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THE ROLE OF CHIRAL SYMMETRY IN HADRONIC PROCESSES

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The role of chiral symmetry in hadronic processes is discussed. Emphasis is given to the cancellation of diagrams in $\pi - \pi$ scattering induced by chiral symmetry and its consequences in the scalar sector.

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1. *A prototype for the spontaneous breakdown of chiral symmetry*

Let us start with a simple example which illustrates the mechanism of spontaneous breaking of chiral symmetry (SBχS): the Hamiltonian of a relativistic fermion in an external field A_μ. The physics of fermions in strong magnetic fields constitutes on its own an active field of research [1]. Here we present a general formalism (in fact a generalization of the methods of Ref. [2] in order to go beyond 2+1 dimensions) simply as an introduction to the issue of SBχS. We have,

$$H = \int \mathrm{d}^2 x \; \bar{\psi}(x) \left[-\mathrm{i}\gamma^j D_j + m \right] \psi(x) \; . \tag{1}$$

The theory is invariant under a $U(2)$ symmetry, which breaks down to $U(1) \times U(1)$ for $m \neq 0$.

Choose the Landau gauge $A_\mu = -By \; \delta_{\mu 1}$, where $B > 0$ is the magnetic field strength. The problem is exactly soluble and the solution in the chiral version has the structure,

$$\psi^B(\mathbf{x}, t) = \begin{pmatrix} \psi_1^B \\ \psi_2^B \end{pmatrix} , \tag{2}$$

where $\psi^B_{1,2}(x)$ are the spinors associated to the two inequivalent representations of the Dirac algebra.

1.1. An example of Bogolioubov-Valatin transformations

Three steps are needed, starting from the free-particle wave function, to construct the wave function of a particle in the presence of a magnetic field. We start from

$$\psi(\mathbf{x}) = \sum_{\mathbf{p}} \frac{1}{\sqrt{L_x L_y}} \left\{ u(\mathbf{p})\ a_{\mathbf{p}} + v(\mathbf{p})\ b^{\dagger}_{-\mathbf{p}} \right\} e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\,, \tag{3}$$

$$u(\mathbf{p}) = \sqrt{\frac{E_{\mathbf{p}}+m}{2E_{\mathbf{p}}}} \begin{bmatrix} 1 \\ \dfrac{p_y - \mathrm{i}p_x}{E_{\mathbf{p}}+m} \end{bmatrix}, \quad v(\mathbf{p}) = \sqrt{\frac{E_{\mathbf{p}}+m}{2E_{\mathbf{p}}}} \begin{bmatrix} -\dfrac{p_y + ip_x}{E_{\mathbf{p}}+m} \\ 1 \end{bmatrix},$$

with $$\left\{a^{\dagger}_{\mathbf{p}}, a_{\mathbf{p}'}\right\} = \left\{b^{\dagger}_{\mathbf{p}},\ b_{\mathbf{p}'}\right\} = \delta_{p_x p'_x}\delta_{p_y p'_y},\ \ E_{\mathbf{p}} = \sqrt{m^2 + |\mathbf{p}|^2}.$$

Step 1: preform a canonical Bogolioubov-Valatin (BV) transformation given by

$$\begin{bmatrix} \tilde{a}_{\mathbf{p}} \\ \tilde{b}^{\dagger}_{-\mathbf{p}} \end{bmatrix} = R_{\phi}(\mathbf{p}) \begin{bmatrix} a_{\mathbf{p}} \\ b^{\dagger}_{-\mathbf{p}} \end{bmatrix}, \ \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = R^{*}_{\phi}(\mathbf{p}) \begin{bmatrix} u(\mathbf{p}) \\ v(\mathbf{p}) \end{bmatrix}, \tag{4}$$

where $$R_{\phi}(\mathbf{p}) = \begin{bmatrix} \cos\phi & -\sin\phi\ (\hat{p}_y + i\hat{p}_x) \\ \sin\phi\ (\hat{p}_y - i\hat{p}_x) & \cos\phi \end{bmatrix},$$

$$\cos\phi = \sqrt{\frac{E_{\mathbf{p}}+m}{2E_{\mathbf{p}}}}, \quad \sin\phi = \sqrt{\frac{E_{\mathbf{p}}-m}{2E_{\mathbf{p}}}} \quad \text{and} \quad \hat{p} = \frac{\mathbf{p}}{|\mathbf{p}|}\,.$$

