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# THE COMPACT STAR IN THE $SU(3)$ SIGMA MODEL

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The linear chiral $SU(3)_\mathrm{L} \times SU(3)_\mathrm{R}$ model is applied to describe properties of the compact star matter inside the quark, protoneutron and neutron star.

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## 1. Introduction

In this presentation we shall investigate the astrophysical properties of the compact stars (protoneutron, neutron and quark stars [1]) within the relativistic mean field (RMF) [2] theory originated from the linear $SU(3)$ sigma model [1 – 3].

The main aim of this work is to show how an effective mean field approximation emerges from the linear $SU(3)$ chiral model and to show comparision to the current RMF approach (Furnstahl – Serot – Tang (FST) model [6]), as well as to present astrophysical applications in the form of the neutron star model. The effective model which includes scalar, vector and scalar-vector interaction terms is applied to describe properties of the quark, protoneutron and neutron star matter.

## 2. The $SU(3)$ sigma model

The chiral $SU(3)$ model was proposed by Papazoglou et al. [3, 4]. In the original form, it describes interaction of the baryons and mesons $SU(3)$ multiplets. Recently, a chiral $SU(3)$ quark model has been proposed by Wang et al. [7]. The basic fields that compose the theory represent the realization of the group $SU(3)_\mathrm{L} \times SU(3)_\mathrm{R}$. The meson content of the model is scalar, pseudoscalar and vector. Naive quark models interpret them as excited $\overline{q}q$ states. Scalar and pseudoscalar mesons can be grouped into

$$\Phi = \Sigma + \mathrm{i}\,\Pi = \frac{1}{\sqrt{2}} T_a \phi_a = \frac{1}{\sqrt{2}} T_a (\sigma_a + \mathrm{i}\,\pi_a)\,, \tag{1}$$

where $\sigma_a$ and $\pi_a$ are members of the scalar and pseudoscalar octet, respectively,

$$\Sigma = \frac{1}{\sqrt{2}} \sigma^a \lambda^a = \left\{ \begin{array}{ccc} \frac{1}{\sqrt{2}}(f_0 + a_0^0), & a_0^+, & K^+ \\ a_0^-, & \frac{1}{\sqrt{2}}(f_0 - a_0^0), & K^0 \\ K^-, & \bar{K}^0, & f_0' \end{array} \right\}, \tag{2}$$

$$\Pi = \frac{1}{\sqrt{2}} \pi^a \lambda^a = \left( \begin{array}{ccc} \frac{1}{\sqrt{2}}(\pi^0 + \eta^8) & \pi^+ & \kappa^+ \\ \pi^- & \frac{1}{\sqrt{2}}(-\pi^0 + \eta^8) & \kappa^0 \\ \kappa^- & \kappa^0 & \zeta \end{array} \right). \tag{3}$$

The vector meson octet is given by

$$V = \frac{1}{\sqrt{2}} v^a \lambda^a = \left( \begin{array}{ccc} \frac{1}{\sqrt{2}}(\omega + \rho_0^0), & \rho_0^+, & K^{*+} \\ \rho_0^-, & \frac{1}{\sqrt{2}}(\omega - \rho_0^0), & K^{*0} \\ K^{*-}, & \overline{K^{*0}}, & \phi \end{array} \right). \tag{4}$$

The most general form of the Lagrangian function can be written as a sum of the following parts

$$\mathcal{L} = \mathcal{L}_{\mathrm{M}} + \mathcal{L}_{\mathrm{F}} + \mathcal{L}_{\mathrm{sb}}\,, \tag{5}$$

where

$$\mathcal{L}_{\mathrm{M}} = \frac{1}{2}\mathrm{Tr}(\partial_\mu \Phi \partial^\mu \Phi) - \frac{1}{2}\mu^2 \mathrm{Tr}(\Phi^2) - \frac{\lambda}{4}\mathrm{Tr}((\Phi^+\Phi)^2) - \frac{\kappa}{4}\mathrm{Tr}(\Phi^+\Phi)^2 \tag{6}$$

is the Lagrangian function which describes scalar and pseudoscalar mesons $\Phi$ ($\Phi = \Sigma + \mathrm{i}\Pi$). The symmetry breaking term $\mathcal{L}_{\mathrm{sb}}$ has the form of

$$\mathcal{L}_{\mathrm{sb}} = \frac{1}{2} c(\mathrm{Det}(\Phi) + \mathrm{Det}(\Phi)^*) + \mathrm{Tr}(H^+\Phi + \Phi^+ H). \tag{7}$$

