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Gall Stereographic Projection and its Generalization

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ABSTRACT. Previous research has shown that for some equal-area, equidistant and conformal cylindrical projections the standard and secant parallels do not match. Furthermore, it was proved that there is no equal-area, equidistant, and conformal cylindrical projection for which the standard and secant parallels would coincide. In this paper, it is shown that there are map projections whose standard and secant parallels coincide. One of them is the Gall stereographic projection. Moreover, a generalization of the Gall stereographic projection is derived, which also has the same property. In addition, the shape of the rectangle in which the world map is displayed can be modified. For example, if we take the latitudes of the standard parallels to be ±62°.1640 we will get a map of the world in a perfect square, and if we take ±35°.0447 as standard parallels, we will get a map of the world in a rectangle whose sides correspond to the landscape A format.

Keywords: map projection, Gall stereographic projection, standard parallel, secant parallel.

1. Introduction

James Gall (1808–1895) was a Scottish clergyman, cartographer, publisher, sculptor, astronomer and author. In cartography he gave his name to three different map projections: Gall stereographic, Gall isographic, and Gall orthographic (Gall-Peters projection) (Freeman 1963, Snyder 1993). Gall presented those three projections to the British Association in 1855 (Gall 1855, 1871, 1885).

In this paper, we will research its stereographic projection and show that it has a special property: its standard and secant parallels coincide.

The normal aspect of cylindrical projection of a sphere of radius 1 is a mapping defined by the formulas

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$$x = n\lambda,\ \ y = y(\varphi), \tag{1}$$

where $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$, $n > 0$ is a constant, and the function $y = y(\varphi)$ is continuous, differentiable, monotone increasing and odd. As with any map projection, x and y are the coordinates of a point in a rectangular (mathematical, right-oriented) coordinate system in a plane. Anyone can see that this is a mapping to the plane, not to the surface of a cylinder. For such a mapping we have coefficients of the first differential fundamental form

$$E = \left(\frac{dy}{d\varphi}\right)^2,\ \ F = 0,\ G = n^2. \tag{2}$$

The local linear scale factors along a meridian and a parallel, respectively, are

$$h = h(\varphi) = \frac{dy}{d\varphi},\ \ k = k(\varphi) = \frac{n}{\cos\varphi}. \tag{3}$$

For the normal aspect cylindrical projection along the standard parallels to which the latitudes $\varphi = \pm\varphi_1$ correspond, it must be

$$h(\varphi_1) = k(\varphi_1) = 1, \tag{4}$$

or considering (3)

$$\frac{dy}{d\varphi}(\varphi_1) = 1 \tag{5}$$

and

$$\frac{n}{\cos\varphi_1} = 1 \text{ i.e. } n = \cos\varphi_1. \tag{6}$$

Thus, two parallels (φ_1 and $-\varphi_1$) are standard parallels of a normal aspect cylindrical projection if (5) and (6) hold. The distance between these two standard parallels on the map is $2y(\varphi_1)$. Let us fold this map into a cylinder surface and mark the radius of that cylinder with r.

The circumference of the circle that is the base of this cylinder will be $2r\pi$, and at the same time equal to the length of the equator, or any parallel in the plane of projection. This amounts to $n \cdot 2\pi$ because for normal aspect cylindrical projections $x = n\lambda$. Hence, it immediately follows that $r = n$. Let us denote by v the height above the equatorial plane in which the cylindrical surface intersects the sphere of radius 1. According to Pythagoras, we have

$$v = \sqrt{1 - r^2} = \sqrt{1 - n^2}. \qquad (7)$$

In previous papers (Lapaine 2022, 2023) it was shown that the following is not valid in general

$$y(\varphi_1) = v = \sqrt{1 - n^2}. \qquad (8)$$

Lapaine (2022) found that standard parallels after bending the map into a cylindrical surface generally cannot overlap with secant parallels in the normal aspect cylindrical projection. He demonstrated this on the examples of one equidistant, one equal-area and one conformal projection. Furthermore, Lapaine (2023) proved that there is no equal-area, equidistant, or conformal cylindrical projection for which the standard and secant parallels coincide. The question arises: is there a normal aspect cylindrical projection in which standard parallels after bending the map into a cylindrical surface can still overlap with secant parallels?

