THE SURJECTIVITY AND THE CONTINUITY OF DEFINABLE FUNCTIONS IN SOME DEFINABLY COMPLETE LOCALLY O-MINIMAL EXPANSIONS AND THE GROTHENDIECK RING OF ALMOST O-MINIMAL STRUCTURES

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Abstract. In this paper, we first show that in a definably complete locally o-minimal expansion of an ordered abelian group $(M, <, +, 0, ...)$ and for a definable subset $X \subseteq M^n$ which is closed and bounded in the last coordinate such that the set $\pi_{n-1}(X)$ is open, the mapping $\pi_{n-1}$ is surjective from $X$ to $M^{n-1}$, where $\pi_{n-1}$ denotes the coordinate projection onto the first $n - 1$ coordinates. Afterwards, we state some of its consequences. Also we show that the Grothendieck ring of an almost o-minimal expansion of an ordered divisible abelian group which is not o-minimal is null. Finally, we study the continuity of the derivative of a given definable function in some ordered structures.

1. Introduction

A locally o-minimal structure $\mathcal{M} = (M, <, ...)$ was first introduced in [10] as a local counterpart of an o-minimal structure. The coordinate projection $\pi_{n-1}$ onto the first $n - 1$ coordinates is a surjective map from $M^n$ to the set $M^{n-1}$. The natural question is that if this map remains surjective from a subset $X \subseteq M^n$ to the set $M^{n-1}$. In this paper, we give a positive answer to this question for a subset which is closed and bounded in the last coordinate such that the set $\pi_{n-1}(X)$ is open and that the subset $X$ is definable in a definably complete locally o-minimal expansion of an ordered abelian group $\mathcal{M} = (M, <, ...)$ to deduce the unboundedness of such subset (see Corollary 3.4 below).

The Grothendieck ring of a model-theoretical structure is built up as a quotient of the definable sets by definable bijections (see below).
In [1] and [11] the following explicit calculations of Grothendieck rings
(denoted by \( K_0 \)) of fields are made: \( K_0(\mathbb{R}, <, \text{ring}) \) is isomorphic to \( \mathbb{Z} \), but
\( K_0(\mathbb{Q}_p, \text{ring}) \) is trivial, where \( p \) is a prime number, \( \mathbb{Q}_p \) is the \( p \)-adic number
field and \( \text{ring} \) is the language \((+, -, 0, 1)\).

By [8], the Grothendieck ring of a structure \( M, K_0(M) \) is nontrivial if
and only if there is no definable set \( A \subseteq M, a \in A \) and an injective definable
map from \( A \) onto \( A \setminus \{a\} \).

In Section 4, we prove the triviality of the Grothendieck ring for an al-
most o-minimal expansion of an ordered divisible abelian group which is not
o-minimal. Finally, we prove that if a definable function in a definably com-
plete locally o-minimal expansion of a densely linearly ordered abelian group
\( M = (M, <, +, 0, ...) \) on an open interval satisfies the intermediate value prop-
erty, then this function is continuous on this whole interval to deduce that a
definable derivable function in an o-minimal expansion of an ordered field is
of class \( C^1 \).

2. Preliminaries

“Definable” will always mean “definable with parameters”.

We recall that a densely linearly ordered set without endpoints \( M =
(M, <, ...) \) is o-minimal if every definable subset \( X \) of \( M \) is a finite union of
points and open intervals.

**Definition 2.1.** A densely linearly ordered structure without endpoints
\( M = (M, <, ...) \) is locally o-minimal if for every definable subset \( X \) of \( M \) and
for every point \( a \in M \) there exists an open interval \( I \) containing the point
\( a \) such that \( X \cap I \) is a finite union of points and open intervals. It is called
almost o-minimal if any bounded definable set in \( M \) is a finite union of points
and open intervals.

**Example 2.2.** Every o-minimal structure is locally and almost o-minimal.

**Definition 2.3.** An expansion of a densely linearly ordered set without
endpoints \( M = (M, <, ...) \) is definably complete if any definable subset \( X \) of
\( M \) has the supremum and infimum in \( M \cup \{\pm \infty\} \).