The vacuum associated to the new operators $\tilde{a}$ and $\tilde{b}$ is given by $|\tilde{0}\rangle = S|0\rangle = \prod_{\boldsymbol{p}}(\cos\phi + \sin\phi\ \ a^{\dagger}_{\boldsymbol{p}} b^{\dagger}_{-\boldsymbol{p}})|0\rangle$, with $\tilde{a}_{\boldsymbol{p}}|\tilde{0}\rangle = 0$, $\tilde{b}_{\boldsymbol{p}}|\tilde{0}\rangle = 0$. Think of $\psi(\mathbf{x}) = \sum_{\mathbf{p}} \frac{1}{\sqrt{L_x L_y}} \left\{ u(\mathbf{p})\ a_{\mathbf{p}} + v(\mathbf{p})\ b^{\dagger}_{-\mathbf{p}} \right\} e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}$ as an *inner* product between the Hilbert space spanned by the spinors $\{u, v\}$ and the Fock space generated by $\{a, b\}$. This inner product is *made invariant* under the BV transformations as any rotation in the Fock space *must engender a counter-rotation* in the Hilbert space.

Step 2: consider the Landau level representation:

$$e^{\mathrm{i}p_y y} = e^{-\mathrm{i}\ell^2 p_x p_y}\sqrt{2\pi}\sum_{n=0}^{\infty} \mathrm{i}^n \omega_n(\xi)\,\omega_n(\ell p_y),\ l = \sqrt{|eB|},$$

$$\omega_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x),\ \xi = \frac{y}{l} + lp_x\,. \tag{5}$$

The wave function can be written in the new basis as

$$\psi(\boldsymbol{x}) = \sum_{n\,p_x} \frac{1}{\sqrt{\ell L_x}} \left\{ \hat{u}_{np_x}(y)\ \hat{a}_{np_x} + \hat{v}_{np_x}(y)\ \hat{b}^\dagger_{n-p_x} \right\} e^{\mathrm{i}p_x x}\,,$$

with

$$\begin{bmatrix} \hat{a}_{np_x} \\ \hat{b}^\dagger_{n-p_x} \end{bmatrix} = \sum_{p_y} \frac{\mathrm{i}^n \sqrt{2\pi\ell}}{\sqrt{L_y}} \begin{bmatrix} \omega_n(\ell p_y) & 0 \\ 0 & -\omega_{n-1}(\ell p_y) \end{bmatrix} \begin{bmatrix} \tilde{a}_p \\ \tilde{b}^\dagger_{-p} \end{bmatrix} \tag{6}$$

and

$$\begin{bmatrix} \hat{u}_{np_x}(y) \\ \hat{v}_{np_x}(y) \end{bmatrix} = \begin{bmatrix} \omega_n(\xi) & 0 \\ 0 & \mathrm{i}\omega_{n-1}(\xi) \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}.$$

The new operators satisfy the usual anticommutation relations and the new vacuum obeys $\hat{a}_{np_x}|\tilde{0}\rangle = 0,\ \hat{b}_{np_x}|\tilde{0}\rangle = 0$.

1.2. An example of the mass gap equation

There are several approaches one can adopt to obtain the mass gap equation (see Ref. [3]): **1** – It can be derived as the condition for the vacuum energy to be at minimum, **2** – to get rid of anomalous Bogolioubov terms, **3** – in the form of a Dyson equation for the fermion propagator, or, finally, **4** – as a Ward identity. Here we use **2**. We finally go through **step 3** and perform one last BV transformation,

$$R_{\theta_n} = \begin{bmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{bmatrix}. \tag{7}$$

The θ_n angles are to be found by imposing the vanishing of the anomalous terms in the Hamiltonian (see Chapter 3). A simple algebraic computation yields the following mass gap equations,

$$\begin{cases} (\ell\, m\cos\theta_0 + \sin\theta_2/\sqrt{2})\sin\theta_0 = 0,\ n=0\ , \\ \ell\, m\sin 2\theta_n - \sqrt{2n}\cos 2\theta_n = 0,\ n>0\ , \end{cases} \qquad \tan 2\theta_n = \frac{\sqrt{2n|eB|}}{m}$$

$$\cos\theta_n = \sqrt{\frac{E_n+m}{2E_n}},\quad \sin\theta_n = \sqrt{\frac{E_n-m}{2E_n}},\quad E_n = \sqrt{m^2+2n|eB|} \tag{8}$$

We obtain ${}_B\langle 0|\psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x})|0\rangle_B = -|eB|/2\pi$. The spontaneous breaking of the $U(2)$ flavour symmetry occurs even in the absence of any additional interaction between fermions. This is an inherent property of the 2+1 dimensional Dirac theory in an external magnetic field. In 3+1 dimensions, these very same 3-steps can be performed and the spontaneous breakdown in a magnetic field can take place only when an "effective" mass term ($m{\neq}0$) is generated [4].