The answer to that question is yes. Namely, conditions (5), (6) and (8) must be met for cylindrical projection (1). From (6) we first conclude that it must be $0 < n < 1$, and then from (5) and (8) that the required function $y = y(\varphi)$ should have the following properties

$$\frac{dy}{d\varphi}(\varphi_1) = 1 \text{ and } y(\varphi_1) = \sin\varphi_1. \qquad (9)$$

2. Gall Stereographic Projection

Among his three projections, Gall preferred his stereographic one for general use (Fig. 1). It is a perspective projection of the globe onto a cylinder from a point on the equator of the globe opposite the meridian being projected. The cylinder is not tangent to the globe, but is secant, cutting the globe at 45° N and S, and giving the projection the same standard parallel as those of his other two projections. It is neither equal-area nor conformal, but its balance of distortion has led to its use in several atlases published in Great Britain, including some by the Bartholomew family (Snyder 1993).

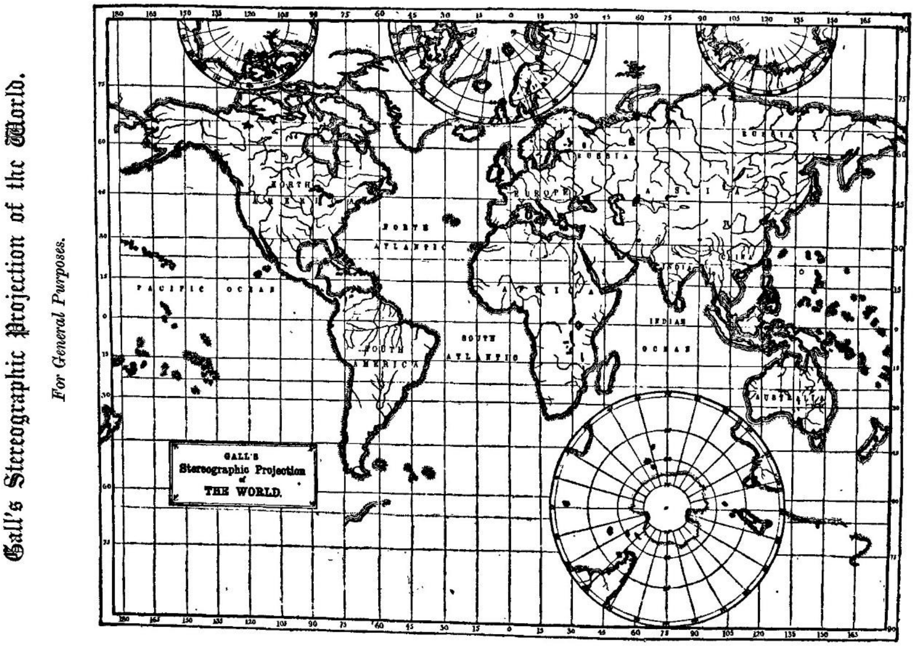

Fig. 1. *Map of the world by James Gall (1885).*

The equations of this projection for the sphere of radius 1 are:

$$x = \frac{\sqrt{2}}{2}\lambda, \;\; y = \left(1 + \frac{\sqrt{2}}{2}\right)\tan\frac{\varphi}{2}, \tag{10}$$

$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$. We have for that projection

$$h(\varphi) = \frac{dy}{d\varphi} = \frac{1+\frac{\sqrt{2}}{2}}{1+\cos\varphi} \tag{11}$$

and

$$k(\varphi) = \frac{n}{\cos\varphi} = \frac{\sqrt{2}}{2\cos\varphi}. \tag{12}$$

It is easy to see that it is

$$h\left(\pm\frac{\pi}{4}\right) = k\left(\pm\frac{\pi}{4}\right) = 1 \tag{13}$$

i.e. the parallels corresponding to latitudes $\pm\frac{\pi}{4}$ are the standard parallels of this projection. Moreover, according to (10) for Gall stereographic projections, the following holds

$$n = \frac{\sqrt{2}}{2} = \cos\left(\pm\frac{\pi}{4}\right) \tag{14}$$

and considering (10)

$$y\left(\pm\frac{\pi}{4}\right) = \pm\left(1+\frac{\sqrt{2}}{2}\right)\tan\frac{\pi}{8} = \pm\sin\frac{\pi}{4} = \pm\frac{\sqrt{2}}{2}. \tag{15}$$

Therefore, for the Gall stereographic projection and $\varphi_1 = \pm\frac{\pi}{4}$, conditions (5), (6) and (8) are fulfilled, i.e. standard parallels are also secant parallels.