**Example 2.4.** Every expansion of \( (\mathbb{R}, <) \) is definably complete.

It is well known thanks to [9, Corollary 1.5] that the definable completeness
is equivalent to \( M \) being definably connected, and also with the validity
of the intermediate value theorem for one variable definable continuous func-
tions.

**Definition 2.5.** Let \( M = (M, <, ...) \) be an expansion of a densely linearly
ordered set without endpoints. A subset \( X \) of \( M^{n+1} \) is called bounded in the
last coordinate if there exists a bounded open interval \( I \) such that \( X \subseteq M^n \times I \).
DEFINITION 2.6. An expansion of a densely linearly ordered group without endpoints \( \mathcal{M} = (M, <, ...) \) has definable bounded multiplication compatible to + if there exist an element \( 1 \in M \) and a map \( \cdot : M \times M \to M \) such that
1. The tuple \( (M, <, 0, 1, +, \cdot) \) is an ordered field.
2. For any bounded open interval \( I \), the restriction \( \cdot|I \times I \) of the product \( \cdot \) to \( I \times I \) is definable in \( M \).

3. SURJECTIVITY OF THE COORDINATE PROJECTION IN A DEFINABLY COMPLETE LOCALLY O-MINIMAL EXPANSION WITHOUT ENDPOINTS OF A DENSELY LINEARLY ORDERED ABELIAN GROUP

In this section, we consider a definably complete locally o-minimal expansion of a densely linearly ordered abelian group \( \mathcal{M} = (M, <, +, 0, ...) \).

Let \( \pi_{n-1} : M^n \to M^{n-1} \) denotes the projection onto the first \( n-1 \) coordinates and let \( X \subseteq M^n \) be a definable subset.

**Lemma 3.1.** Consider a definably complete locally o-minimal expansion of a densely linearly ordered abelian group \( \mathcal{M} = (M, <, +, 0, ...) \). Let \( X \) be a definable subset of \( M^n \) which is not closed. Take a point \( a \in \bar{X} \setminus X \). There exist a small positive \( \epsilon \) and a definable continuous map \( \gamma : [0, \epsilon] \to X \) such that \( \lim_{t \to 0^+} \gamma(t) = a \).

**Proof.** By [5, Corollary 3.2], we know that this lemma holds true for a DCULOAS structure; by following that proof literally, we only use Lemma 3.1 (definable choice), Proposition 2.2(7) and Lemma 2.3 of [5]. By [5, Lemma 3.1], Lemma 3.1 holds true in a definably complete expansion of a densely linearly ordered abelian group. According to [4], Proposition 2.2(7) and Lemma 2.3 of [5] hold true for all definably complete locally o-minimal structures satisfying the property (a). Finally, any definably complete locally o-minimal structure satisfies the property (a) by [6, Theorem 2.5].

**Theorem 3.2.** Let \( X \subseteq M^n \) be a definable subset in a definably complete locally o-minimal expansion of a densely linearly ordered abelian group \( \mathcal{M} = (M, <, +, 0, ...) \) which is closed and bounded in the last coordinate such that the set \( \pi_{n-1}(X) \) is open. Then the mapping \( \pi_{n-1} \) is surjective from \( X \) to \( M^{n-1} \).

**Proof.** Assume for contradiction that we can take a point \( x \) in the frontier of \( \pi_{n-1}(X) \). By Lemma 3.1, there exists a continuous curve \( \gamma : (0, \epsilon) \to \pi_{n-1}(X) \) definable in \( \mathcal{M} \) such that \( \lim_{t \to 0^+} \gamma(t) = x \). Define \( f_n : (0, \epsilon) \to \pi_{n-1}(X) \) (\( \pi_{-1} \) denotes the projection onto the last coordinate) by \( f_n(t) = \sup \{ y \in M; (\gamma(t), y) \in X \} \). The set \( \{(t, y) \in [0, \epsilon] \times M; (\gamma(t), y) \in X \} \) is definable because \( X \) is definable. Therefore, as \( X \) is bounded in the last coordinate, the function \( f_n \) is definable in \( \mathcal{M} \). We may assume that \( f_n \) is continuous and monotone by the monotonicity theorem ([6, Theorem 5.1]) and...
by taking a sufficiently small $\epsilon > 0$ if necessary. The limit $y = \lim_{t \to 0^+} f_u(t)$ exists because the definable function $f_u$ is bounded and monotone. We have $(x, y) \in X$ because $X$ is closed in $M^n$, so $x \in \pi_{n-1}(X)$, a contradiction. So $\pi_{n-1}(X)$ is closed in $M^{n-1}$. By [9, Corollary 1.5], $M^{n-1}$ is a definably connected set, and we deduce that $\pi_{n-1}(X) = M^{n-1}$.