2. A class of Hamiltonians

We now consider the simplest Hamiltonian containing the ladder-Dyson-Schwinger machinery for chiral symmetry. See Ref. [3], and references therein, for more complicated treatments. In any case, as most of the results presented here *do not depend on the choice of kernel*, this simple example will do. For the zero current quark masses we have

$$H = \int \mathrm{d}^3x\, q^+(x) \left(-\mathrm{i}\overrightarrow{\alpha}.\overrightarrow{\nabla}\right) q(x) + \int \frac{\mathrm{d}^3x, y}{2} J^a_\mu(x) K^{ab}_{\mu\nu}(x-y) J^b_\nu(y)$$

$$J^a_\mu(x) = \overline{q}(x)\gamma_\mu \frac{\lambda^a}{2} q(x),\ K^{ab}_{\mu\nu}(x-y) = \delta^{ab} K_{\mu\nu}(|\overrightarrow{x} - \overrightarrow{y}|). \tag{9}$$

This class of Hamiltonians has a rich structure enabling the study of a variety of hadronic phenomena controlled by global symmetries. **1** – it is chiral compliant: The fermions know about the kernel; **2** – it reproduces in a non-trivial manner the low energy properties of pion physics like, for instance, $\pi\pi$ scattering; **3** – it possesses the mechanism of pole-doubling in what concerns scalar decays.

2.1. More on BV transformations

As in the case of the Hamiltonian of Eq.(1), we look again for a BV transformation in order to obtain the new Fock space operators $\widetilde{\widehat{b}}$ and $\widetilde{\widehat{d}}$ from the old $\widehat{b}$ and $\widehat{d}$ operators,

$$\left[\begin{array}{c} \widetilde{\widehat{b}} \\ \widetilde{\widehat{d}}^+ \end{array}\right]_s = \left[\begin{array}{ll} \cos\phi & -\sin\phi M_{ss'} \\ \sin\phi M^\star_{ss'} & \cos\phi \end{array}\right] \left[\begin{array}{c} \widehat{b} \\ \widehat{d}^+ \end{array}\right]_{s'}, \tag{10}$$

where $$M_{ss'} = -\sqrt{8\pi} \sum_{m_l m_s} \left[\begin{array}{ccc} 1 & 1 & |0 \\ m_l & m_s & |0 \end{array}\right] \left[\begin{array}{ccc} 1/2 & 1/2 & |1 \\ s & s' & |m_s \end{array}\right] y_{1 m_l}(\theta, \phi)$$

The functions $\Phi(p)$ classify the infinite set of possible Fock spaces. $M_{ss'}(\theta, \phi)$ represent the $^3\mathrm{P}_0$ quark-antiquark pair. Then, requiring the invariance of $\Psi_{fc}(\overrightarrow{x})$ under the Fock space rotations is equivalent to the requirement of a counter-rotation of the spinors u and v,

$$\left[\begin{array}{c} u \\ v \end{array}\right] = \left[\begin{array}{ll} \cos\phi & -\sin\phi M^*_{ss'} \\ \sin\phi M_{ss'} & \cos\phi \end{array}\right] \left[\begin{array}{c} u \\ v \end{array}\right]. \tag{11}$$

The $\{u,v\}$ contain now the information on the angle $\phi(p)$.

3. *Chiral symmetry*

Consider the transformation $\Psi \to \exp\left\{-\mathrm{i}\alpha^a \frac{1}{2} T^a \gamma_5\right\}$. Then the Hamiltonian in Eq. (9) transforms, for non-zero current quark masses, like

$$H\,[m_q] \to H[m_q \cos\left(\frac{\alpha^2}{2}\right) - m_q \sin\left(\frac{\alpha^2}{2}\right) \mathrm{i}\gamma_5]. \tag{12}$$

H is chirally symmetric if $m_q = 0$. Now assume that ϕ exists and construct $Q_5^a = \int \mathrm{d}^3x\, \overline{\Psi}\gamma_0\gamma_5\Psi$. We obtain

$$Q_5^a = \int \mathrm{d}^3p \cos(2\phi) \left[\overrightarrow{p}.\overrightarrow{\sigma}\, \widehat{b}^+(p)\widehat{b}(p) + (\widehat{d}^+\widehat{d}) \right] + \sin(2\phi) \left[\underbrace{\mu_{ss'}\widehat{b}^+(p)\widehat{d}^+(-p) + (\widehat{d}\widehat{b})}_{\text{Anomalous terms}} \right]. \tag{13}$$