3. Generalization of the Gall Stereographic Projection

In the previous section we saw that in the original Gall version, the parallels to which the latitudes $\frac{\pi}{4}$ and $-\frac{\pi}{4}$ correspond are standard and secant parallels. Let us show that by appropriate modification of the Gall stereographic projection a projection can be obtained for which the standard and secant parallels coincide for any predetermined value of latitude φ_1. For this purpose, we mark $n = \cos\varphi_1$ and look for the value of A such that the mapping given by formulas

$$x = n\lambda,\ \ y = A\tan\frac{\varphi}{2}, \tag{16}$$

$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$, has the required properties. Let us calculate first

$$h(\varphi) = \frac{dy}{d\varphi} = \frac{A}{1+\cos\varphi} \tag{17}$$

and

$$k(\varphi) = \frac{n}{\cos\varphi}, \tag{18}$$

and then

$$h(\varphi_1) = \frac{A}{1+\cos\varphi_1} = \frac{A}{1+n} \tag{19}$$

$$k(\varphi_1) = \frac{n}{\cos\varphi_1} = 1. \tag{20}$$

For a parallel corresponding to latitude φ_1 to be a standard one, it obviously needs $A = 1 + n$. Furthermore

$$y(\varphi_1) = (1+n)\tan\frac{\varphi_1}{2} = (1+n)\sqrt{\frac{1-\cos\varphi_1}{1+\cos\varphi_1}} = \sqrt{1-n^2} = \sin\varphi_1. \quad (21)$$

Thus, according to (9), the parallels which correspond to latitudes $\pm\varphi_1$ are also secant parallels. This way we proved that for any given latitude φ_1 a generalized Gall stereographic cylindrical projection

$$x = \lambda\cos\varphi_1,\ \ y = (1+\cos\varphi_1)\tan\frac{\varphi}{2}, \quad (22)$$

$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$, has standard parallels $\pm\varphi_1$ which are at the same time secant parallels.

Local linear scale factors along the meridians and parallels, respectively are

$$k(\varphi) = \frac{\cos\varphi_1}{\cos\varphi},\ h(\varphi) = \frac{1+\cos\varphi_1}{1+\cos\varphi}. \quad (23)$$

By choosing the standard parallels of a certain projection, we influence the distribution of distortions of that projection. This can be clearly seen from the formulas (23) for the generalized Gall stereographic projection. It is interesting that by the selection of standard parallels, the relationship of the sides of the map in the form of a rectangle can be indirectly changed. The length of any parallel in that projection is $2\pi\cos\varphi_1$, and the length of any meridian is $2(1+\cos\varphi_1)$. Let us denote by q the ratio of the length and height of the map made in the generalized Gall stereographic projection

$$q = \frac{2\pi\cos\varphi_1}{2(1+\cos\varphi_1)} = \frac{\pi}{2}\left(1-\tan^2\frac{\varphi_1}{2}\right). \quad (24)$$

In Table 1, we give the values of the ratio q of the length and height of the map made in the generalized Gall stereographic projection.