**Remark 3.3.** Theorem 3.2 still holds if we replace the assumption that $\pi_{n-1}(X)$ is open with that for all $x \in X$, there exists an open box $B$ in $M^n$ containing the point $x$ such that $B \cap X$ is the graph of a continuous map defined on $\pi_{n-1}(B)$ (i.e. $X$ is locally the graph of a continuous map). In fact, let $a$ be in $\pi_{n-1}(X)$ and fix $b$ such that $(a, b)$ is in $X$. By assumption, there is an open box $B$ such that $(a, b)$ is in $B$ and $B \cap X$ is the graph of a continuous map defined on $\pi_{n-1}(B)$. In particular, $\pi_{n-1}(B)$ is in $\pi_{n-1}(X)$, and contains $a$, and $\pi_{n-1}(X)$ is also open (as $\pi_{n-1}(B)$ is an open box). So every point in $\pi_{n-1}(X)$ is contained in an open set that is contained in $\pi_{n-1}(X)$, so $\pi_{n-1}(X)$ is open.

**Corollary 3.4.** If $X \subseteq M^n$ is a definable subset as in Theorem 3.2, then $X$ is unbounded.

**Proof.** Assume that $X$ is closed and bounded, so $X$ is bounded in the last coordinate. By Theorem 3.2, we deduce that $\pi_{n-1}(X) = M^{n-1}$. If $X$ is bounded, then by [9, Lemma 1.7], the set $M^{n-1}$ is bounded, which is a contradiction.

4. **The Grothendieck ring of an almost o-minimal expansion of an ordered divisible abelian group**

We begin this section by recalling the notion of the Grothendieck ring of a given structure.

**Definition 4.1.** Let $\mathcal{M} = (M, <, \ldots)$ be a structure. The notation $\text{Defn}(\mathcal{M})$ denotes the family of all definable subsets of $M^n$. The Grothendieck group of a structure $\mathcal{M}$ is the abelian group $K_0(\mathcal{M})$ generated by symbols $[X]$, where $X \in \text{Defn}(\mathcal{M})$ with the relations $[X] = [Y]$ if $X$ and $Y$ are definably isomorphic, and $[U \cup V] = [U] + [V]$ where $U, V \in \text{Defn}(\mathcal{M})$, and $U \cap V = \emptyset$. The ring structure is defined by $[X][Y] = [X \times Y]$, where $X \times Y$ is the Cartesian product of definable sets. The ring $K_0(\mathcal{M})$ with this multiplication is called Grothendieck ring of the structure $\mathcal{M}$.

**Proposition 4.2.** Consider an almost o-minimal expansion $\mathcal{M}$ of an ordered divisible abelian group whose underlying set is $M$, and assume that this expansion is not o-minimal. Then the Grothendieck ring of this expansion is the zero ring $\{0\}$.

**Proof.** Let $\mathcal{M}$ be such a structure. By [3, Lemma 2.31] there exists an unbounded discrete $\mathcal{M}$-definable set $D$. Without loss of generality, we may
assume that \( D \cap [0, \infty) \) is an infinite set, so \( D' := D \cap [0, \infty) \) is an infinite discrete definable set. By [3, Lemma 2.18], the definable set \( D' \) is closed.

By [3, Corollary 4.6], the structure \( \mathcal{M} \) is definably complete.

As the structure \( \mathcal{M} \) is definably complete, the set \( D' \) admits an infimum in \( \mathcal{M} \) which we denote by \( m \). Take a sufficiently bounded open interval \( I \) containing the point \( m \). The set \( I \cap D' \) is finite, so \( m \in D' \), otherwise if \( m \notin D' \), there exists the smallest element \( n \in I \cap D' \) with \( m \neq n \). Since \( m = \inf D', m < n \). There are no elements of \( G' \) between \( m \) and \( n \) because \( m \) and \( n \) are contained in the open interval \( I \). It contradicts the fact that \( m \) is the infimum of \( D' \). The successor map \( s_{D'} : D' \setminus \{m\} \to D' \) defined in [2, Definition 3] is a definable bijection.