In the mean-field approximation, the chiral symmetry is broken and the meson fields gain non-vanishing vacuum expectation values $(\sigma, \chi)$

$$< \Phi > = < \Sigma > = \left\{ \begin{array}{ccc} \frac{1}{\sqrt{2}}\sigma, & 0, & 0 \\ a_0^-, & \frac{1}{\sqrt{2}}\sigma, & 0 \\ 0, & 0, & \chi \end{array} \right\}.$$

The effective potential $U_{\mathrm{eff}}(\sigma, \chi) = - < L >$ (Fig. 1) determines the scale of the chiral symmetry breaking. Shifting the meson field $\Phi = \bar{\Phi} + < \Phi >$ ($f_0 = \bar{f}_0 + \sigma$,

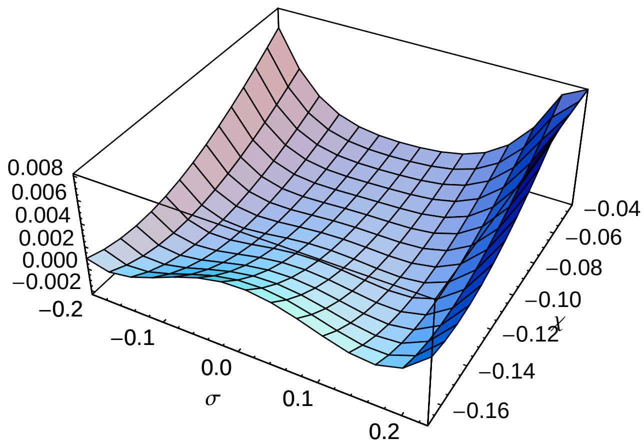


*Fig. 1. The effective potential $U_{\mathrm{eff}}(\sigma, \chi)$ in the $SU(3)$ sigma model.*

$f_0' = \bar{f}_0' + \chi$) and diagonalizing the square mass matrix $m^2_{\mathrm{a,b}} = \partial^2 U_{\mathrm{eff}}/(\partial\bar{\sigma}_{\mathrm{a}}\partial\bar{\sigma}_{\mathrm{b}})$ produce the physical meson fields

$$\left\{ \begin{array}{c} \varphi \\ \varphi_* \end{array} \right\} = \left( \begin{array}{cc} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{array} \right) \left\{ \begin{array}{c} f_0 \\ f_0' \end{array} \right\}. \tag{8}$$

Fitting the model to the observed masses of mesons allows to determine its parameters (similar to the case in Ref. [8] of the explicit chiral symmetry breaking with $U(1)_{\mathrm{A}}$ anomaly). It gives $\mu = 552.274$ MeV, $\lambda = 45.11$, $\kappa = -8.917$ and $c = 3412.25$. This fitting gives meson masses (e.g., $\pi$, $K$, $\varphi$, $\varphi_*$, $\delta = a_0$, etc.) including sigma meson mass $m_\sigma = 502.27$ MeV. The only uncertain thing is the sigma meson mass. The light scalar meson $\sigma$ (denoted here as $\varphi$) is an elusive subject of classification.

The fermion content of the model consists of either quarks or baryons (QMF or RMF model).

