\pi\pi} = -i4N_c\, g^2 g_\rho \int đ^4p\,(p^2 - \hat{m}^2)^{-2} = g_\rho\,,$$

by virtue of the LDGE [10]. Then, folding in the mesonic π-σ-π loop changes the VMD prediction only slightly to [12] $g_{\rho\pi\pi} = g_\rho + g_{\rho\pi\pi}/6 = (6/5)g_\rho$, compatible with the observed couplings $g_{\rho\pi\pi} \approx 6.04$ and $g_\rho \approx 5.01$, since (for $p_{CM} = 358$ MeV)

$$\Gamma_{\rho\pi\pi} = \frac{p_{CM}^3\, g_{\rho\pi\pi}^2}{6\pi m_\rho^2} = 149.2 \pm 0.7\,\text{MeV} \implies g_{\rho\pi\pi} \approx 6.04 \tag{11}$$

$$\Gamma_{\rho ee} = \frac{e^4\, m_\rho}{12\pi g_\rho^2} = 6.85 \pm 0.11\,\text{keV} \implies g_\rho \approx 5.01\,, \tag{12}$$

with $e \approx 0.3028$ (i.e., $\alpha \approx 1/137$). Also, the quark-loop VPV or PVV (see Fig. 5) amplitudes are [41], using $\Gamma_{VPV} = p^3|F_{VPV}|^2/12\pi$,

$$|F(\rho\to\pi\gamma)| = \frac{eg_\rho}{8\pi^2 f_\pi} \approx 0.207\,\text{GeV}^{-1}\,,\quad |F(\omega\to\pi\gamma)| = \frac{eg_\omega}{8\pi^2 f_\pi} \approx 0.704\,\text{GeV}^{-1}\,,$$

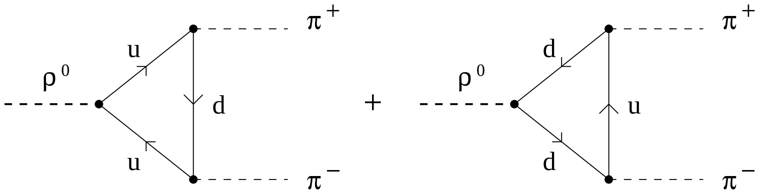

Fig. 4. Vector-mesonic VPP quark triangle graphs.

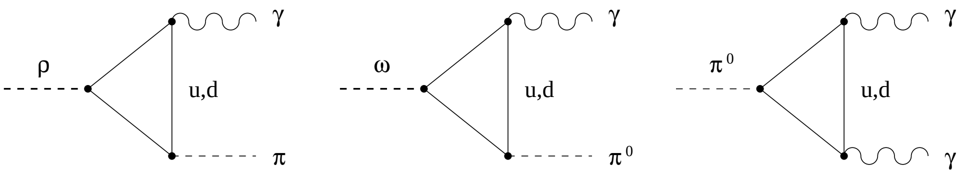

Fig. 5. PVV quark triangle graphs for $\rho \to \pi\gamma$, $\omega \to \gamma\pi^0$, and $\pi^0 \to \gamma\gamma$.

$$\left|F(\pi^0 \to 2\gamma)\right| = \frac{\alpha}{\pi f_\pi} = \frac{e^2}{4\pi^2 f_\pi} \approx 0.025\,\mathrm{GeV}^{-1}\,, \tag{13}$$

for $g_\rho \approx 5.01$ and $g_\omega \approx 17.06$, very close to the data 0.222 ± 0.012 GeV^{-1} [15], 0.698 ± 0.014 GeV^{-1} [42] and 0.0252 ± 0.0009 GeV^{-1} [15], respectively. Equivalently, VMD predicts at tree level $|F_{\rho\pi\gamma}|\,e/g_\rho = |F_{\omega\pi\gamma}|\,e/g_\omega = |F_{\pi^0\gamma\gamma}|/2$, then compatible with the LσM quark loops in Eq. (13).