**Problem 4.1.** Let \( \mathcal{M} \) be the structure as in Proposition 4.2. By [3, Theorem 2.13], there exists an o-minimal expansion \( \mathcal{R} \) of the ordered group having the same underlying set \( \mathcal{M} \) such that any definable set in \( \mathcal{R} \) is definable in \( \mathcal{M} \). By [7, Theorem 1], the Grothendieck ring of the structure \( \mathcal{R} \) is isomorphic to the ring \( \mathbb{Z}[T]/(T^2 + T) \) because there is no definable bijection in \( \mathcal{R} \) between a bounded interval and an unbounded one, and this structure is the reduct of the structure \( \mathcal{M} \) whose Grothendieck ring is null by Proposition 4.2. The Grothendieck ring of the structure \( \mathcal{R} \) is contained in that of the structure \( \mathcal{M} \). Here the open question rises: Under what additional conditions do we have this inclusion?

5. The continuity of the derivative of a definable function in some ordered expansions of a given field

We know by [9, Corollary 1.5] that a continuous definable function in a definably complete structure satisfies the intermediate value property. Fortunately the converse of the intermediate value property in a definably complete locally o-minimal expansion of a densely linearly ordered abelian group holds true, which is the aim of the following proposition.

**Proposition 5.1.** Let \( \mathcal{M} = (M, <, ...) \) be a definably complete locally o-minimal expansion of a densely linearly ordered abelian group \( \mathcal{M} = (M, <, +, 0, ...) \), \( I \) be an open interval of \( \mathcal{M} \), and \( f : I \to M \) be a definable function. Suppose that for all \( a, b \in I \), and all \( y \) between \( f(a) \) and \( f(b) \), there exists \( x \in [a, b] \) such that \( f(x) = y \) (i.e., \( f \) satisfies the intermediate value property). Then \( f \) is continuous on \( I \).

**Proof.** We demonstrate this proposition by contraposition. By Theorem 2.3 of [6], there exists a mutually disjoint definable partition \( I = X_d \cup X_o \cup X_+ \cup X_- \) satisfying the following conditions:

1. the definable set \( X_d \) is discrete and closed;
2. the definable set \( X_o \) is open and \( f \) is locally constant on \( X_o \);
(3) the definable set $X_+$ is open and $f$ is locally strictly increasing and continuous on $X_+$.

(4) the definable set $X_-$ is open and $f$ is locally strictly decreasing and continuous on $X_-.$

Let $c$ be a point at which $f$ is discontinuous. We have $c \in X_d.$ Take $a, b \in I$ sufficiently close to $c$ such that $a < c < b.$ By local o-minimality, the interval $[a, c)$ is contained exactly in one of $X_c, X_+$ and $X_d.$ It is the same for the interval $(c, b].$ By definable completeness and uniform monotonicity of the functions $f|_{[a,c)}$ and $f|_{(c,b]}$, the left/right limits $f_-(c) := \lim_{x \to c^-} f(x)$, $f_+(c) := \lim_{x \to c^+} f(x)$ exist in $M \cup \{\pm \infty\}$. Since $f$ is discontinuous at $c$, we have three cases.

Case 1. $f_-(c) = f(c)$ and $f_+(c) \neq f(c)$. 