Table 1. *The ratio q of the length and height of the map made in the generalized Gall stereographic projection.*

φ_1	q
0°	1.57
30°	1.46
45°	1.30
60°	1.05

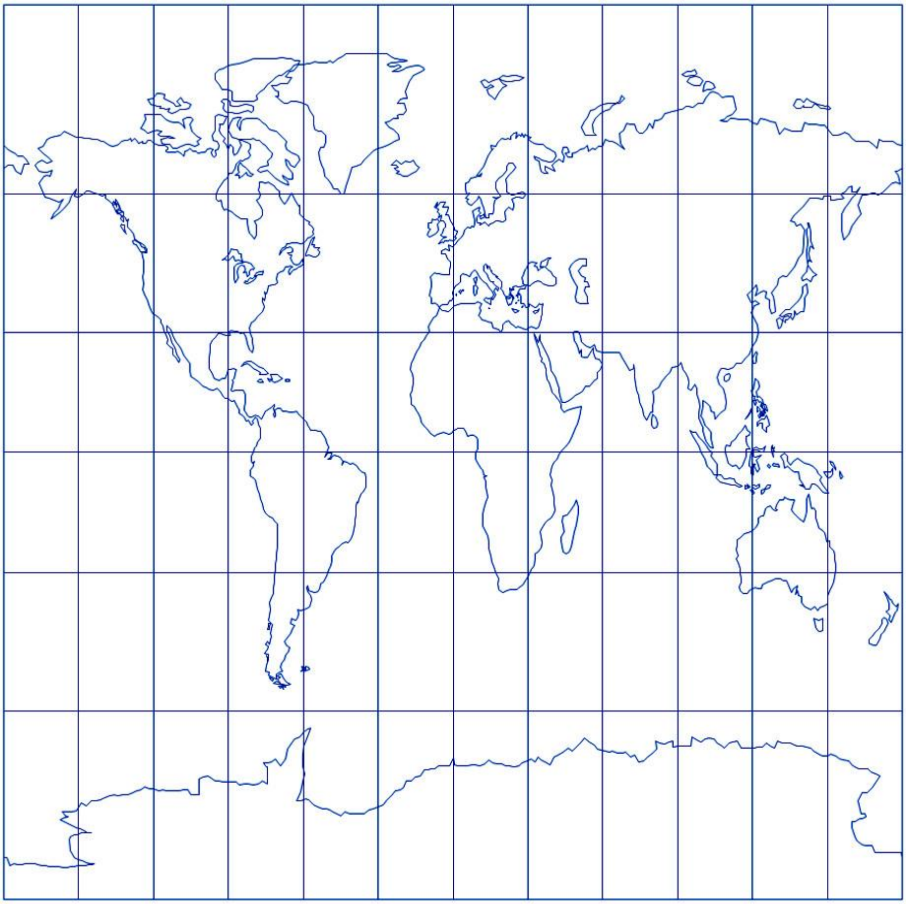

Fig. 2. *World map in a perfect square in the generalized Gall stereographic projection. Standard and secant parallels at latitudes ±62°.1640 coincide.*

It is interesting that a map made in the generalized Gall projection can have the shape of a perfect square. For this purpose, it is necessary to take $\varphi_1 = 62°.1640$, the value obtained when $q = 1$ is put in (24). A world map in a perfect square produced in the generalized Gall stereographic projection is shown in Fig. 2.

If, on the other hand, we want a map in that projection that is best adapted to the A4 landscape paper format, then $q = \frac{297}{210} = \sqrt{2} = 1.4143$, so from equation (24) $\varphi_1 = 35°.0447$ can be calculated (Fig. 3).