The chiral $SU(3)$ quark mean-field model has been applied to describe the quark matter or nucleon matter. In the chiral limit, the quark field $q = \{u, d, s\}$ with three flavors can be decomposed into left and right-handed parts $q = q_{\mathrm{L}} + q_{\mathrm{R}}$. The quarks are described by the Lagrange function

$$\mathcal{L}_{\mathrm{F}} = \mathrm{i}\,\bar{q}\gamma^\mu D_\mu q - \bar{q}m_0 q + g_{\mathrm{s}}\bar{q}\Phi q - \chi_{\mathrm{c}}(r)\bar{q}q\,,$$

where $\chi_{\mathrm{c}}(r)$ is the quark confining potential. Solving the Dirac equation for quark in the confining potential, one can calculate the baryon masses

$$M_{\mathrm{eff,N}}(\varphi, \varphi_*) = M_{\mathrm{N}} - g_\sigma(\varphi)\,\varphi - g_{\sigma*}\varphi_* = M_{\mathrm{N}} - g_\sigma\,\varphi + \frac{1}{2}g_\sigma C'(0)\,\varphi^2 + ...$$

with $g_\sigma(\varphi) = g_\sigma - \frac{1}{2}g_\sigma C'(0)\,\varphi = g_\sigma - \frac{1}{2}a\,\varphi$. The last nonlinear term indicates the inner nucleon structure.

## 3. *The effective RMF approach*

The nuclear relativistic mean field approach describes the nuclear interactions due to the mesons exchange between baryons (p, n, $\Lambda$, $\Sigma$, $\Xi$). Baryons are grouped into the isospin and hipercharge representations $(\frac{1}{2}, 1)$, $(\frac{1}{2}, -1)$, $(1, 0)$

$$\Lambda,\ N = \begin{pmatrix} p \\ n \end{pmatrix},\ \Xi = \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix},\ \Sigma = \begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix}. \tag{9}$$

The RMF Lagrange functions

$$\mathcal{L}_{\mathrm{RMF}} = \mathcal{L}_{\mathrm{B}} + \mathcal{L}_{\mathrm{M}} \tag{10}$$

describe baryons ($B = \{\mathrm{b}, \mathrm{n}, \Lambda,\ \Sigma,\ \Xi\}$)

$$\mathcal{L}_{\mathrm{B}} = \sum_{\mathrm{B}} \mathrm{i}\, \overline{\psi}_{\mathrm{B}} \gamma^{\mu} D_{\mu} \psi_{\mathrm{B}} - \sum_{B} M_{\mathrm{B}}(\varphi, \varphi_*) \overline{\psi}_{\mathrm{B}} \psi_{\mathrm{B}} \tag{11}$$

and mesons

$$\mathcal{L}_{\mathrm{M}} = \mathcal{L}_{\mathrm{Ms}} + \mathcal{L}_{\mathrm{Mv}}. \tag{12}$$

Mesons can be divided into scalar mesons ($\varphi$, $\varphi_*$, $\delta$) described by $\mathcal{L}_{\mathrm{Ms}}$ and vector mesons ($\omega$, $\rho$, $\phi$) described by $\mathcal{L}_{\mathrm{Mv}}$.

$$\begin{aligned} \mathcal{L}_{\mathrm{Ms}} &= \tfrac{1}{2}\partial_{\mu}\varphi\partial^{\mu}\varphi + \tfrac{1}{2}\partial_{\mu}\delta^{a}\partial^{\mu}\delta^{a} \\ &+ \tfrac{1}{2}\partial_{\mu}\varphi_*\partial^{\mu}\varphi_* - U_{\mathrm{S,eff}}(\varphi, \varphi_*, \delta), \end{aligned} \tag{13}$$

$$\begin{aligned} \mathcal{L}_{\mathrm{Mw}} = &-\tfrac{1}{4}\Omega_{\mu\nu}\Omega^{\mu\nu} + \tfrac{1}{2}m_{\omega}^{2}\omega_{\mu}\omega^{\mu} - \tfrac{1}{4}R^{a}_{\mu\nu}R^{a\mu\nu} + \tfrac{1}{2}m_{\rho}^{2}\rho^{a}_{\mu}\rho^{a\mu} \\ &-\tfrac{1}{4}\Phi_{\mu\nu}\Phi^{\mu\nu} + \tfrac{1}{2}m_{\phi}^{2}\phi_{\mu}\phi^{\mu} + U_{\mathrm{V,eff}}(\omega, \rho)\,, \end{aligned} \tag{14}$$

where

$$\Omega_{\mu\nu} = \partial_{\mu}\omega_{\nu} - \partial_{\nu}\omega_{\mu}\ , \qquad \Phi_{\mu\nu} = \partial_{\mu}\phi_{\nu} - \partial_{\nu}\phi_{\mu} \tag{15}$$

$$\text{and} \quad R_{\mu\nu} = R^{a}_{\mu\nu}T^{a} = \partial_{\mu}R_{\nu} - \partial_{\nu}R_{\mu} - \mathrm{i}\, g_{\rho}[R_{\mu}, R_{\nu}]. \tag{16}$$