We consider the case in which $f(c) < f_+(c).$ The proof is similar when $f_+(c) < f(c).$ We take $y$ between $f(c)$ and $f_+(c).$ Since $f(c) < f_+(c)$ and $(M, <)$ is a densely linearly ordered set without endpoints, we can take such $y$ (even when $f_+(c) = +\infty$). When $f_+(c) = +\infty$, the restriction of $f$ to $(c, b]$ is strictly decreasing and continuous by the assumption. If we retake $b$ sufficiently close to $c$, $f(b) > y$. We have $y \notin f((c, b))$ and $y < f(b)$ in this case. When $f_+(c) \in M$, the function given by

$$g(x) = \begin{cases} f(x) & \text{if } c < x \leq b \\ f_+(c) & \text{if } x = c \end{cases}$$

is continuous. Take $\alpha, \beta$ in $M$ so that $\alpha < f_+(c) < \beta$ and $y < \alpha$. It is possible because $(M, <)$ is a densely linearly ordered set without endpoints. If we retake $b$ sufficiently close to $c$, we have $g((c, b]) \subseteq (\alpha, \beta)$ because $g$ is continuous at $c$. In particular, $f((c, b])$ does not contain the point $y$ and $y < f(b)$.

In both cases, we have $y \notin f((c, b])$ and $y < f(b)$.

Take $\alpha', \beta'$ in $M$ so that $\alpha < f(c) < \beta'$ and $y > \beta'$. Because the restriction of $f$ to $[a, c]$ is continuous at $c$, if we retake the point $a$ closer to $c$, we have $f([a, c]) \subseteq (\alpha', \beta')$. It implies that $f([a, c])$ does not contain the point $y$ and $y > f(a)$. Consequently, we get $y \notin f([a, b])$ and $f(a) < y < f(b)$.

Case 2. $f_+(c) = f(c)$ and $f_-(c) \neq f(c)$. Similar to Case 1.

Case 3. $f_+(c) \neq f(c)$ and $f_-(c) \neq f(c)$.

a) Either $f(c) < f_+(c)$. When $f_+(c) \in M$, take $y \in M$ such that $f(c) < y < f_+(c).$ The function $g: [c, b] \to M$ given by

$$g(x) = \begin{cases} f(x) & \text{if } c < x \leq b \\ f_+(c) & \text{if } x = c \end{cases}$$

is continuous. Take $\alpha, \beta$ in $M$ so that $\alpha < f_+(c) < \beta$ and $y < \alpha$. If we retake $b$ sufficiently close to $c$, we have $g([c, b]) \subseteq (\alpha, \beta)$ because $g$ is continuous at $c$. In particular, $f((c, b])$ does not contain the point $y$ and $y < f(b)$.
Set \( a = c \). We have \( f([a, b]) = f(c) \cup f((c, b]) \). We get \( y \notin f([a, b]) \) and \( f(a) = f(c) < y < f(b) \).

When \( f_+(c) = +\infty \). Let \( f(c) < y \) and \( y > 0 \), the restriction of \( f \) to \((c, b)\) is strictly decreasing and continuous. If we retake \( b \) sufficiently close to \( c \), \( y < f(b) \). If \( y \in f((c, b]), y = f(d) \) where \( c < d < b \). As \( f \) is strictly decreasing, \( f(b) < f(d) = y \), which is a contradiction. Set \( a = c \), \( f([a, b]) = f(c) \cup f((c, b]) \). We get \( y \notin f([a, b]) \).

b) Or \( f(c) > f_+(c) \). When \( f_+(c) \in M \), the proof is similar to Case 3(a). When \( f_+(c) = -\infty \), let \( y < 0 < f(c) \). If we retake \( b \) sufficiently close to \( c \), \( f(b) < y < f(c) \). The restriction of \( f \) to \((c, b)\) is strictly increasing and continuous, if \( y \in f((c, b]), y = f(d) \) where \( c < d < b \), we have \( y = f(d) < f(b) \) which is absurd. Set \( a = c \), \( f([a, b]) = f(c) \cup f((c, b]) \). We get \( y \notin f([a, b]) \).

c) Or \( f(c) < f_-(c) \). When \( f_-(c) \in M \), then take \( y \in M \) such that \( f(c) < y < f_-(c) \). The function \( g : [b, c] \to M \) given by

\[
g(x) = \begin{cases} f(x) & \text{if } b \leq x < c \\ f_-(c) & \text{if } x = c 
\end{cases}
\]

is continuous. Take \( \alpha, \beta \) in \( M \) so that \( \alpha < f_-(c) < \beta \) and \( y < \alpha \). If we retake \( b \) sufficiently close to \( c \), we have \( g([b, c]) \subseteq (\alpha, \beta) \) because \( g \) is continuous at \( c \). In particular, \( f([b, c]) \) does not contain the point \( y \) and \( y < f(b) \).