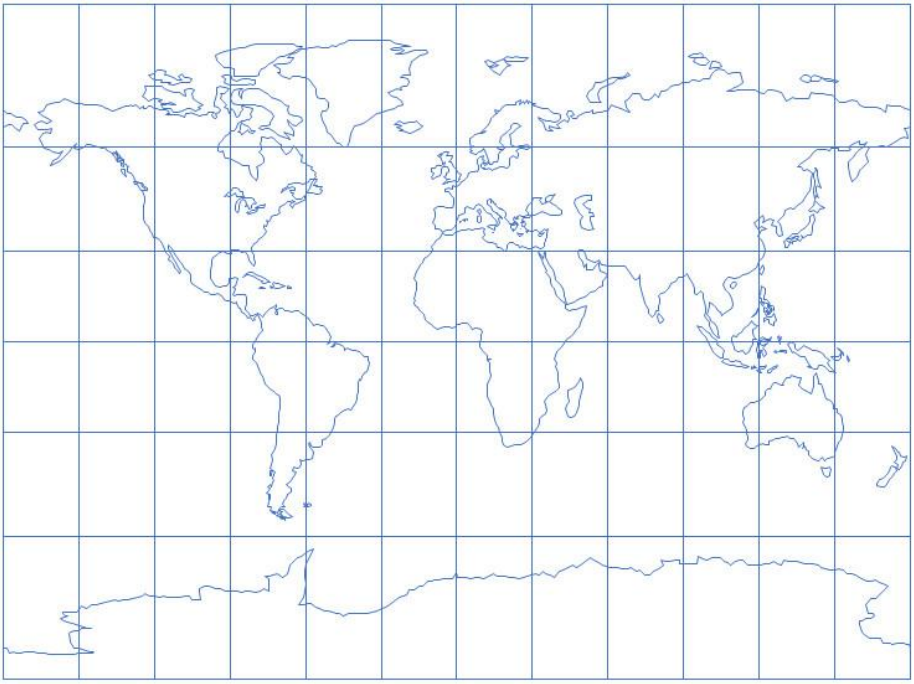

Fig. 3. *World map in a format landscape rectangle in the generalized Gall stereographic projection. Standard and secant parallels at latitudes ±35°.0447 coincide.*

In the theory of map projections, some special cases of the generalized Gall projection are known. Carl Braun (1831−1907) proposed the stereographic cylindrical projection

$$x = R\lambda,\ \ y = 2R\tan\frac{\varphi}{2}, \tag{25}$$

where R is a radius of the sphere (Braun 1868). It is easy to see that this projection for $R = 1$ is a special case of the generalized Gall projection if we put $\cos\varphi_1 = 1$, that is, $\varphi_1 = 0$ in equations (22). It could also be said that the Gall stereographic projection is a generalization of the Braun projection.

In 1929, V. A. Kamenetskiy created an overview map of the population density of the USSR, taking $\varphi_1 = 55°$ (Bugaevsky and Snyder 1995).

Several maps in a volume of the B.S.A.M. (*Bol'shoy sovetskij atlas mira*, Great Soviet World Atlas, 1937) were produced using $\varphi_1 = 30°$. The latter is known as the BSAM cylindrical projection (Bugaevsky and Snyder 1995).

4. Conclusion

Standard and secant parallels are often considered identical, but some papers demonstrated that widely accepted facts about these parallels are wrong and need to be revised (Lapaine 2022, 2023). To prevent misunderstandings in the theory of map projections and their teaching, it was recommended to avoid the use of developable surfaces as intermediate surfaces, and thus secant parallels and projections. This requires a critical approach to the established customs in teaching and researching map projections.

Only for some cylindrical projections the distances between the secant parallels on the sphere and the standard parallels in the projection plane are equal. This applies, for example, to the Gall stereographic projection and some modifications of it, such as those presented in this paper.

By choosing standard parallels, we influence the distribution of distortions, but also the shape of the rectangle in which the world map is displayed. For example, if we take the latitudes of the standard parallels to be ±62°.1640, we will get a map of the world in a perfect square.

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Gallova stereografska projekcija i njezina generalizacija

SAŽETAK. Prethodna istraživanja pokazala su da se za neke ekvivalentne, ekvidistantne i konformne cilindrične projekcije standardne i presječne paralele ne poklapaju. Nakon toga je dokazano da ne postoji ni jedna ekvivalentna, ekvidistantna i konformna cilindrična projekcija za koju bi se standardne i presječne paralele podudarale. U ovom je radu pokazano da ipak postoje kartografske projekcije čije se standardne i presječne paralele poklapaju. Jedna od njih je Gallova stereografska projekcija. Štoviše, izvedena je generalizacija te projekcije, koja ima isto svojstvo. Osim toga, oblik pravokutnika u kojem je prikazana karta svijeta može se mijenjati. Na primjer, ako uzmemo da su geografske širine standardnih paralela ±62°.1640, dobit ćemo kartu svijeta u kvadratu, a ako uzmemo za standardne paralele ±35°.0447, dobit ćemo kartu svijeta u pravokutniku čije stranice odgovaraju položenom formatu A.

Ključne riječi: *kartografska projekcija, Gallova stereografska projekcija, standardna paralela, presječna paralela.*

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