7. *Nonleptonic weak* $\mathrm{K}_{2\pi}$ $\Delta I = 1/2$ *rule and* σ, κ *mesons*

The well-known [15] $\Delta I = 1/2$ rule $\Gamma(\mathrm{K_S} \to \pi^+\pi^-)/\Gamma(\mathrm{K}^+ \to \pi^+\pi^0) \approx 450$ for nonleptonic weak $\mathrm{K}_{2\pi}$ decays suggests [43] that the parity-violating (PV) amplitude $\langle 2\pi|H_w^{pv}|K_\mathrm{S}\rangle$ could be dominated by the $\Delta I = 1/2$ weak transition $\langle\sigma|H_w^{pv}|K_\mathrm{S}\rangle$. The σ-pole graph of Fig. 6, with LσM coupling $\langle 2\pi|\sigma\rangle = m_\sigma^2/2f_\pi$ for m_σ near m_K and $\Gamma_\sigma \sim m_\sigma$, predicts [44]

$$|\langle 2\pi|H_w^{pv}|K_\mathrm{S}\rangle| = \left|\frac{2\,\langle 2\pi|\sigma\rangle\,\langle\sigma|H_w^{pv}|K_\mathrm{S}\rangle}{m_\mathrm{K}^2 - m_\sigma^2 + \mathrm{i}\,m_\sigma\Gamma_\sigma}\right| \approx \frac{1}{f_\pi}\,|\langle\sigma|H_w^{pv}|K_\mathrm{S}\rangle|\,. \tag{14}$$

But pion PCAC (manifest in the LσM) requires, using the weak chiral commutator $[Q_5 + Q, H_w] = 0$,

$$|\langle 2\pi|H_w^{pv}|K_\mathrm{S}\rangle| \to f_\pi^{-1}\,|\langle\pi|[Q_5^\pi, H_w]|K_\mathrm{S}\rangle| \approx f_\pi^{-1}\left|\langle\pi^0|H_w^{pc}|K_\mathrm{L}\rangle\right|\,, \tag{15}$$

with both pions being consistently reduced in Ref. [45]. To reconfirm Eq. (15), one considers the $\Delta I = 1/2$ weak tadpole graph, giving

$$|\langle 2\pi|H_w^{pv}|K_\mathrm{S}\rangle| = m_\mathrm{K}^{-2}\,|\langle 0|H_w|K_\mathrm{S}\rangle\,\langle K_\mathrm{S}\,2\pi|K_\mathrm{S}\rangle|\,, \tag{16}$$

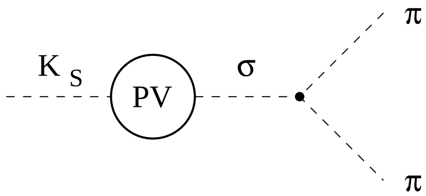

Fig. 6. Parity-violating two-pion decay of K_S dominated by σ pole.

and then one invokes the Weinberg-Osborn [46] strong chiral coupling $|\langle K_S\ 2\pi|K_S\rangle| = m^2_{K_S}/2f^2_\pi$, together with the usual PCAC relation $|\langle 0|H^{pv}_w|K_S\rangle| = |2f_\pi \langle\pi^0|H^{pc}_w|K_L\rangle|$, to recover Eq. (15) [47].

In either case, equating Eqs. (14) and (15) gives $|\langle\sigma|H^{pv}_w|K_S\rangle| \approx |\langle\pi^0|H^{pc}_w|K_L\rangle|$, suggesting that the π and σ mesons are "chiral partners", at least for nonleptonic weak interactions. But of course, Secs. 1–6 above also show that the π and the σ are chiral partners for strong, e.m., and semileptonic weak interactions, as well. To compare this chiral-partner $K \to \pi$ transition with $K_{2\pi}$ data, we return to the PCAC equation (15) to write, for $f_\pi \approx 93$ MeV,

$$|\langle 2\pi|H^{pv}_w|K_S\rangle| \approx \frac{1}{f_\pi}|\langle\pi^0|H^{pc}_w|K_L\rangle| \approx 38\times 10^{-8}\ \mathrm{GeV}\,, \tag{17}$$

midway between the observed $K_S \to \pi^+\pi^-$ and $K_S \to \pi^0\pi^0$ amplitudes

$$|M^{+-}_{K_S\to\pi\pi}|_{PDG} = m_{K_S}\left[\frac{8\pi\Gamma^{K_S}_{+-}}{q}\right]^{\frac{1}{2}} = (39.1\pm 0.1)\times 10^{-8}\ \mathrm{GeV}\,, \tag{18}$$