For $m_q = 0$, $[Q_5^a, H] = 0$. Thence, for an arbitrary ϕ, $Q_5^a|0> \neq 0$, so that Q_5^a, acting in the vacuum, *creates a state*, which will turn out to be the pion ($\mu_{ss'}$ is the spin wave function for a spin zero bound state, made out of two spin 1/2's). In general, the Hamiltonian of Eq. (9) can be written as

$$\widehat{H} = \widehat{H}_{\text{normal}}[\phi] + \widehat{H}_{\text{anomalous}}[\phi],\ \widehat{H}|0>= \widehat{H}_{\text{anomalous}}[\phi]|0> \neq 0, \tag{14}$$

so that we must find ϕ_0 such as to have $\widehat{H}_{\text{anomalous}}[\phi_0]|0>= 0$. **This is the mass gap equation**. It happens that in general we cannot *simultaneously* get rid of *both* the anomalous terms for $\widehat{H}$ and $\widehat{Q_5^a}$. $\widehat{Q_5^a}$ will remain anomalous: $\widehat{Q_5^a}|0> = |\pi>$. Because of $[H, Q_5] = 0$, π is *massless* . It turns out that we have several ways of arriving at the mass gap equation: **1** - $\widehat{H}_2$[anomalous] $= 0$, **2** - by minimization of $H_0 : \delta H_0/\delta\varphi = 0$, or **3** - by the use of the Ward identities. As a by product, we can obtain the renormalized fermion propagators.

4. Pion Salpeter amplitude

From the renormalized propagator, we can construct the Bethe-Salpeter equation for mesonic states. We can proceed via two identical formalisms: **1** - either work in the Dirac space or **2** - work in the spin representation. To go from the Dirac representation to the spin representation, it is sufficient to construct the spin wave functions like for instance, $\chi^{++}_{\alpha\beta}(k) = u_{s1;\alpha}(k)\Phi_{s1s2}(k)\overline{v}_{s2;\beta}(-k)$. We get

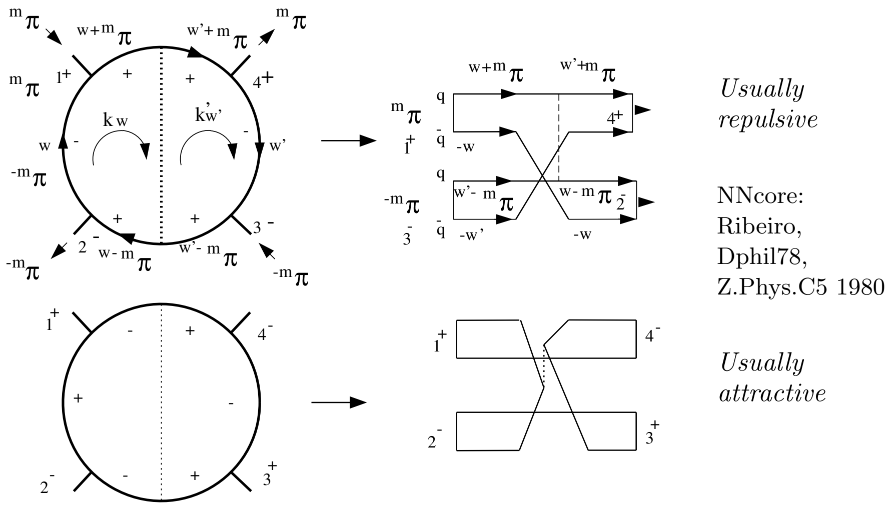
$$H|\Phi\rangle = \begin{bmatrix} H^{++} & H^{+-} \\ H^{-+} & H^{--} \end{bmatrix} \begin{bmatrix} \Phi^+ \\ \Phi^- \end{bmatrix} = m_\pi\, \sigma_3 \begin{bmatrix} \Phi^+ \\ \Phi^- \end{bmatrix}. \tag{15}$$

Therefore, in the space of pions we can write the Hamiltonian of Eq. (9) as,

$$H = \sigma_3 \begin{bmatrix} \Phi^+ \\ \Phi^- \end{bmatrix} m_\pi \left[\Phi^+,\ \Phi^-\right] \sigma_3 + \sigma_3 \begin{bmatrix} \Phi^- \\ \Phi^+ \end{bmatrix} m_\pi \left[\Phi^-,\ \Phi^+\right] \sigma_3 \tag{16}$$