The scalar meson interaction is enourmously nonlinear. It comes from the shifting in the potential $U_{\mathrm{eff}}(\sigma, \chi)$ according to the prescription (Eq. (8)). In the simplest approximation, this procedure generates the polynomial scalar interaction (Fig. 2) of the RMF approach

$$U_0(\varphi) = U_{\mathrm{eff}}(\sigma_0 + \varphi, \chi_0) = U_{\mathrm{eff}}(\varphi, 0) = \frac{1}{2}m_{\sigma}^{2}\varphi^{2} + \frac{1}{3}g_2\varphi^{3} + \frac{1}{4}g_3\varphi^{4} \tag{17}$$

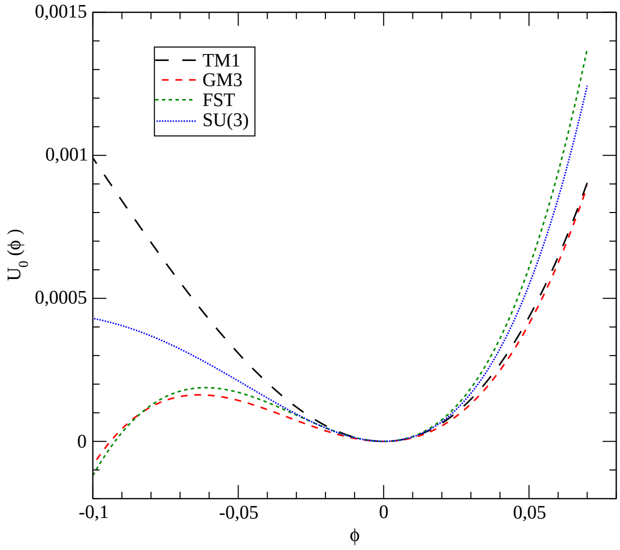


*Fig. 2. The potential $U_0$ for the scalar meson $\varphi$ of the effective RMF theory.*

The form of the potential function was first introduced by Boguta and Bodmer [9] in order to get the correct value of the compressibility $K$ of nuclear matter at saturation density (see Table 1). The simplest Walecka model (L2) (*linear* Walecka model, $g_2 = g_3 = 0$) brings a very large, unrealistic value of the parameter $K$ [2]. Figure 3 depicts the effective nucleon masses obtained for different parameter set functions of baryon number density $n_{\mathrm{B}}$. The parameters describing the nucleon-nucleon interactions in the RMF approach are chosen in order to reproduce the properties of the symmetric nuclear matter at saturation such as the binding energy, symmetry energy and incompressibility. In the chiral $SU(3)$ model, they are generally calculable from the starting ones ($\mu^2$, $\lambda$, $\kappa$) which are fitted to the meson spectroscopy. The appropriate parameter set is constrained not only by the value of physical scalar meson masses, but also by the properties of nuclear matter at saturation. For symmetric nuclear matter, the nucleon density equals $n_0 = 2.5\ 10^{14}$ $\mathrm{g\,cm^{-3}} = 0.15$ nucleons/fm$^3$ = 140 MeV fm$^{-3}$. The obtained results are collected in Table 1.