$$|M^{00}_{K_S\to\pi\pi}|_{PDG} = m_{K_S}\left[\frac{16\pi\Gamma^{K_S}_{00}}{q}\right]^{\frac{1}{2}} = (37.1\pm 0.2)\times 10^{-8}\ \mathrm{GeV}\,, \tag{19}$$

suggesting $|\langle\pi^0|H^{pc}_w|K_L\rangle| \approx 3.58\times 10^{-8}\ \mathrm{GeV}^2$. In fact, when one statistically averages *eleven* first-order weak data sets for $K_S \to 2\pi$, $K \to 3\pi$, $K_L \to 2\gamma$, $K_L \to \mu^+\mu^-$, $K^+ \to \pi^+e^+e^-$, $K^+ \to \pi^+\mu^+\mu^-$, and $\Omega^- \to \Xi^0\pi^-$, one finds [48]

$$|\langle\pi^0|H^{pc}_w|K_L\rangle| = |\langle\pi^+|H^{pc}_w|K^+\rangle| = (3.59\pm 0.05)\times 10^{-8}\ \mathrm{GeV}^2\,. \tag{20}$$

To induce theoretically at the quark level the ΔI=1/2 $s \to d$ single-quark-line (SQL) transition scale β_w in a model-independent manner, one considers the second-order weak (see Fig. 7) $K_L - K_S$ mass difference Δm_{LS} diagonalized to [49]

$$2\beta^2_w = \frac{\Delta m_{LS}}{m_K} = (0.70126\pm 0.00121)\times 10^{-14}$$
$$\Longrightarrow |\beta_w| \approx (5.9214\pm 0.0051)\times 10^{-8}\,. \tag{21}$$

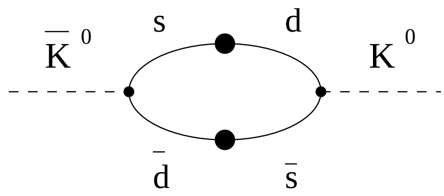

Fig. 7. $\bar{\mathrm{K}}^0 \leftrightarrow \mathrm{K}^0$ *SQL graph. Each dot represents the SQL weak scale* β_w.

Then using Eq. (21), one predicts from the soft-meson theorem, or from Cronin's chiral Lagrangian [50],

$$\left|\langle \pi^0 | H_w^{pc} | K_\mathrm{L} \rangle\right| \;=\; 2\beta_w m_{\mathrm{K_L}}^2 f_\pi^{-1} f_\mathrm{K} \;=\; (3.5785 \pm 0.0031) \times 10^{-8}\,\mathrm{GeV}^2\,, \qquad (22)$$

given $f_\mathrm{K}/f_\pi \approx 1.22$. This SQL scale β_w in Eq. (21) and the $\mathrm{K} \to \pi$ weak amplitude in Eq. (22) (or in Eq. (20)), correspond to a "truly weak" interaction, which Weinberg [51] shows cannot be transformed away in the electroweak standard model. To test the latter weak scale (22) (or the similar data averages (20)), we first re-express the neutral chiral-partner relation (extended to the κ transition [44]) as

$$\langle \pi^0 | H_w^{pc} | K^0 \rangle = \langle \sigma | H_w^{pv} | K^0 \rangle = \langle \pi^0 | H_w^{pv} | \kappa^0 \rangle = \frac{3.58}{\sqrt{2}} \times 10^{-8}\,\mathrm{GeV}^2\,.$$