5. The Weinberg formula for $\pi\pi$ scattering

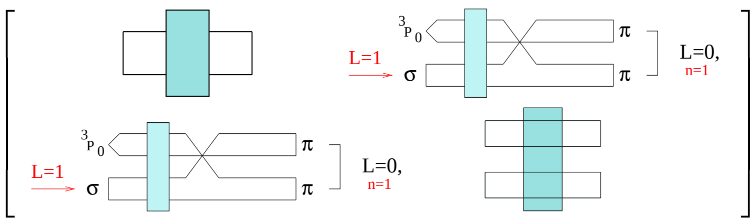
Below we depict two diagrams (among many others) contributing to $\pi - \pi$ scattering. Eqs. (15) and (16) are then used to map all these diagrams into a single pion Salpeter amplitude (see Refs. [5] for details).



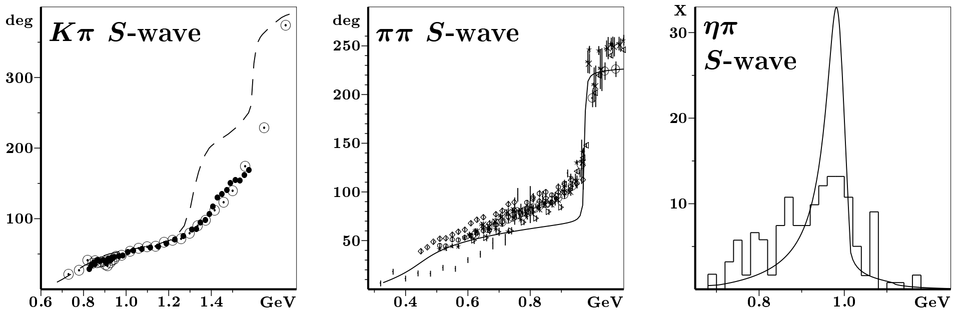
In the $\pi\pi$ case and *only in the chiral limit*, the above two types of diagrams *cancel exactly* when the relative momenta $\to$ 0. Outside the chiral limit they cease to cancel, yielding a net result proportional to m_π. Using $\{\Phi^{\pm} = \sin\varphi/a \pm a\Delta,\ a = \sqrt{2/3} f_\pi m_\pi\}$ and the normalization $\int(\Phi^{+^2} - \Phi^{-^2}) = 1$, we obtain the Weinberg results in the point-like limit ($\sin\varphi \to 1$): $\left\{-7/2\, m_\pi^2/f_\pi^2,\quad 1\, m_\pi^2/f_\pi^2\right\}$.

6. *Low-energy scalar resonances*

The $\pi\pi$ cancellation of diagrams in the chiral limit illustrates the fact that the mechanism for $q\bar{q}$ pair creation, ultimately responsible for the existence of hadronic coupled channels, has to be calculated in a chiral-consistent way once a quark kernel is given. This has been done in Ref. [6] for the case of vector mesons with the intermediate mesons in a relative *P-wave*. Here we outline the general consequences of $q\bar{q}$ pair creation in the scalar sector. We have

$H =$  (17)

In Eq. 17, the off-diagonal overlaps are evaluated using the graphical rules of Ref. [7]. They yield a potential well *just in the region of existence of the intermediate mesons relative s-wave* (sometimes) strong enough to support *another* pole *not* *present* in the bare case. In Z. Phys C 30 (1986) 615, we have obtained the results shown in the following diagrams



$K\pi$ invariant mass **$\pi\pi$ invariant mass** **$\eta\pi$ invariant mass**

κ pole: $727-i263$ MeV **σ pole: $470-i208$ MeV** **a_0 pole: $962-i28$ MeV**

f_0 pole: $994-i20$ MeV

Since then, this effect has been found in several approaches [8].

7. *Summary*

The role of chiral symmetry in hadronic physics is: **1** - To ensure the existence of a Goldstone pion. **2** - To ensure that any microscopic calculation of $\pi\pi$ scattering must get the Weinberg results, and more generally, to control pion mediated reactions. **3** - To ensure that hadronic wave functions cannot have a fixed number of valence quarks because it sets constraints between quark annihilation (and creation) and exchange diagrams. **4** - To fix the strength and form of mesonic coupled channels, responsible, among other effects, for the existence of light scalars.

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ULOGA KIRALNE SIMETRIJE U HADRONSKIM PROCESIMA

Raspravljamo ulogu kiralne simetrije u hadronskim procesima. Naglašavamo poništenje dijagrama u $\pi\pi$ raspršenju uzrokovano kiralnom simetrijom i njene posljedice u skalarnom sektoru.