*TABLE 1. Properties of the nuclear matter at saturation for the symmetric nuclear matter.*

| Parameter | GM3[10] | TM1[11] | FST[6] | $SU(3)$ |
|---|---|---|---|---|
| $E_0$ (MeV) | −16.35 | −16.26 | −16.38 | −16.31 |
| $\delta_0$ | 0.793 | 0.659 | 0.661 | 0.761 |
| $n_0$ (fm$^{-3}$) | 0.153 | 0.145 | 0.155 | 0.145 |
| $K$ (MeV) | 241.12 | 281.53 | 219.5 | 194.9 |
| $J$ (MeV) | 32.44 | 36.82 | 38.17 | 34.70 |

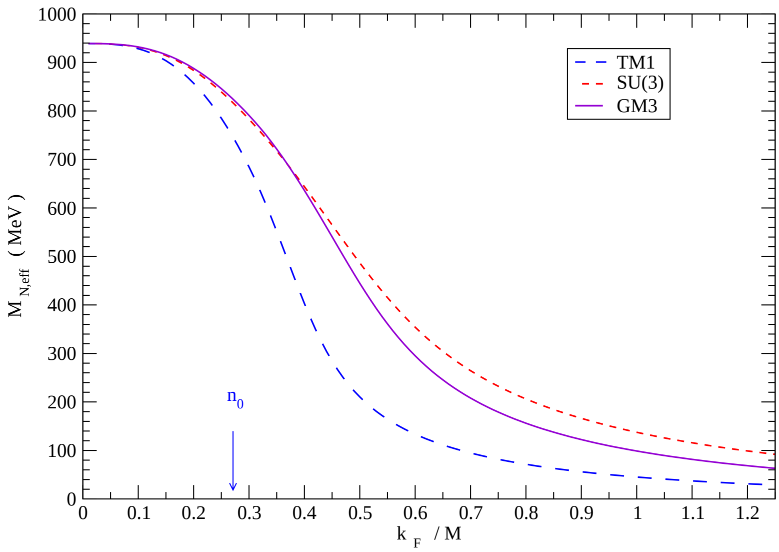


*Fig. 3. The effective nucleon masses for different parameter sets as a function of the nucleon Fermi momentum $k_F$.*

## 4. *The neutron star in the $SU(3)$ sigma model*

The compact star is a result of the equilibrium between gravitational collapse and the pressure generated by the nuclear or quark matter. The matter is in $\beta$ equilibrium. In the neutron star, weak interactions are responsible for $\beta$ decay

$$\mathrm{n} + \nu_\mathrm{e} \leftrightarrow \mathrm{p} + \mathrm{e}, \tag{18}$$

$$\mu + \nu_\mathrm{e} \leftrightarrow \mathrm{e} + \nu_\mu. \tag{19}$$

These reactions produce appropriate relation among the chemical potentials of neutrinos in a protoneutron star where they are trapped

$$\mu_\mathrm{p} = \mu_\mathrm{n} + \mu_{\nu_\mathrm{e}} - \mu_\mathrm{e} \tag{20}$$

$$\mu_{\nu_\mathrm{e}} = \mu_\mathrm{e} + \mu_\mathrm{p} - \mu_\mathrm{n} \tag{21}$$

$$\mu_{\nu_\mu} = \mu_{\nu_\mathrm{e}} - \mu_\mathrm{e} + \mu_\mu\,. \tag{22}$$

In a protoneutron star, when the neutrino is trapped, its chemical potential strongly depends on nuclear asymmetry

$$\mu_{\nu_\mathrm{e}} = \mu_\mathrm{e} + \varepsilon_\mathrm{p} - \varepsilon_\mathrm{n} + g_\rho r_0, \tag{23}$$

$$\text{where} \quad \epsilon_{\mathrm{p,n}} = \sqrt{k_{\mathrm{F,B}}^2 + M_\mathrm{B}^2}|_{\mathrm{B=n,p}} \tag{24}$$

and $r_0 =< \rho_0^3 >$ is the expected value for the $\rho$ meson in medium.