We fix this $\kappa^0 \to \pi^0$ weak transition to the weak PV K^0 tadpole graph of Fig. 8, via the $\mathrm{K}^0 \to$ vacuum PCAC scale, as $\left|\langle 0 | H_w^{pv} | K^0 \rangle\right| = 2f_\pi^2 \left|\langle 2\pi^0 | H_w^{pv} | K^0 \rangle\right| /(1 - m_\pi^2/m_\mathrm{K}^2) = 0.51 \times 10^{-8}$ GeV^3, using $\left|\langle 2\pi^0 | H_w^{pv} | K^0 \rangle\right| = 26.26 \times 10^{-8}$ GeV from data, while eliminating the 4% $\Delta I = 3/2$ component (see Ref. [51], third paper). Then Fig. 8 predicts the amplitude magnitude $\left|\langle \pi^0 | H_w^{pv} | \kappa^0 \rangle\right| = \left|\langle 0 | H_w^{pv} | K^0 \rangle\right| g_{\kappa^0 \mathrm{K}^0 \pi^0}/m_{\mathrm{K}^0}^2 \approx 2.53 \times 10^{-8}$ GeV^2, scaled to the latter neutral chiral-partner relation, *provided* one uses the LσM coupling, for $f_\pi = 92.4$ MeV [15], $|g_{\kappa^0 \mathrm{K}^0 \pi^0}| = m_\kappa^2 - m_\mathrm{K}^2/4f_\pi = 1.229$ GeV, corresponding to a κ mass of 838 MeV. This value is not too distant from our earlier $m_\kappa = 730$–800 MeV predictions [37, 38], and the very recent E791 observed mass $m_\kappa \approx 800$ MeV [52]. Moreover, the $SU(3)$ analogue $|g_{\sigma\pi\pi}| = (m_\sigma^2 - m_\pi^2)/2f_\pi$ suggests $m_\sigma = 687$ MeV, reasonably near the predicted CL-LσM value [3, 53] $m_\sigma = 650$ MeV.

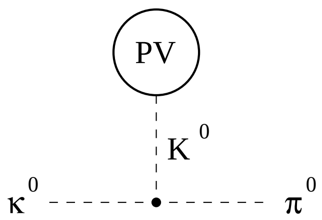

Fig. 8. Parity-violating weak K^0 *tadpole graph for* $\kappa^0 \to \pi^0$ *transition.*

8. *Summary and conclusions*

In Sec. 1 we reviewed the solution of the LσM at the quark-loop level. In Sec. 2 we used $SU(2)$, $SU(3)$ Goldberger–Treiman quark relations to normalize the π and K form factors to unity at $k^2 = 0$, after which we differentiated these form factors to predict the LσM charge radii, both being compatible with data. Next, in Sec. 3, we briefly reviewed e.m. charged-pion polarizabilities for $\gamma\gamma \to \pi\pi$, and compared them with LσM predictions. In Sec. 4 we used quark loops to match the observed form factor $f_+(0)$. In Sec. 5 we showed that the LσM form factors F_V^π, R^π, $F_V^{\rm K}+F_A^{\rm K}$, and the ratio F_A^π/F_V^π are all in agreement with the measured values. Then in Sec. 6 we compared tree-level VMD with LσM VPP and PVV quark loops. Both theories agree well with data. Finally, in Sec. 7 we successfully extended this LσM picture to nonleptonic weak decays, in particular to the $\Delta I = 1/2$-dominated $\rm K_{2\pi}$ decays, and inferred $\sigma(687)$ and $\kappa(838)$ masses.

In view of the good agreement with experiment for practically all studied observables, and the absence of any freely adjustable parameters, we believe the quark-level LσM is an excellent candidate for an effective theory of low-energy QCD [54 – 60].

Acknowledgements

The authors are indebted to A. E. Kaloshin for valuable discussions. This work was partly supported by the *Fundação para a Ciência e a Tecnologia* of the *Ministério do Ensino Superior, Ciência e Tecnologia* of Portugal, under Grant no. PRAXIS XXI/BPD/20186/99 and under contract number CERN/P/FIS/43697/-2001.

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MEZONSKI FAKTORI OBLIKA U LINEARNOM σ MODELU NA KVARKOVSKOJ RAZINI

Primjenjujemo linearni σ model (LσM) na kvarkovskoj razini za računanje niza elektromagnetskih i slabih veličina za lake mezone, kao što su pionski i kaonski faktori oblika i nabojski polumjeri, polarizivost nabijenih piona, poluleptonski slab $K_{\ell 3}$ raspad, poluleptonski slabi radijativni pionski i kaonski faktori oblika, radijativni raspadi vektorskih mezona i neleptonski slabi raspadi $K_{2\pi}$. Slaganje svih predviđanja s ishodima mjerenja je izvanredno. Usput, pokazujemo blisku vezu LσM sa prevladavanjem vektorskih mezona. Izvodima neke zaključke o LσM u odnosu na pristupe QCDu s rešetkama i renormalizacijskom grupom.