Set \( a = c \). We have \( f([b, a]) = f(c) \cup f([b, c]) \). We get \( y \notin f([b, a]) \) and \( f(a) = f(c) < y < f(b) \).

When \( f(c) < f_-(c) = +\infty \). If we retake \( b \) sufficiently close to \( c \), \( f(c) < y \) with \( y > 0 \). We have \( f(b) > y \). The restriction of \( f \) to \([b, c)\) is strictly increasing and continuous, if \( y \in f([b, c)), y = f(d) \) where \( b < d < c \), we have \( f(b) < y = f(d) \) which is absurd. Set \( a = c \), \( f([b, a]) = f(c) \cup f([b, c]) \). We get \( y \notin f([b, a]) \).

d) Or \( f(c) > f_-(c) \). When \( f_-(c) \in M \), the proof is similar to Case 3(c). When \( f_-(c) = -\infty \), if we retake \( b \) sufficiently close to \( c \), \( f(c) > y \) and \( f(b) < y < 0 \). The restriction of \( f \) to \([b, c)\) is strictly decreasing and continuous. If \( y \in f([b, c)), y = f(d) \) where \( b < d < c \), we have \( y = f(d) < f(b) \), which is absurd. Set \( a = c \), \( f([b, a]) = f(c) \cup f([b, c]) \). We get \( y \notin f([b, a]) \).

\[ \Box \]

**Corollary 5.2.** Let \( \mathcal{R} = (R, <, +, \cdot, - \ldots) \) be an \( o \)-minimal expansion of an ordered field \( R \), \( I \) be an open interval in \( R \), and \( f : I \to R \) be a definable derivable function. Then this function is of class \( C^1 \) on \( I \).

**Proof.** Darboux’s theorem for definable functions holds true. In fact, we can prove it by following the classical proof in real analysis of Darboux’s theorem, as the Corollary (Max-min theorem) in [9] holds true for a definably complete structure. Therefore, \( f' \) satisfies all assumptions of Proposition 5.1, and by applying this proposition, we get that the function \( f' \) is continuous on the interval \( I \), so \( f \) is of class \( C^1 \) on \( I \). \[ \Box \]
We end this paper by concluding that the converse of [6, Lemma 3.7] holds true under the local o-minimality assumption.

**Corollary 5.3.** Consider a locally o-minimal expansion $\mathcal{R} = (\mathbb{R}, <, +, 0, \ldots)$ of the ordered group of reals having definable bounded multiplication compatible to $+$. Let $I$ be a closed and bounded interval and $f : I \to \mathbb{R}$ be a definable function. Then $f$ is a $C^1$ function if and only if its derivative is a definable function.

**Proof.** If $f$ is a $C^1$ function, then by [6, Lemma 3.7], its derivative $f'$ is a definable function. Conversely, if $f'$ is a definable function on a closed bounded interval $I$, by Darboux’s theorem, $f'(I)$ is an interval and therefore $f'$ satisfies the intermediate value property. By Proposition 5.1, $f'$ is continuous on $I$. $\square$

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**References**

Surjektivnost i neprekidnost definabilnih funkcija u nekim definabilnim potpunim lokalno o-minimalnim proširenjima i Grothendieckovom prstenu gotovo o-minimalnih struktura

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Sažetak. U ovom članku najprije pokazujemo da u definabilnom potpunom lokalno o-minimalnom proširenju uređene abelove grupe $(M,<,+,0,...)$ i za definabilni podskup $X \subseteq M^n$ koji je zatvoren i ograničen u zadnjoj koordinati tako da je skup $\pi_{n-1}(X)$ otvoren, preslikavanje $\pi_{n-1}$ je surjekcija sa $X$ u $M^{n-1}$, gdje $\pi_{n-1}$ označava koordinatnu projekciju na prvi $n - 1$ koordinata. Nakon toga navodimo neke posljedice. Također pokazujemo da Grothendieckov prsten gotovo o-minimalnog proširenja uređene djeljive abelove grupe koja nije o-minimalna je nul-prsten. Konačno, proučavamo neprekidnost derivacije dane definabilne funkcije u nekim uređenim strukturama.

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