The equilibrium conditions with respect to the $\beta$ decay between baryonic (including hyperons) and leptonic species lead to the following relations among their chemical potentials and constrain the species fraction in the star interior

$$\mu_{\rm p} = \mu_{\Sigma^+} = \mu_{\rm n} - \mu_{\rm e}\,, \qquad \mu_\Lambda = \mu_{\Sigma^0} = \mu_{\Xi^0} = \mu_{\rm n}\,, \tag{25}$$

$$\mu_{\Sigma^-} = \mu_{\Xi^-} = \mu_{\rm n} + \mu_{\rm e}\,, \qquad \mu_\mu = \mu_{\rm e}\,. \tag{26}$$

At higher density, a greater number of hyperon species are expected to appear. They can be formed both in leptonic and baryonic processes. In the latter one, the strong interaction process such as

$$\rm n + n \to n + \Lambda \tag{27}$$

proceeds. There are other relevant strong reactions that establish the hadron population in neutron star matter, e.g.

$$\Lambda + {\rm n} \to \Sigma^- + {\rm p} \qquad \Lambda + \Lambda \to \Xi^- + {\rm p} \tag{28}$$

The final result is the equation of the state (Fig. 4). All these lead to the neutron star model with the value of maximum mass close to 1.5 $M_\odot$ with the reduced value of proton fraction and very compact hyperon core. The obtained form of the equation of state serves as an input to the Oppenheimer-Volkoff equations and

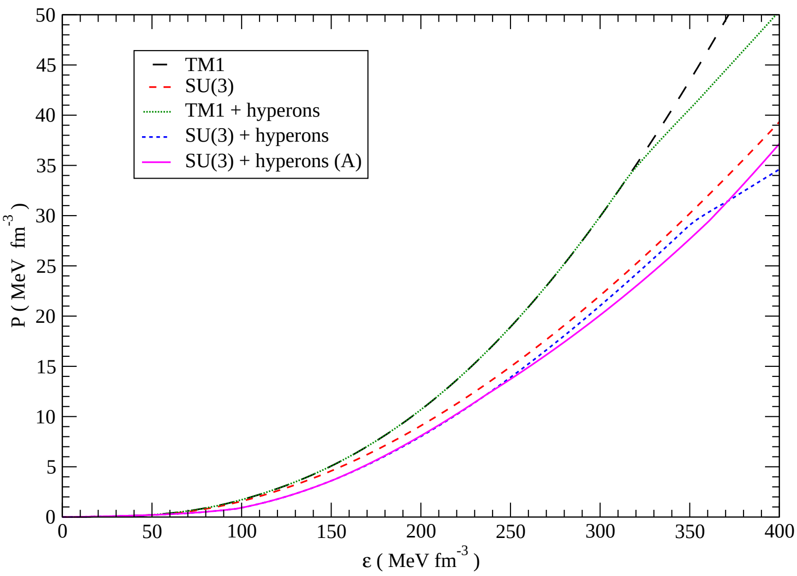


*Fig. 4. The nuclear matter equation of state for different parameter sets.*

determines the structure of spherically symmetric stars

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}\,\frac{\Big(1+P(r)/\rho(r)\Big)\Big(1+4\pi r^3 P(r)/m(r)\Big)}{1-2\,Gm(r)/r}\,, \tag{29}$$

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r)\,. \tag{30}$$

The gravitational binding energy of a relativistic star is defined as a difference between its gravitational and baryon masses

$$E_{\mathrm{b,g}} = \Big(M_{\mathrm{p}} - m(R)\Big)\, c^2\,, \tag{31}$$

where

$$M_{\mathrm{p}} = 4\pi \int\limits_0^R \mathrm{d}r\, r^2 \left(1 - \frac{2\,Gm(r)}{c^2 r}\right)^{-1/2} \rho(r)\,. \tag{32}$$

The numerical solution of the above equation is of considerable relevance to the selected EOS. Numerical solutions of these equations allow to construct the mass-radius relations of the neutron star (Fig. 5) and the gravitational final energy (Fig. 6).

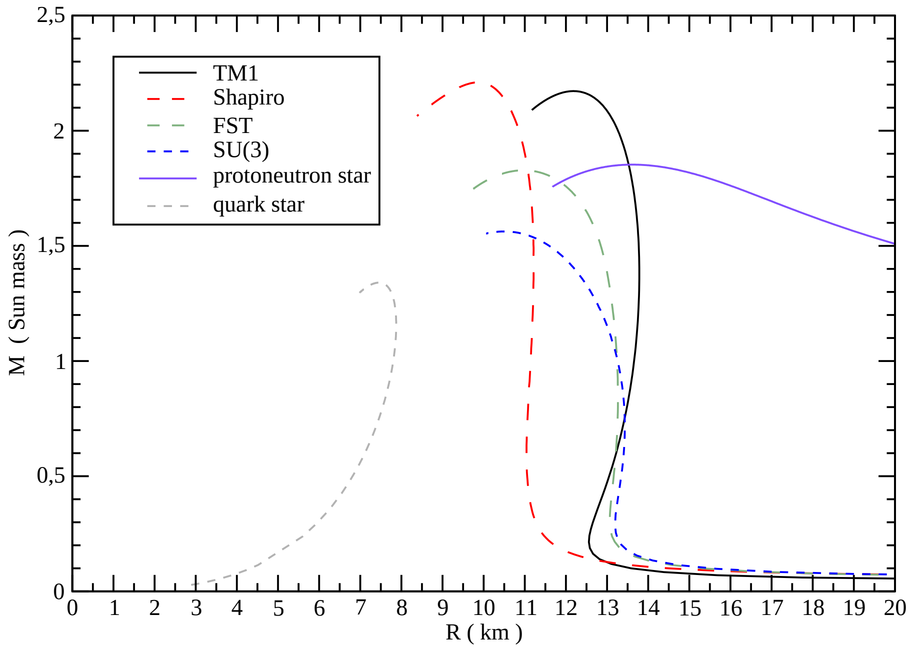


*Fig. 5. The mass radius relation for various quark and neutron stars.*

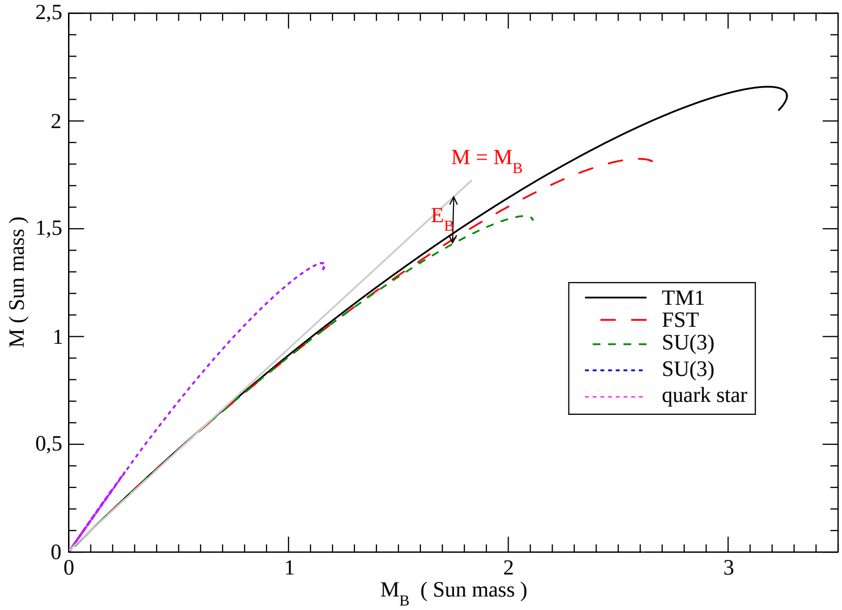


*Fig. 6. The gravitational binding energy for quark and neutron stars.*

## 5. *Conclusions*

The astrophysical properties of the compact stars are strongly related to the properties of quark and nuclear matter. Their mass, radius and binding energy depend on inner physics of elementary particles and nuclei. The linear sigma model predicts properties of the compact stars quite convicingly. Only the quark star seems to look as an unphysical object, i.e. with the positive binding energy (see Fig. 6).

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## ZVIJEZDE VELIKE GUSTOĆE I $SU(3)$ SIGMA MODEL

Primjenjuje se linearan kiralni model $SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}}$ za opis svojstava guste zvjezdane tvari u kvarkovskoj, protoneutronskoj i neutronskoj zvijezdi.